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RAISING MULTIWAVELET APPROXIMATION ORDER THROUGH LIFTING*

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Abstract. Given a pair of biorthogonal, compactly supported multiwavelets, we present an algorithm for raising their approximation orders to any desired level, using one lifting step and one dual lifting step. Free parameters in the algorithm are explicitly identified, and can be used to optimize the result with respect to other criteria.

Key words. wavelets, multiwavelets, lifting, approximation order

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1. Introduction.

A *refinable function vector of multiplicity* r *and dilation factor* m is a vector $\boldsymbol{\phi}^{(0)}$ of r real-valued functions

$$\boldsymbol{\phi}^{(0)}(x) = \begin{pmatrix} \phi_1^{(0)}(x) \\ \vdots \\ \phi_r^{(0)}(x) \end{pmatrix}, \qquad x \in \boldsymbol{R}, \tag{1.1}$$

which satisfies a *matrix refinement equation*

$$\boldsymbol{\phi}^{(0)}(x) = \sqrt{m} \sum_{k \in \boldsymbol{Z}} h_k^{(0)} \boldsymbol{\phi}^{(0)}(mx - k). \tag{1.2}$$

The sequence $\mathcal{H}^{(0)} = \{h_k^{(0)}\}_{k \in \boldsymbol{Z}}$ of coefficient matrices is called the *mask* of the function. We assume that only finitely many $h_k^{(0)}$ are nonzero and that all $\phi_j^{(0)}$ have compact support.

We call $\boldsymbol{\phi}^{(0)}$ a *multiscaling function* if it generates a *multiresolution approximation* (MRA) [21] of $L^2(\boldsymbol{R})$. This means that there exists a sequence of subspaces V_j, $j \in \boldsymbol{Z}$, of $L^2(\boldsymbol{R})$ with the following properties:

1. $V_j \subset V_{j+1}$,
2. $\bigcap_j V_j = \{0\}$, $\quad \overline{\bigcup_j V_j} = L^2(\boldsymbol{R})$,
3. $f(x) \in V_j \Leftrightarrow f(x - m^{-j}k) \in V_j, \quad k \in \boldsymbol{Z}$,
4. $f(x) \in V_j \Leftrightarrow f(mx) \in V_{j+1}$,
5. $\{\phi_j^{(0)}(x-k) : j, k \in \boldsymbol{Z}\}$ forms a Riesz basis of V_0.

In detail, property 5 means that there exist constants $0 < A \le B$ so that

$$A \sum_j \|\boldsymbol{c}_j\|_2^2 \le \left\| \sum_j \boldsymbol{c}_j^* \boldsymbol{\phi}^{(0)}(x-j) \right\|_2^2 \le B \sum_j \|\boldsymbol{c}_j\|_2^2 \tag{1.3}$$

for any sequence of coefficient vectors $\{\boldsymbol{c}_j\}$ with $\sum_j \|\boldsymbol{c}_j\|_2^2 < \infty$. The superscript $*$ denotes the transpose.

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Other function vectors $\boldsymbol{\phi}^{(\nu)}$, $\nu = 1, \ldots, m-1$, are called *multiwavelet functions* if $\{\phi_j^{(\nu)}(x-k) : j, k \in \boldsymbol{Z}\}$ form Riesz bases of other spaces $W_0^{(\nu)}$ so that

$$V_0 \oplus W_0^{(1)} \oplus \cdots \oplus W_0^{(m-1)} = V_1$$

and

$$\{m^{\ell/2}\phi_j^{(\nu)}(m^\ell x - k) : j, k, \ell \in \boldsymbol{Z}, \nu = 1, \ldots, m-1\}$$

forms a Riesz basis of $L^2(\boldsymbol{R})$.

These multiwavelet functions satisfy refinement equations

$$\boldsymbol{\phi}^{(\nu)}(x) = \sqrt{m}\sum_k h_k^{(\nu)}\boldsymbol{\phi}^{(0)}(mx-k). \tag{1.4}$$

We again assume that all coefficient sequences are finite and all multiwavelet functions have compact support.

In the standard literature, the word "wavelet" sometimes means an individual wavelet function, sometimes scaling function and wavelets together. To avoid ambiguity, we refer to the entire collection $\boldsymbol{\phi} = \{\boldsymbol{\phi}^{(\nu)} : \nu = 0, \ldots, m-1\}$ as a multiwavelet and to the individual $\boldsymbol{\phi}^{(\nu)}$ as multiscaling functions or multiwavelet functions.

