THE SINH^{-1} TRANSFORMATION IN CARDINAL APPROXIMATION

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Abstract. This paper discusses the Sinc transformation $x \to \phi(x) = \sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$, yielding a new class of formulas for approximation and numerical integration on $(-\infty, \infty)$. Errors decay like $O(\exp(-cN^{1/2}))$ for functions f that are analytic and bounded in the region enclosed by the two branches of the hyperbola $\{z = x + iy : y^2/\sin^2 d - x^2/\cos^2 d = 1\}$, $d < \pi/2$, and for which f has $O(|x|^{-\alpha})$ decay as $|x| \to \infty$ for interpolation (resp. $O(|x|^{-\alpha-1})$ decay for quadrature), where $\alpha > 0$. If f decays like $\exp(-\alpha|x|)$, errors decay at the faster rate $O(\exp(-cN/\log N))$. A significant practical value is in the computation of solutions of scattering problems, where integrals containing the Green's function $(4\pi|\vec{r} - \vec{r}\,'|)^{-1} \exp(ik|\vec{r} - \vec{r}|)$ must be evaluated. The transformation also yields a new n-point linear approximate of $|x|^\alpha$ on $[-1, 1]$, $0 < \alpha_0 \le \alpha \le \alpha_1$, for which the error decreases at a uniform $O(\exp(-\pi N \log N))$ rate.

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1. Introduction.

Let f be a function in $L^2(\mathbb{R})$ and analytic in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$. Whittaker's cardinal function $C(f, h)$ is defined by

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) S(k,h)(x),$$

whenever this series converges. Here h is a positive constant, and

$$S(k,h)(x) = \frac{\sin[(\pi/h)(x - kh)]}{(\pi/h)(x - kh)}.$$

The function $C(f, h)$ was introduced by E. T. Whittaker [10] and studied extensively by J. M. Whittaker [11], [12]. The use of $C(f, h)$ and the truncated series $C_N(f, h)$ to approximate a function f, its integral $\int_{-\infty}^{\infty} f(x)\,dx$ and other functions and quantities related to f, was studied by F. Stenger and others (see [7] for a survey). Many formulas with explicit error bounds are known and collectively referred to as Sinc methods.

Usually, the error decays like $O(\exp(-cN^{1/2}))$ as the number of points N goes to infinity. If f is defined on a domain other than a neighborhood of the real line, application of a conformal map will transform the problem into the original form, providing corresponding formulas for other curves.

The present paper investigates a particular transform which has the following properties. In the original Sinc formulas, f must be analytic in a strip along R and decay to zero exponentially in order to achieve the asymptotic error bound $O(\exp(-cN^{1/2}))$. The $\sinh^{-1}$ transformation requires only polynomial decay, but a larger region of analyticity for the same bound. If f decays exponentially and also has a larger region of analyticity, one can achieve the better asymptotic bound $O(\exp(-cN/\log N))$.

A particular application for which this method is very suitable is the numerical solution of integral equations containing a kernel such as $(4\pi|\vec{r} - \vec{r}\,'|)^{-1} \exp(ik|\vec{r} - \vec{r}\,'|)$.

The $\sinh^{-1}$ transformation may also be applied after another transformation has mapped a finite or semi-infinite interval onto the real line. The so-called Double Exponential or DE-rule has been proposed by several Japanese authors for numerical integration on $[-1,1]$ (see [3] for an easily accessible reference). It consists essentially of the $\sinh^{-1}$ transformation composed with the mapping $\log[(1+x)/(1-x)]$ of the interval $[-1,1]$ onto the real line.

The $\sinh^{-1}$ transformation itself has been mentioned in the context of numerical integration (see for instance [9]). However, these references do not provide the exact error bounds derived below. They also do not consider the problem of approximation of a given function.

Sections 2 and 3 of this paper summarize known results about cardinal approximation. Sections 4 and 5 introduce the $\sinh^{-1}$ transform and contain exact statements and proofs of the results mentioned above. Section 6 contains an application of these results to approximation theory, and section 7 has some numerical examples.

2. Interpolation. Most results in this section are well-known and summarized in Stenger [7]. However, various estimates derived separately before are now contained in a single proof. Also, this proof considers a more general approximation process before specializing to the case of Sinc aproximation, which points the way to possible other methods. For example, a method of approximation by rational functions (Stenger [8]) is derived in a similar way.

Let D_d, $d>0$, be the domain $D_d=\{x+iy : |y|<d\}$ in the complex plane $\mathbb{C}$. Let $B_p(D)$, $1\le p<\infty$, be the family of all functions which are analytic in D_d and such that

$$N_p(f,D_d)=\lim_{y\to d^-}\left\{\left[\int_{\mathbb{R}}|f(x+iy)|^p\,dx\right]^{1/p}+\left[\int_{\mathbb{R}}|f(x-iy)|^p\,dx\right]^{1/p}\right\}<\infty$$

and such that

$$\int_{-d}^{d}|f(x+iy)|\,dy\le M \qquad \text{as} \quad x\to\pm\infty \tag{2.1}$$

for some constant M.

Let $C(f,h)$ be the cardinal approximation of f with stepsize h

$$C(f,h)=\sum_{k=-\infty}^{\infty}f(kh)S(k,h) \tag{2.2}$$

and $C_N(f,h)$ the truncated cardinal approximation

$$C_N(f,h)=\sum_{k=-N}^{N}f(kh)S(k,h), \tag{2.3}$$

where

$$S(k,h)(x)=\frac{\sin[(\pi/h)(x-kh)]}{(\pi/h)(x-kh)}.$$

The corresponding error functions are

$$\begin{aligned} E(f,h) &= f-C(f,h)\\ E_N(f,h) &= f-C_N(f,h)\end{aligned}$$

and the truncation error $T_N(f,h)$ is given by

$$T_N(f,h) = E_N(f,h) - E(f,h) = \sum_{|k|\geq N+1} f(kh)S(k,h).$$

Theorem 2.1. *(a) If $f \in B_p(D_d)$, $1 \leq p < \infty$, then for $p \leq r \leq \infty$ (if $p > 1$) or $1 < r \leq \infty$ (if $p = 1$)*

$$\|E(f,h)\|_r \leq C_1 \frac{N_p(f,D_d)}{\sinh(\pi d/h)},$$

where C_1 depends on p, r, d, but not on f or h. If $1/q = 1 + 1/r - 1/p$, $1 \leq q \leq \infty$, then

- *if $1 < q < \infty$, then $C_1 = \dfrac{1}{2\pi d^{1/p}}\left[\dfrac{\Gamma((q-1)/2)\Gamma(1/2)}{\Gamma(q/2)}\right]^{1/q}$*
- *if $q = \infty$, then $C_1 = 1/2\pi d$,*
- *if $q = 1$, then $C_1 = (1+H_p)/2$, where H_p is the norm of the Hilbert transform as a mapping from L^p to L^p. That is,*

$$H_p = \begin{cases} \tan(\pi/2p), & 1 < p \leq 2 \\ 1/\tan(\pi/2p), & 2 \leq p < \infty \end{cases}$$

In particular if $p = 2$, then $C_1 = 1$.

(b) If $f \in B_p(D_d)$, $1 \leq p < \infty$, and $|f(x)| \leq ce^{-\alpha|x|}$, $x \in \mathbb{R}$, where α, c are positive constants, then by choosing $h = (\pi d/\alpha N)^{1/2}$ one obtains

$$\|E_N(f,h)\|_\infty \leq C_2 N^{1/2} e^{-(\pi d\alpha N)^{1/2}},$$

where C_2 depends only on $N_p(f,D_d)$, p, d, c, α.

Proof. (a) Consider first a more general interpolation process. Let $B(x)$ be analytic in D_d, with simple zeros at a (finite or infinite) set of points $x_k \in R$. Then

$$F(x) = \sum_k \frac{f(x_k)B(x)}{(x-x_k)B'(x_k)} \tag{2.4}$$

interpolates f at the points x_k, provided that the series converges. If f is analytic in D_d, then so is the function

$$g(z) = \frac{f(z)}{(z-x)B(z)}, \qquad x \neq x_k,$$

except for simple poles at the points x and x_k. Let $\Gamma(N,y)$ be the positively oriented rectangular contour connecting $\pm N \pm iy$, $0 < y < d$, $N > |x|$, $N \neq |x_k|$. Integrating $g(z)$ around this contour, one finds by the residue theorem

$$f(x) = \sum_{|x_k|<N} \frac{f(x_k)B(x)}{(x-x_k)B'(x_k)} + \frac{B(x)}{2\pi i}\int_{\Gamma(N,y)} \frac{f(z)}{(z-x)B(z)}\,dz \tag{2.5}$$

If the contribution of the vertical parts of the contour integral goes to zero as $N \to \infty$, then

$$f(x) = F(x) + E(f,h),$$

where

$$E(f,h)(x) = \frac{B(x)}{2\pi i}\int_{\mathbb{R}} \left\{ \frac{-f(t+iy)}{(t+iy-x)B(t+iy)} + \frac{f(t-iy)}{(t-iy-x)B(t-iy)} \right\} dt \tag{2.6}$$