The properties of refinable function vectors, multiscaling functions, and multiwavelet functions with dilation factor $m = 2$ are discussed in many papers. Some of the earliest occurrences are [1], [9], [11], [12]; more recent treatments include [4], [6], [14], [18], [25], [26], [28]. It is straightforward to extend these results to the case of general m, following the one-dimensional case which is discussed, for example, in [16], [27], [35].

Two multiwavelets $\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}$ form a biorthogonal pair if they satisfy the *biorthogonality conditions*

$$\int \phi_k^{(\mu)}(x)\tilde{\phi}_\ell^{(\nu)}(x-j)\,dx = \delta_{\mu\nu}\delta_{0j}\delta_{k\ell}, \tag{1.5}$$

where δ is the Kronecker delta.

One of the properties of a multiscaling function which has great practical interest is the *approximation order* [15], [17], [22], [23]. $\boldsymbol{\phi}^{(0)}$ has approximation order $p \geq 1$ if all powers of x up to $p-1$ can be locally written as linear combinations of its integer translates. That is, there exist vectors $\boldsymbol{y}_k^{(j)} \in \boldsymbol{R}^r$ such that for $j = 0, \ldots, p-1$

$$x^j = \sum_k \boldsymbol{y}_k^{(j)*}\boldsymbol{\phi}^{(0)}(x-k). \tag{1.6}$$

Since we assume compact support, the sum is finite for each fixed x, and there are no convergence problems.

This paper considers the following problem: Given a biorthogonal multiwavelet pair $\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}$ and integers $p \geq 1$, $\tilde{p} \geq 1$, find an algorithm to generate from them new multiwavelets $\boldsymbol{\phi}_{\text{new}}$, $\tilde{\boldsymbol{\phi}}_{\text{new}}$ with approximation orders p, $\tilde{p}$, respectively.

One known way to raise approximation order is through the use of two-scale similarity transforms (TSTs) [24], [25], [28], [29], [30]. Our approach uses *lifting.* As a systematic strategy for creating new multiwavelet functions, this approach dates back to [5], and in the more general context of stable multiscale representations to [2]. Under the name "lifting," these techniques were later applied in [3], [7], [8], [20], [33], [34]. Details can be found in section 3.

Compared to TSTs, the lifting approach has the following advantages:

1. lifting produces a complete new multiwavelet pair; TST produces only a new multiscaling function;
2. lifting uses no matrix division or singular matrices;
3. lifting generally produces shorter new masks than the TST algorithm.

The outline of this paper is as follows.

Sections 2, 3 introduce notation and summarize needed results from the literature. The main result can be found in Theorem 4.1 at the end of section 4. The proof is constructive, and forms the basis for a numerical algorithm. An alternative approach, based on a suggestion in [34], is presented in section 5. Implementation details for both algorithms are stated in section 6. Section 7 contains some examples.

2. Representations of multiwavelet masks. The results in this section are well known. Proofs or appropriate references can be found, e.g., in [2], [5], [6], or [25].

Throughout this paper, all calculations are based on the masks $\mathcal{H}$, $\tilde{\mathcal{H}}$ alone. In terms of masks, the biorthogonality conditions (1.5) are represented as

$$\sum_k h_k^{(\mu)} \tilde{h}_{k+mj}^{(\nu)T} = \delta_{\mu\nu}\delta_{0j} I. \tag{2.1}$$

Here and in the remainder of this paper, I denotes an identity matrix of appropriate size.

The existence of a biorthogonal pair of masks does not automatically guarantee the existence of a corresponding pair of MRAs. This raises the question of whether the new masks produced by our algorithm actually represent multiwavelets in the sense described in the introduction, or merely the coefficients of filter banks for signal processing applications. We will address this question in section 6.3.3, using the notation of [2], [5], which we briefly introduce at this point. (The multiscale representations discussed in [2] actually cover more general cases than described here.)

There are various conditions given in the literature (see, for example, [4], [10], [14], [19], [25]) which can be checked to see whether a given $\mathcal{H}^{(0)}$ gives rise to a refinable function vector $\boldsymbol{\phi}^{(0)}$ and an MRA. This corresponds to the concept of a (uniformly) *stable basis* in [2].

Given $\boldsymbol{\phi}^{(0)}$, the multiwavelet functions $\boldsymbol{\phi}^{(\nu)}$, $\nu = 1, \ldots, m-1$, always exist by (1.4). They form a *stable completion* of $\boldsymbol{\phi}^{(0)}$ if

$$\{\phi_j^{(\nu)}(x-k) : \nu = 0, \ldots, m-1, j, k \in \boldsymbol{Z}\}$$

forms a Riesz basis of V_1.

If

$$\{m^{\ell/2}\phi_j^{(\nu)}(m^\ell x - k) : j, k, \ell \in \boldsymbol{Z}, \nu = 1, \ldots, m-1\}$$

forms a Riesz basis of $L^2(\boldsymbol{R})$, this is called *stability over all levels*.

In section 6.3.3, we will refer to these concepts as stability of $\boldsymbol{\phi}^{(0)}$, stability of $\boldsymbol{\phi}$, and stability over all levels, respectively.

The information contained in a mask $\mathcal{H}$ can be represented in various forms. We present here the two forms used in this paper.

The *symbol* of a function mask $\mathcal{H}^{(\nu)}$ is defined as

$$H^{(\nu)}(\xi) = \frac{1}{\sqrt{m}} \sum_k h_k^{(\nu)} e^{-ik\xi}, \qquad \xi \in \boldsymbol{R}. \tag{2.2}$$

In terms of symbols, the biorthogonality conditions (2.1) can be expressed as either

$$\sum_{k=0}^{m-1} H^{(\nu)}\left(\xi+\frac{2\pi}{m}k\right)\tilde{H}^{(\mu)*}\left(\xi+\frac{2\pi}{m}k\right)=\delta_{\nu\mu}I \tag{2.3}$$

or

$$\sum_{\nu=0}^{m-1} \tilde{H}^{(\nu)*}\left(\xi+\frac{2\pi}{m}k\right)H^{(\nu)}\left(\xi+\frac{2\pi}{m}\ell\right)=\delta_{k\ell}I, \tag{2.4}$$

where for complex-valued functions the superscript $*$ stands for the complex conjugate transpose.

A mask $\mathcal{H}$ satisfies *condition E* if $H^{(0)}(0)$ has a simple eigenvalue of 1, with all other eigenvalues less than 1 in modulus. Condition E is automatically satisfied if the mask generates an MRA of $L^2(\boldsymbol{R})$ with compactly supported basis functions [19], [25].

The *polyphase representation* $P(\xi)$ is the block matrix

$$P(\xi)=\begin{pmatrix} H_0^{(0)}(\xi) & H_1^{(0)}(\xi) & \dots & H_{m-1}^{(0)}(\xi) \\ H_0^{(1)}(\xi) & H_1^{(1)}(\xi) & \dots & H_{m-1}^{(1)}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ H_0^{(m-1)}(\xi) & H_1^{(m-1)}(\xi) & \dots & H_{m-1}^{(m-1)}(\xi) \end{pmatrix}, \tag{2.5}$$

where the *polyphase symbols* $H_\mu^{(\nu)}(\xi)$ are defined by

$$H_\mu^{(\nu)}(\xi)=\sum_k h_{mk+\mu}^{(\nu)}e^{-ik\xi}. \tag{2.6}$$

The normalization is chosen so that biorthogonality is equivalent to

$$P(\xi)\tilde{P}(\xi)^*=I. \tag{2.7}$$

The determinants of $P(\xi)$, $\tilde{P}(\xi)$ are trigonometric polynomials. If $\mathcal{H}$, $\tilde{\mathcal{H}}$ both have finite length, the determinants must be monomials.

If a multiwavelet has approximation order p, as defined in (1.6), then necessarily (see [15], [22], [23])

$$\boldsymbol{y}_k^{(j)}=\sum_{\ell=0}^{j}\binom{j}{\ell}k^{j-\ell}\boldsymbol{y}^{(\ell)} \tag{2.8}$$

for some vectors $\boldsymbol{y}^{(j)}\in\boldsymbol{R}^r$, $\boldsymbol{y}^{(0)}\neq\boldsymbol{0}$, and

$$\sum_{\ell=0}^{j}\binom{j}{\ell}(-i)^{j-\ell}m^\ell\boldsymbol{y}^{(\ell)*}D^{j-\ell}H^{(0)}\left(\frac{2\pi}{m}k\right)=\delta_{0k}\boldsymbol{y}^{(j)*} \tag{2.9}$$

for $j=0,\dots,p-1$ and $k=0,\dots,m-1$. D denotes the differentiation operator. We take (2.9) as the definition of approximation order for masks.