If $|B(x)| \le M_1$ for $x \in \mathbb{R}$, and $|B(t+iy)| \ge M_2$ for $t \in \mathbb{R}$, then

$$|E(f,h)(x)| \le \frac{M_1}{2\pi M_2}\left\{ |f_1 * g_1(x)| + |f_2 * g_2(x)| \right\}$$

where $f_1(t) = f(t+iy)$, $f_2(t) = f(t-iy)$, $g_1(t) = (t+iy)^{-1}$, $g_2(t) = (t-iy)^{-1}$. Thus if $1/p + 1/q = 1 + 1/r$, then by Young's inequality

$$\|E(f,h)\|_r \le \frac{M_1}{2\pi M_2}\left(\|f_1\|_p \|g_1\|_q + \|f_2\|_p \|g_2\|_q \right),$$

and one calculates

$$\begin{aligned} \|g_1\|_q = \|g_2\|_q &= \left[\frac{\Gamma((q-1)/2)\Gamma(1/2)}{y\Gamma(q/2)} \right]^{1/q}, \qquad 1 < q < \infty, \\ \|g_1\|_\infty = \|g_2\|_\infty &= 1/d. \end{aligned}$$

In the case $q = 1$ (that is, $p = r$), we have

$$\begin{aligned} f_1 * g_1(x) &= \int_{\mathbb{R}} \frac{f_1(x-t)}{t+iy}\, dt \\ &= \int_{\mathbb{R}} f_1(x-t)\frac{t}{t^2+y^2}\, dt - i\int_{\mathbb{R}} f_1(x-t)\frac{y}{t^2+y^2}\, dt \\ &= \pi P_y(Hf_1)(x) - i\pi P_y f_1(x) \end{aligned}$$

where P_y, H denote the Poisson integral and Hilbert transform, respectively. (The last equality is lemma VI.1.5 in Stein/Weiss [6]). Since the norm of P_y as an operator from L^p to L^p is 1, the rest follows.

The cardinal approximation of f is now obtained by using $B(x) = \sin(\pi x/h)$. If $N \to \infty$ through the points $(n+1/2)h$, $n \in N$, then $|B(x+iy)| = \cosh(\pi y/h)$ is bounded below by 1 on the line segments $z = \pm(n+\frac{1}{2}) + iy$, $-d < y < d$, independent of N. It then follows from (2.1) that the contribution of these segments to the integral in (2.5) goes to zero, so that formula (2.6) holds. On the lines $z = t \pm iy$, $t \in \mathbb{R}$, similarly $|B(x+iy)| \ge \sinh(\pi y/h)$, while obviously $|B(x)| \le 1$ on the real line. After letting $y \to d$, we obtain the estimate in (a).

(b) The truncation error $T_N(f,h)$ can be estimated by

$$\begin{aligned} |T_N(f,h)(x)| &\le \sum_{|k| \ge N+1} |f(kh)|\,|S(k,h)(x)| \le 2\sum_{k=N+1}^{\infty} ce^{-\alpha kh} \\ &\le 2c\int_N^\infty e^{-\alpha ht}\, dt = \frac{2c}{\alpha h}e^{-\alpha Nh}. \end{aligned}$$

Choosing $h = (\pi d/\alpha N)^{1/2}$,

$$\|T_N(f,h)\|_\infty \le \frac{2c}{a(\alpha\pi d)^{1/2}} N^{1/2} e^{-(\pi d\alpha N)^{1/2}}.$$

From (a),

$$\begin{aligned}\|E(f,h)\|_\infty &\leq C_1\frac{N_p(f,D_d)}{\sinh(\pi d/h)} \leq C_1' N_p(f,D_d)e^{-\pi d/h}\\ &= C_1' N_p(f,D_d)e^{-(\pi d\alpha N)^{1/2}},\end{aligned}$$

so

$$\|E_N(f,h)\|_\infty \leq \|E(f,h)\|_\infty + \|T_N(f,h)\|_\infty \leq C_2 N^{1/2}e^{-(\pi d\alpha N)^{1/2}}.$$

$\square$

Let now D be a simply connected domain in the complex plane and ϕ a conformal map from D onto D_d, for some $d > 0$. Denote the boundary of D by ∂D and let a, $b \in \partial D$ be the points such that $\phi(a) = -\infty$, $\phi(b) = \infty$. Let $\psi = \phi^{-1}$ be the inverse map and set $\Gamma = \psi((-\infty,\infty))$. If F is a function analytic in D, one can approximate F on Γ by considering the function $f(z) = F(\psi(z))$ on D_d and using the above approach. The details are as follows.