If $Y(\xi)$ is any vector of trigonometric polynomials with

$$D^jY(0)=i^{-j}\boldsymbol{y}^{(j)},\qquad j=0,\dots,p-1, \tag{2.10}$$

then another way to express (2.9) is

$$D^j \left[H^{(0)*}(\xi + \frac{2\pi}{m}k)Y(m\xi) \right]\Big|_{\xi=0} = \delta_{0k} D^j Y(0) = \delta_{0k} i^{-j} y^{(j)}. \tag{2.11}$$

The following theorem states that the approximation order of $\tilde{\mathcal{H}}$ can be determined by examining $\mathcal{H}$. It is a crucial step for the development in section 4. A partial result (the "only if" part for $\nu = 0$) was earlier derived in [32]. A similar result for multivariate wavelets can also be found in [13].

THEOREM 2.1. *Assume $\mathcal{H}$, $\tilde{\mathcal{H}}$ are biorthogonal masks. Then $\tilde{\mathcal{H}}$ has approximation order $\tilde{p}$ with vectors $\tilde{\boldsymbol{y}}^{(j)}$ if and only if*

$$\sum_{s=0}^{j} \binom{j}{s} i^{j-s} D^{j-s} H^{(\nu)}(0) \tilde{\boldsymbol{y}}^{(s)} = \delta_{0\nu} m^j \tilde{\boldsymbol{y}}^{(j)} \tag{2.12}$$

for $j = 0, \ldots, \tilde{p} - 1$, $\nu = 0, \ldots, m - 1$.

Proof. Assume $\tilde{Y}(\xi)$ satisfies (2.11) for the dual mask:

$$D^j \left[\tilde{H}^{(0)*}(\xi + \frac{2\pi}{m}k)\tilde{Y}(m\xi) \right]\Big|_{\xi=0} = \delta_{0k} D^j \tilde{Y}(0). \tag{2.13}$$

Take $\mu = 0$ in (2.3) and multiply by $\tilde{Y}(m\xi)$:

$$\sum_{k=0}^{m-1} H^{(\nu)} \left(\xi + \frac{2\pi}{m}k \right) \tilde{H}^{(0)*} \left(\xi + \frac{2\pi}{m}k \right) \tilde{Y}(m\xi) = \delta_{0\nu} \tilde{Y}(m\xi). \tag{2.14}$$

Differentiate j times and evaluate at $\xi = 0$:

$$\begin{aligned} \delta_{0\nu} m^j D^j \tilde{Y}(0) &= \sum_{s=0}^{j} \sum_{k=1}^{m-1} \binom{j}{s} D^{j-s} H^{(\nu)}(\frac{2\pi}{m}k) D^s \left[H^{(0)*}(\xi + \frac{2\pi}{m}k)\tilde{Y}(m\xi) \right]\Big|_{\xi=0} \\ &= \sum_{s=0}^{j} \sum_{k=1}^{m-1} \binom{j}{s} D^{j-s} H^{(\nu)}(0) D^s \tilde{Y}(0), \end{aligned} \tag{2.15}$$

which simplifies to (2.12). ∎

Remark. If the dual multiscaling function $\tilde{\boldsymbol{\phi}}^{(0)}$ has approximation order $\tilde{p}$, this implies that the multiwavelet functions have $\tilde{p}$ vanishing continuous moments, that is,

$$\int x^j \phi_k^{(\nu)}(x)\, dx = 0 \tag{2.16}$$

for $j = 0, \ldots, \tilde{p} - 1$, $k = 1, \ldots, r$, and $\nu = 1, \ldots, m - 1$.

Equation (2.12) for $\nu = 1, \ldots, m - 1$ can also be derived from the vanishing moment condition (2.16). Thus, Theorem 2.1 is a strictly algebraic version of the statement "the dual multiscaling function has approximation order $\tilde{p}$ if and only if the multiwavelet functions have $\tilde{p}$ vanishing moments."

3. Lifting. The following theorem forms the basis for the lifting procedure.

THEOREM 3.1. *If P_1, P_2 are polyphase matrices for two multiwavelets with the same multiscaling function, they are related by*

$$P_2(\xi) = \begin{pmatrix} I & 0 \\ L(\xi) & M(\xi) \end{pmatrix} P_1(\xi), \tag{3.1}$$

where L is of size $(m-1)r \times r$, M is of size $(m-1)r \times (m-1)r$. If both masks have finite length, $\det(M(\xi))$ *is a monomial.*

For wavelets of multiplicity 1 and dilation factor 2, this theorem dates back to [36].

As a general technique for creating stable refinable bases, the theorem was first used (in a periodic setting) in [5]. The most general version is given in [2], in the context of stable multiscale representations. A multiscale representation generalizes the concept of MRA by allowing each of the nested subspaces V_j to have its own basis Φ_j, not necessarily generated by translates and dilations from a small number of scaling functions. It is shown that any two stable completions of the same Φ_j are related in a manner similar to (3.1). (Polyphase matrices are not available in the general multiscale case, so the notation is different).

In the scalar case, Sweldens called (3.1) with $M = 1$ a *lifting step* [7], [33], [34], and showed that any wavelet can be built from the trivial polyphase matrix $P(\xi) = I$ by a finite combination of lifting steps and dual lifting steps:

$$P_{\text{new}}(\xi) = \begin{pmatrix} 1 & L(\xi) \\ 0 & 1 \end{pmatrix} P(\xi). \tag{3.2}$$

We also ignore M, since it has no effect on the scaling functions or dual scaling functions and their approximation orders. Thus, we define a multiwavelet lifting step as

$$P_{\text{new}}(\xi) = \begin{pmatrix} I & 0 \\ L(\xi) & I \end{pmatrix} P(\xi) = \begin{pmatrix} I & 0 & \dots & 0 \\ L^{(1)}(\xi) & I & & \\ \vdots & & \ddots & \\ L^{(m-1)}(\xi) & & & I \end{pmatrix} P(\xi), \tag{3.3}$$

where each $L^{(\nu)}(\xi)$ is an $r \times r$ matrix trigonometric polynomial. The effect on the dual is

$$\tilde{P}_{\text{new}}(\xi) = \begin{pmatrix} I & -L(\xi)^* \\ 0 & I \end{pmatrix} \tilde{P}(\xi). \tag{3.4}$$

In terms of the function symbols, our definition of multiwavelet lifting is equivalent to

$$\begin{aligned} H^{(0)}_{\text{new}}(\xi) &= H^{(0)}(\xi), \\ H^{(\nu)}_{\text{new}}(\xi) &= H^{(\nu)}(\xi) + L^{(\nu)}(m\xi)H^{(0)}(\xi), && \nu = 1,\dots,m-1, \\ \tilde{H}^{(0)}_{\text{new}}(\xi) &= \tilde{H}^{(0)}(\xi) - \textstyle\sum_{\nu=1}^{m-1} L^{(\nu)*}(m\xi)\tilde{H}^{(\nu)}(\xi), \\ \tilde{H}^{(\nu)}_{\text{new}}(\xi) &= \tilde{H}^{(\nu)}(\xi), && \nu = 1,\dots,m-1. \end{aligned} \tag{3.5}$$

In terms of multiscaling and multiwavelet functions, we have

$$\begin{aligned} \boldsymbol{\phi}^{(0)}_{\text{new}}(x) &= \boldsymbol{\phi}^{(0)}(x), \\ \boldsymbol{\phi}^{(\nu)}_{\text{new}}(x) &= \boldsymbol{\phi}^{(\nu)}(x) + \textstyle\sum_k L^{(\nu)}_k \boldsymbol{\phi}^{(0)}(x-k), && \nu = 1,\dots,m-1, \\ \tilde{\boldsymbol{\phi}}^{(0)}_{\text{new}}(x) &= \tilde{\boldsymbol{\phi}}^{(0)}(\xi) - \textstyle\sum_{\nu=1}^{m-1}\sum_k L^{(\nu)*}_{-k} \tilde{\boldsymbol{\phi}}^{(\nu)}(x-k), \\ \tilde{\boldsymbol{\phi}}^{(\nu)}_{\text{new}}(x) &= \tilde{\boldsymbol{\phi}}^{(\nu)}(x), && \nu = 1,\dots,m-1. \end{aligned} \tag{3.6}$$

Different but related multiwavelet lifting procedures are described in [8], [32]. Lifting for multivariate wavelets is discussed in [3] and [20].