Let $F \circ \psi \in B(D_d)$. The cardinal approximations to F on Γ are

$$\begin{aligned}C(F,h)\circ\phi &= \sum_{k=-\infty}^{\infty} F(z_k)S(k,h)\circ\phi\\ C_N(F,h)\circ\phi &= \sum_{k=-N}^{N} F(z_k)S(k,h)\circ\phi \qquad (2.7)\end{aligned}$$

where $z_k = \psi(kh)$ and

$$S(k,h)\circ\phi(x) = \frac{\sin[(\pi/h)(\phi(x)-kh)]}{(\pi/h)(\phi(x)-kh)}.$$

The corresponding error functions are

$$\begin{aligned}E(F,h)\circ\phi &= F - C(F,h)\circ\phi\\ E_N(F,h)\circ\phi &= F - C_N(F,h)\circ\phi\\ T_N(F,h)\circ\phi &= E_N(F,h)\circ\phi - E(F,h)\circ\phi\end{aligned}$$

One obtains immediately

THEOREM 2.2. *(a) If $F\circ\psi \in B_p(D_d)$, $1 \leq p < \infty$, then for $p \leq r \leq \infty$ (if $p > 1$) or $1 < r \leq \infty$ (if $p = 1$)*

$$\|E(f,h)\circ\phi\|_r \leq C_1\frac{N_p(F\circ\psi,D_d)}{\sinh(\pi d/h)},$$

where C_1 depends on p, r, d, but not on f or h. (The values are given in theorem 2.1).

(b)If $F\circ\psi \in B_p(D_d)$, $1 \leq p < \infty$, and $|F(z)| \leq ce^{-\alpha|\phi(z)|}$, $z \in \Gamma$, where α, c are positive constants, then by choosing $h = (\pi d/\alpha N)^{1/2}$ one obtains

$$\|E_N(f,h)\circ\phi\|_\infty \leq C_2 N^{1/2}e^{-(\pi d\alpha N)^{1/2}},$$

where C_2 depends only on $N_p(F\circ\psi,D_d)$, p, d, c, α.

EXAMPLES: A summary of frequently used contours, with mapping functions, discretization points and a sufficient condition to achieve $\exp(-cN^{1/2})$ convergence is as follows.

- $\Gamma = [0,1]$, $D = \{z : |\arg(z/(1-z))| < d\}$
  $\phi(z) = \log \dfrac{z}{1-z}, \quad \psi(w) = \dfrac{e^w}{1+e^w}, \quad z_k = \dfrac{e^{kh}}{1+e^{kh}}$
  $|F(z)| \le cz^\alpha (1-z)^\alpha, \alpha > 0$
- $\Gamma = [-1,1]$, $D = \{z : |\arg((1+z)/(1-z))| < d\}$
  $\phi(z) = \log \dfrac{1+z}{1-z}, \quad \psi(w) = \tanh(w/2), \quad z_k = \tanh(kh/2)$
  $|F(z)| \le c(1+z)^\alpha (1-z)^\alpha, \alpha > 0.$
- $\Gamma = [0,\infty)$, $D = \{z : |\arg z| < d\}$,
  $\phi(z) = \log z, \quad \psi(w) = e^w, \quad z_k = e^{kh}$
  $|F(z)| \le \left\{ \begin{array}{ll} cz^\alpha, & 0 \le z \le 1 \\ cz^{-\alpha}, & 1 \le z < \infty \end{array} \right\}$

$\square$

3. Numerical Integration. Most of the contents of this section can also be found in [7]. By integrating (2.2), (2.3) and using

$$\int_{\mathbb{R}} S(k,h)(x)\,dx = h,$$

one obtains the approximate integration formulas

$$\int_{\mathbb{R}} f(x)\,dx \approx h \sum_{k=-\infty}^{\infty} f(kh)$$

$$\int_{\mathbb{R}} f(x)\,dx \approx h \sum_{k=-N}^{N} f(kh)$$

with errors

$$\begin{aligned} \eta(f,h) &= \int_{\mathbb{R}} f(x)\,dx - h \sum_{k=-\infty}^{\infty} f(kh) = \int_{\mathbb{R}} E(f,h)(x)\,dx \\ \eta_N(f,h) &= \int_{\mathbb{R}} f(x)\,dx - h \sum_{k=-N}^{N} f(kh) = \int_{\mathbb{R}} E_N(f,h)(x)\,dx \\ \tau_N(f,h) &= \eta_N(f,h) - \eta(f,h) = h \sum_{|k| \ge N+1} f(kh) \end{aligned}$$

THEOREM 3.1. *(a) If* $f \in B_1(D_d)$*, then*

$$|\eta(f,h)| \le \frac{e^{-\pi d/h}}{2\sinh(\pi d/h)} N_1(f, D_d)$$

(b) If $f \in B_1(D_d)$ *and* $|f(x)| \le ce^{-\alpha|x|}$, $x \in R$, *where* α, c *are positive constants, then by choosing* $h = (2\pi d/\alpha N)^{1/2}$ *one obtains*

$$|\eta_N(f,h)| \le Ce^{-(2\pi d\alpha N)^{1/2}},$$

where C depends only on $N_1(f, D_d)$, d, c, α.