4. Raising approximation order by lifting. In this section, we show how a single lifting step can be used to raise the approximation order of the dual multiscaling function to any desired level, while leaving the multiscaling function and its approximation order invariant.

In the scalar case, the idea of using lifting to raise the dual approximation order goes back to Sweldens' original papers [33], [34]. In the multiwavelet setting, different implementations appear in [8], [32]. Similar ideas can also be found in [13] (for multivariate wavelets) and [2] (for general multiscale approximations).

Let $\mathcal{H}$, $\tilde{\mathcal{H}}$ be a biorthogonal pair of masks, with $\mathcal{H}$ satisfying condition E.

Remark. As pointed out above, condition E is automatically satisfied for compactly supported stable $\boldsymbol{\phi}^{(0)}$, so it is a desirable property anyway. This is the reason why we impose condition E instead of the slightly weaker conditions we actually need.

Given any trigonometric matrix polynomial

$$L(\xi) = \sum_k L_k \, e^{-ik\xi}, \tag{4.1}$$

we define its *discrete moments* as

$$\Lambda_j = \sum_k k^j L_k = i^j D^j L(0), \qquad j = 0, 1, \dots . \tag{4.2}$$

If the coefficients L_k are nonzero only for $k = k_0, \dots, k_0 + n - 1$, and $N \geq 1$ is arbitrary, then L_k and Λ_k are related by

$$(\Lambda_0, \dots, \Lambda_{N-1}) = (L_{k_0}, \dots, L_{k_0+n-1}) \, A, \tag{4.3}$$

where A is a block Vandermonde matrix with blocks of size $r \times r$

$$A = \begin{pmatrix} k_0^0 I & k_0^1 I & \cdots & k_0^{N-1} I \\ (k_0+1)^0 I & (k_0+1)^1 I & \cdots & (k_0+1)^{N-1} I \\ \vdots & \vdots & \ddots & \vdots \\ (k_0+n-1)^0 I & (k_0+n-1)^1 I & \cdots & (k_0+n-1)^{N-1} I \end{pmatrix}. \tag{4.4}$$

Let $M_j^{(\nu)}$ denote the jth discrete moment of $H^{(\nu)}$. It follows from differentiating (3.5) and evaluating at $\xi = 0$ that the new moments after lifting are given by

$$\begin{aligned} M_{\text{new},j}^{(0)} &= M_j^{(0)}, \\ M_{\text{new},j}^{(\nu)} &= M_j^{(\nu)} + \sum_{s=0}^{j} \binom{j}{s} m^s \Lambda_s^{(\nu)} M_{j-s}^{(0)}. \end{aligned} \tag{4.5}$$

We want to satisfy the dual approximation order conditions (2.12) of order $\tilde{p}$. We do this first for $\nu = 0$, where the conditions are

$$\sum_{s=0}^{j} \binom{j}{s} M_{j-s}^{(0)} \tilde{\boldsymbol{y}}^{(s)} = m^j \tilde{\boldsymbol{y}}^{(j)}, \qquad j = 0, \dots, \tilde{p} - 1. \tag{4.6}$$

We can rewrite this in the form

$$\tag{4.7}\begin{aligned}\tilde{\boldsymbol{y}}^{(0)} &= M_0^{(0)}\tilde{\boldsymbol{y}}^{(0)},\\ \tilde{\boldsymbol{y}}^{(j)} &= \left(m^j I - M_0^{(0)}\right)^{-1}\sum_{s=0}^{j-1}\binom{j}{s} M_{j-s}^{(0)}\tilde{\boldsymbol{y}}^{(s)}, \qquad j=1,2,\ldots,\tilde{p}-1.\end{aligned}$$

Condition E is sufficient to guarantee solvability.

After the $\tilde{\boldsymbol{y}}^{(j)}$ have been determined, define

$$\tag{4.8}\boldsymbol{z}_j^{(\nu)} = m^{-j}\sum_{s=0}^{j}\binom{j}{s} M_{j-s}^{(\nu)}\tilde{\boldsymbol{y}}^{(s)}.$$

By (4.6),

$$\tag{4.9}\boldsymbol{z}_j^{(0)} = \tilde{\boldsymbol{y}}^{(j)}$$

for $j=0,\ldots,\tilde{p}-1$. By (2.12), $\tilde{\mathcal{H}}$ has existing approximation order $\tilde{q}$ if and only if

$$\tag{4.10}\boldsymbol{z}_j^{(\nu)} = \mathbf{0}$$

for $\nu=1,\ldots,m-1$ and $j=0,\ldots,\tilde{q}-1$.

Next, we satisfy the remaining conditions (2.12) for $\nu=1,\ldots,m-1$, which are

$$\tag{4.11}\sum_{s=0}^{j}\binom{j}{s} M_{\mathrm{new},j-s}^{(\nu)}\tilde{\boldsymbol{y}}^{(s)} = \mathbf{0}.$$

Substitute (4.5) to obtain

$$\tag{4.12}\begin{aligned}-\sum_{s=0}^{j}\binom{j}{s} M_{j-s}^{(\nu)}\tilde{\boldsymbol{y}}^{(s)} &= \sum_{s=0}^{j}\sum_{\ell=0}^{j-s}\binom{j}{s}\binom{j-s}{\ell} m^\ell \Lambda_\ell^{(\nu)} M_{j-s-\ell}^{(0)}\tilde{\boldsymbol{y}}^{(s)}\\ &= \sum_{\ell=0}^{j}\sum_{s=0}^{j-\ell}\binom{j}{\ell}\binom{j-\ell}{s} m^\ell \Lambda_\ell^{(\nu)} M_{j-s-\ell}^{(0)}\tilde{\boldsymbol{y}}^{(s)}\\ &= \sum_{\ell=0}^{j}\binom{j}{\ell} m^\ell \Lambda_\ell^{(\nu)}\sum_{s=0}^{j-\ell}\binom{j-\ell}{s} M_{j-s-\ell}^{(0)}\tilde{\boldsymbol{y}}^{(s)}\\ &= m^j\sum_{\ell=0}^{j}\binom{j}{\ell}\Lambda_\ell^{(\nu)}\tilde{\boldsymbol{y}}^{(j-\ell)},\end{aligned}$$

or

$$\tag{4.13}\sum_{\ell=0}^{j}\binom{j}{\ell}\Lambda_\ell^{(\nu)}\tilde{\boldsymbol{y}}^{(j-\ell)} = -\boldsymbol{z}_j^{(\nu)}.$$

For each fixed ν, this can be solved by choosing $\Lambda_j^{(\nu)}$ successively for $j=0,\ldots,\tilde{p}-1$ to satisfy

$$\tag{4.14}\Lambda_j^{(\nu)}\tilde{\boldsymbol{y}}^{(0)} = -\boldsymbol{z}_j^{(\nu)} - \sum_{\ell=0}^{j-1}\binom{j}{\ell}\Lambda_\ell^{(\nu)}\tilde{\boldsymbol{y}}^{(j-\ell)}.$$

The solution is not unique, except in the scalar case.

The matrix A in (4.3) is nonsingular for $N = n = \tilde{p}$, so we can always find a trigonometric polynomial $L^{(\nu)}(\xi)$ of length $\tilde{p}$ or less with an arbitrary starting index k_0 and prescribed moments $\Lambda_j^{(\nu)}$, $j = 0, \ldots, \tilde{p} - 1$.

We summarize the results of this section in the following theorem.

THEOREM 4.1. *Assume $\mathcal{H}$, $\tilde{\mathcal{H}}$ are biorthogonal masks with approximation orders q, $\tilde{q}$, respectively, with $\mathcal{H}$ satisfying condition E. Then for any $\tilde{p} \geq 1$ it is possible to find trigonometric polynomials $L^{(\nu)}(\xi)$ of length at most $\tilde{p}$, so that the new masks $\mathcal{H}_{new}$, $\tilde{\mathcal{H}}_{new}$ produced by the lifting process* (3.3) *have approximation orders q, $\tilde{p}$, respectively.*

If $\tilde{\mathcal{H}}_{new}$ satisfies condition E, we can follow the first lifting step with a dual lifting step that produces new masks with approximation orders p, $\tilde{p}$, respectively, for any $p \geq 1$.

It is shown in section 6.3.3 below that if $\tilde{\mathcal{H}}$ satisfies condition E, it is always possible to preserve it during the lifting step.

5. A modified approach. In the procedure in the previous section, it is necessary to impose all $\tilde{p}$ conditions, even if the original $\tilde{\mathcal{H}}$ already has some approximation order $\tilde{q}$. A modified algorithm, suggested in the scalar case in [34], can be adapted to the multiwavelet case.