Proof. From (2.6),

$$\eta(f,h) = \int_{\mathbb{R}} E(f,h)(x)\,dx$$

$$= \int_{\mathbb{R}} \frac{\sin(\pi x/h)}{2\pi i} \int_{\mathbb{R}} \left\{ \frac{f(t-iy)}{(t-iy-x)\sin[(\pi/h)(x-iy)]} - \frac{f(t+iy)}{(t+iy-x)\sin[(\pi/h)(x+iy)]} \right\} dt\,dx$$

$$= \frac{1}{2\pi i} \left\{ \int_{\mathbb{R}} \left\{ \frac{f(t-iy)}{\sin[(\pi/h)(t-iy)]} \int_{\mathbb{R}} \frac{\sin(\pi x/h)}{t-iy-x} - \frac{f(t+iy)}{\sin[(\pi/h)(t+iy)]} \int_{\mathbb{R}} \frac{\sin(\pi x/h)}{t+iy-x} \right\} dx. \right\}$$

Using

$$\int_{\mathbb{R}} \frac{\sin(\pi x/h)}{t \pm iy - x}\,dx = -\pi e^{-(\pi y/h) \pm i(\pi t/h)}$$

([1] 3.723-2 and -3), and $|\sin[(\pi/h)(t-iy)]| \geq \sinh(\pi y/h)$, one obtains

$$|\eta(f,h)| \leq \frac{1}{2\pi} \pi e^{-\pi y/h} \frac{1}{\sinh(\pi y/h)} \int_{\mathbb{R}} \{|f(x-iy)| + |f(x+iy)|\}\,dx$$

and after letting $y \to d$,

$$|\eta(f,h)| \leq \frac{e^{-\pi d/h}}{2\sinh(\pi d/h)} N_1(f, D_d).$$

The proof of (b) is similar to that of 2.1(b). ∎

If D is a simply connected domain in the complex plane, ϕ a conformal map of D onto D_d, with $\psi = \phi^{-1}$, Γ, a, b as before, one obtains the approximate integration formulas

$$\int_{\Gamma} F(z)\,dz \approx h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\phi'(z_k)} \tag{3.1}$$

$$\int_{\Gamma} F(z)\,dz \approx h \sum_{k=-N}^{N} \frac{F(z_k)}{\phi'(z_k)}$$

where $z_k = \psi(kh)$ as before.

The corresponding errors are

$$\begin{aligned} \eta(F,h) \circ \phi &= \int_{\Gamma} F(z)\,dz - h \sum_{k=-\infty}^{\infty} \frac{F(z_k)}{\phi'(z_k)} \\ \eta_N(F,h) \circ \phi &= \int_{\Gamma} F(z)\,dz - h \sum_{k=-N}^{N} \frac{F(z_k)}{\phi'(z_k)} \\ \tau_N(F,h) \circ \phi &= \eta_N(F,h) \circ \phi - \eta(F,h) \circ \phi = h \sum_{|k| \geq N+1} \frac{F(z_k)}{\phi'(z_k)}. \end{aligned}$$

THEOREM 3.2. *(a)If* $(F\circ\psi)\psi' \in B_1(D_d)$, *then*

$$|\eta(F,h)\circ\phi| \le \frac{e^{-\pi d/h}}{2\sinh(\pi d/h)} N_1((F\circ\psi)\psi', D_d)$$

(b) If $(F\circ\psi)\psi' \in B_1(D_d)$, *and* $|F(x)| \le c|\phi'(x)|e^{-\alpha|\phi(x)|}$, $x \in \Gamma$, *where* α, c *are positive constants, then by choosing* $h = (2\pi d/\alpha N)^{1/2}$ *one obtains*

$$|\eta_N(f,h)| \le Ce^{-(2\pi d\alpha N)^{1/2}},$$

where C *depends only on* $N_1((F\circ\psi)\psi', D_d)$, d, c, α.

EXAMPLES: The curves Γ and mappings ϕ, ψ in these examples are the same as before. The only changes are the conditions that the functions $F(z)$ have to satisfy in order to achieve the error bounds given in theorem 3.2.

- $\Gamma = [0,1]$, $\quad |F(z)| \le cz^{\alpha-1}(1-z)^{\alpha-1}, \alpha > 0$
- $\Gamma = [-1,1]$, $\quad |F(z) \le c(1+z)^{\alpha-1}(1-z)^{\alpha-1}, \alpha > 0.$
- $\Gamma = [0,\infty)$, $\quad |F(z)| \le \begin{cases} cz^{\alpha-1}, & 0 \le z \le 1 \\ cz^{-\alpha-1}, & 1 \le z < \infty \end{cases}$

□

4. The transform $\phi = \sinh^{-1}$**.** From now on, let D be the domain

$$D = \{z = u+iv : \frac{v^2}{\sin^2 d} - \frac{u^2}{\cos^2 d} \le 1\}, \qquad 0 < d < \pi/2$$

and let $\phi(z) = \sinh^{-1} z = \log(z+\sqrt{1+z^2})$, $\psi(w) = \phi^{-1}(x) = \sinh(w)$. ϕ maps D conformally onto the strip D_d by using the branches of $\sqrt{z}$, $\log(z)$ which are real on the positive real axis and slitted along the negative real axis. The real axis is mapped into itself, thus $\Gamma = R$. The points z_k are given by $z_k = \sinh(kh)$.

The condition $|F(x)| \le ce^{-\alpha|\phi(x)|}$, $x \in \mathbb{R}$, in theorem 2.2(b) becomes

$$|F(x)| \le c\exp(-\alpha|\log(x+\sqrt{1+x^2})|) \le c'\frac{1}{1+|x|^\alpha}.$$

This can be seen for $x > 0$ from

$$\exp(-\alpha |\log(x + \sqrt{1+x^2})|) = (x + \sqrt{1+x^2})^{-\alpha} \le \frac{c''}{1 + |x|^\alpha}$$

since both functions are positive, continuous for $x \ge 0$ and have the same behaviour as $x \to \infty$. For $x < 0$, the fact that $\sinh^{-1}$ is odd yields the same estimate.

Likewise, the condition $|F(x) \le c|\phi'(x)|e^{-\alpha|\phi(x)|}$, $x \in \Gamma$, in theorem 3.2(b) becomes

$$|F(x)| \le c\frac{1}{1 + |x|^{\alpha+1}}.$$

Formulas (2.7), (3.1) specialize to

$$C(F, h) \circ \sinh^{-1} = \sum_{k=-\infty}^{\infty} F(\sinh(kh)) S(k, h) \circ \sinh^{-1}$$

$$\int_{\mathbb{R}} F(x)\,dx \approx h \sum_{k=-\infty}^{\infty} \cosh(kh) F(\sinh(kh))$$

Theorems 2.2(b), 3.2(b) specialize to

Theorem 4.1. *(a) If* $F \circ \sinh \in B_p(D_d)$, $1 \le p < \infty$, *and* $|F(x)| \le c/(1 + |x|^\alpha)$, $x \in R$, *where* c, α *are positive constants, then by choosing* $h = [(\pi d)/(\alpha N)]^{1/2}$, *the interpolation error satisfies*

$$\|E_N(F, h) \circ \sinh^{-1}\|_\infty \le C_1 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}$$

where C_1 *depends only on* $N_p(F \circ \sinh, D_d)$, p, d, α, c.

(b) If $(F \circ \sinh) \cosh \in B_1(D_d)$ *and* $|F(x)| \le c/(1 + |x|^{\alpha+1})$, $x \in R$, *where* c, α *are positive constants, then by choosing* $h = (2\pi d/\alpha N)^{1/2}$ *the numerical integration error satisfies*

$$|\eta_N(F, h) \circ \sinh^{-1}| \le C_2 e^{-(2\pi d \alpha N)^{1/2}},$$

where C_2 *depends only on* $N_1((F \circ \sinh)\cosh, D_d)$, d, c, α.

The practical significance of this is the following. In the interpolation or numerical integration of a function f defined on R the Sinc method gives an error of order $O(e^{-cN^{1/2}})$, provided f is analytic in a strip along R and decays to zero at an exponential rate. The new transformation gives the same convergence rate for functions f which decay to zero at a polynomial rate only, but are analytic in the larger region D. A typical example is $F(z) = 1/(1 + z^2)$.

Note that a similar phenomenon occurs when the transformation $\phi(x) = \log(x)$ for $\Gamma = [0, \infty)$ (example 3. above) is replaced by $\phi(x) = \log(\sinh(x))$ (see Stenger [7]).

5. A New Method with Faster Convergence.

Theorem 5.1. *(a) If* $F \circ \sinh \in B_p(D_d)$, $1 \le p < \infty$, *and* $|F(x)| \le ce^{-\alpha|x|}$, $x \in R$, *where* c, α *are positive constants, then by choosing* $h = \log N/N$, *the interpolation error satisfies*

$$\|E_N(F, h) \circ \sinh^{-1}\|_\infty \le C_1 e^{-\pi d N/\log N}$$

where C_1 *depends only on* $N_p(F \circ \sinh, D_d)$, p, d, α, c.

(b) If $(F\circ\sinh)\cosh \in B_1(D_d)$ *and* $|F(x)| \le ce^{-\alpha|x|}$, $x \in R$, *where* c, α *are positive constants, then by choosing* $h = \log N/N$, *the numerical integration error satisfies*

$$|\eta_N(F,h)\circ\sinh^{-1}| \le C_2 e^{-2\pi dN/\log N}$$

where C_2 *depends only on* $N_1((F\circ\sinh)\cosh, D_d)$, d, c, α.

Proof. (a) The truncation error T_N is

$$\begin{aligned} |T_N(F,h)\circ\sinh^{-1}(x)| &\le \sum_{|k|\ge N+1} |F(z_k)|\,|S(k,h)\circ\sinh^{-1}(x)| \\ &\le 2c\sum_{k=N+1}^{\infty} e^{-\alpha\sinh(kh)} \\ &\le \frac{2c}{h\alpha\cosh(Nh)}\int_N^\infty h\alpha\cosh(hx)e^{-\alpha\sinh(hx)}\,dx \\ &= \frac{2c}{h\alpha\cosh(Nh)}e^{-\alpha\sinh(Nh)}, \end{aligned}$$

while the interpolation error is

$$\|E(f,h)\circ\sinh^{-1}\|_\infty \le \frac{1}{2\pi d\sinh(\pi d/h)}N_p(F\circ\sinh, D_d)$$

as before.

If $h = \log N/N$, then for large N,

$$\cosh(Nh) \approx \sinh(Nh) \approx N/2,$$

thus asymptotically

$$\|T_N(F,h)\circ\sinh^{-1}\|_\infty \le \frac{4c}{\alpha\log N}e^{-\alpha N/2},$$

$$\|E(f,h)\circ\sinh^{-1}\|_\infty \le C_3 e^{-\pi d/h} = C_3 e^{-\pi dN/\log N}.$$

As $N\to\infty$, $\|T_N\|_\infty \to 0$ much faster than $\|E\|_\infty$, so the total error is asymptotically bounded by $\|E\|_\infty$.

(b) Here

$$|\tau_N(F,h)\circ\sinh^{-1}| \le h\sum_{|k|\ge N+1} |F(\sinh(kh))|\cosh(kh)$$

$$\le 2ch\sum_{k=N+1}^{\infty}\cosh(kh)e^{-\alpha\sinh(kh)}$$

$$\le \frac{2c}{\alpha}\int_N^\infty \alpha h\cosh(hx)e^{-\alpha\sinh(hx)}\,dx = \frac{2c}{\alpha}e^{-\alpha\sinh(Nh)}$$

and

$$|\eta(F,h)\circ\sinh^{-1}| \le \frac{e^{-\pi d/h}}{2\pi d\sinh(\pi d/n)} N_1((F\circ\sinh)\cosh, D_d).$$

If $h = \log N/N$, then

$$|\eta| \le C_4 e^{-2\pi d/h} = C_4 e^{-2\pi dN/\log N}$$

$$|\tau_N| \le \frac{2c}{\alpha} e^{-\alpha N/2}$$

and again $|\eta|$ dominates. $\square$

Thus, if f decays to zero at an exponential rate and is also analytic in the larger region D, the error in interpolation and numerical integration is $O(e^{-cN/\log N})$, which is asymptotically much better than $O(e^{-cN^{1/2}})$.

Examples for such functions can be found in section 7.

6. An Application To Approximation Theory. As an example, consider the approximation of the function $|x|^\alpha$, $\alpha > 0$, on $[-1,1]$. Through the transformation

$$z = \cosh^{-1}(1/|x|)\mathrm{sign}(x) \quad \leftrightarrow \quad x = (1/\cosh(z))\mathrm{sign}(z)$$

this is equivalent to the approximation of $F(z) = 1/\cosh^\alpha(z)$ on the real line. F is analytic in the complex plane slitted from i to $i\infty$ and from $-i$ to $-i\infty$, and decays like $e^{-\alpha|x|}$ on the real line.

The interpolation formula reads in this case

$$\frac{1}{\cosh^\alpha z} \approx \sum_{k=-N}^{N} \frac{1}{\cosh^\alpha(\sinh(kh))} S(k,h)\circ\sinh^{-1}(z)$$

or

$$|x|^\alpha \approx \sum_{k=-N}^{N} \frac{1}{\cosh^\alpha(\sinh(kh))} S(k,h)\circ\sinh^{-1}(\cosh^{-1}(1/|x|))$$

(The $\mathrm{sign}(x)$ term is unnecessary, since the right-hand side is an even function of x).

If we take $d = 1$, we find that on the lines $\{x \pm i : x \in \mathbb{R}\}$, the function $1/|\cosh(z)|$ is greater than 1 (but finite) in a neighborhood of the imaginary axis, and less than 1 otherwise. Also, the function is bounded above by $|\sinh(\sinh x\cos 1)|$, so the integral clearly exists. If $0 < \alpha_0 \le \alpha \le \alpha_1$, we can find a uniform bound on $N_1(1/|\cosh^\alpha(\sinh z)|, D_1)$, by using the α_1-power for comparison where the integrand is greater than 1, the α_0-power otherwise.

An inspection of the proof of theorem 5.1(a) shows that for $N \ge N_0$, where N_0 may depend on α_0, α_1, $\|T_N\|_\infty \le \|E\|_\infty$.

This implies that for fixed $0 < \alpha_0 < \alpha_1$, there is a constant C and a number N_0 so that for each α between α_0 and α_1 and each $N \ge N_0$, the error of best approximation by linear combinations of basis functions

$$B_{N,k,h}(x) = S(k,h)\circ\sinh^{-1}(\cosh^{-1}(1/|x|)), \qquad k = -N,\ldots,N$$

is bounded by $C\exp(-\pi N/\log N)$.

In comparison, the error of the best N-point approximation by polynomials decays like $N^{-\alpha}$ (Meinardus [2], section 5.6).

7. Numerical Examples. The choice $h = \log N/N$, while giving asymptotically optimal estimates, is unlikely to be useful for small values of N, while an accurate estimate of the optimal h requires in general more calculation than the original problem. The heuristic method proposed in [4] is still quite adequate for the new method. The idea is as follows: The truncation error T_N depends mainly on Nh, the interpolation error E only on h. Given an initial h and desired accuracy ϵ, increase N until T_N is small enough, then keep doubling N and halving h until E is small enough. An added advantage is that at each step, the results from the previous step can be used.

We calculated the values of the modified Bessel function K_1 at the points $x = 0.1, 1, 10$ using three different formulas ([1], 8-432)

$$K_1(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\sqrt{1+t^2}}\,dt$$

$$K_1(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh t}\cosh t\,dt$$

$$K_1(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh(\sinh t)}\cosh(\sinh t)\sinh(t)\,dt$$

All three integrals were evaluated using the trapezoidal rule (i.e. the usual Sinc method). This is equivalent to evaluating (a) in three different ways: by the trapezoidal rule, the $\sinh^{-1}$ transformation, and the repeated $\sinh^{-1}$ transformation. The integration was performed by the INTHP program published in [5]. Note that the number of points could be cut in half everywhere if the evenness of the integrands were taken into account.

As a simple example related to numerical integration of the Green's function $(4\pi|\vec{r}-\vec{r}'|)^{-1}\exp(ik|\vec{r}-\vec{r}'|)$, we also evaluated

$$\int_{-\infty}^{\infty} \frac{\cos(\sqrt{1+t^2})}{\sqrt{1+t^2}}e^{-\sqrt{1+t^2}}\,dt$$

$$\int_{-\infty}^{\infty} \cos(\cosh(t))e^{-\cosh(t)}\,dt$$

Again, the trapezoidal rule for (e) coresponds to the $\sinh^{-1}$ transformation applied to (d).

The results are contained in table 7. The values of ϵ given in the top row are the desired accuracies requested from the program. For each of these, the actual error and the number of points needed is given in the table.

A comparison of (b) and (c) shows that the $\sinh^{-1}$ transformation does not always improve convergence. The constants (notably the N_1-norm of the integrand) for the $\sinh^{-1}$ transformation may be much larger than those for the standard trapezoidal rule, even though asymptotically the $\sinh^{-1}$ transformation would still perform better.

A numerical example for the $\sinh^{-1}$ transformation composed with a mapping from $[-1, 1]$ to $(-\infty, \infty)$ (the so-called double exponential of DE-rule) is found in [3].

TABLE 1
Numerical Results

	$\epsilon = 10^{-6}$		$\epsilon = 10^{-9}$		$\epsilon = 10^{-12}$	
	$\lvert\eta_N\rvert$	N	$\lvert\eta_N\rvert$	N	$\lvert\eta_N\rvert$	N
$x = 0.1$						
(a)	$1.8_{10} - 6$	621	$1.8_{10} - 9$	1793	$1.8_{10} - 12$	2345
(b)	$3.6_{10} - 15$	65	$3.6_{10} - 15$	73	$3.6_{10} - 15$	73
(c)	$1.5_{10} - 10$	161	$1.5_{10} - 10$	161	$3.3_{10} - 10$	321
$x = 1$						
(a)	$1.6_{10} - 7$	69	$3.2_{10} - 11$	193	$2.1_{10} - 13$	249
(b)	$4.2_{10} - 8$	25	0	49	0	57
(c)	$2.0_{10} - 9$	65	$2.0_{10} - 9$	81	$1.1_{10} - 16$	161
$x = 10$						
(a)	$4.2_{10} - 8$	17	$4.2_{10} - 8$	17	$3.2_{10} - 13$	41
(b)	$1.7_{10} - 9$	13	$1.7_{10} - 9$	17	$5.0_{10} - 19$	33
(c)	$1.3_{10} - 9$	13	$1.3_{10} - 9$	17	$6.3_{10} - 13$	33
(d)	$2.3_{10} - 8$	121	$2.1_{10} - 11$	177	$2.6_{10} - 14$	449
(e)	$1.4_{10} - 9$	49	$2.8_{10} - 17$	97	$2.8_{10} - 17$	113

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