The motivation is the following. As stated in (3.6) above, the effect of lifting on the multiwavelet functions is described by

$$\boldsymbol{\phi}_{\text{new}}^{(\nu)}(x) = \boldsymbol{\phi}^{(\nu)}(x) + \sum_k L_k^{(\nu)} \boldsymbol{\phi}^{(0)}(x - k). \tag{5.1}$$

Sweldens [34] proposes to replace this by

$$\boldsymbol{\phi}_{\text{new}}^{(\nu)}(x) = \boldsymbol{\phi}^{(\nu)}(x) + \sum_k T_k^{(\nu)} \boldsymbol{\phi}^{(\nu)} \left(\frac{x}{m} - k\right), \tag{5.2}$$

since this preserves existing vanishing moment conditions (2.16).

In our setting, this suggestion amounts to choosing

$$L^{(\nu)}(\xi) = T^{(\nu)}(\xi) H^{(\nu)}(\xi) \tag{5.3}$$

for some shorter trigonometric polynomials $T^{(\nu)}$. It is easy to verify directly that this approach will preserve the existing approximation orders for masks.

THEOREM 5.1. *If $\mathcal{H}$, $\tilde{\mathcal{H}}$ are biorthogonal masks, and $\mathcal{H}_{new}$, $\tilde{\mathcal{H}}_{new}$ are produced by lifting with*

$$L^{(\nu)}(\xi) = T^{(\nu)}(\xi) H^{(\nu)}(\xi), \tag{5.4}$$

then $\tilde{\mathcal{H}}_{new}$ has at least the same approximation order as $\tilde{\mathcal{H}}$.

Proof. Assume that $\tilde{\mathcal{H}}$ has approximation order $\tilde{q}$ or, equivalently (see (4.10))

$$\boldsymbol{z}_j^{(\nu)} = \mathbf{0} \tag{5.5}$$

for $\nu = 1, \ldots, m - 1$ and $j = 0, \ldots, \tilde{q} - 1$.

Differentiate (5.3) and evaluate at $\xi = 0$ to get

$$\Lambda_\ell^{(\nu)} = \sum_{s=0}^{\ell} \binom{\ell}{s} \Upsilon_s^{(\nu)} M_{\ell-s}^{(\nu)}, \tag{5.6}$$

where $\Upsilon_s^{(\nu)}$ are the moments of $T^{(\nu)}(\xi)$. Thus, for $\nu = 1, \ldots, m-1$ and $j = 0, \ldots, \tilde{q}-1$,

$$\begin{aligned}\sum_{\ell=0}^{j} \binom{j}{\ell} \Lambda_{j-\ell}^{(\nu)} \tilde{\boldsymbol{y}}^{(\ell)} &= \sum_{\ell=0}^{j} \sum_{s=0}^{j-\ell} \binom{j}{\ell} \binom{j-\ell}{s} \Upsilon_s^{(\nu)} M_{j-\ell-s}^{(\nu)} \tilde{\boldsymbol{y}}^{(\ell)} \\ &= \sum_{s=0}^{j} \sum_{\ell=0}^{j-s} \binom{j}{s} \binom{j-s}{\ell} \Upsilon_s^{(\nu)} M_{j-s-\ell}^{(\nu)} \tilde{\boldsymbol{y}}^{(\ell)} \\ &= \sum_{s=0}^{j} \binom{j}{s} m^{j-s} \Upsilon_s^{(\nu)} \boldsymbol{z}_{j-s}^{(\nu)} = \mathbf{0} = -\boldsymbol{z}_j^{(\nu)},\end{aligned} \tag{5.7}$$

so (4.13) is satisfied. □

The calculations in (5.7) also provide the equations for determining $\Upsilon_j^{(\nu)}$. As in (4.13), we need

$$\begin{aligned}-\boldsymbol{z}_j^{(\nu)} &= \sum_{\ell=0}^{j} \binom{j}{\ell} \Lambda_{j-\ell}^{(\nu)} \tilde{\boldsymbol{y}}^{(\ell)} \\ &= \sum_{s=0}^{j} \binom{j}{s} m^{j-s} \Upsilon_s^{(\nu)} \boldsymbol{z}_{j-s}^{(\nu)} \\ &= \sum_{s=0}^{j-\tilde{q}} \binom{j}{s} m^{j-s} \Upsilon_s^{(\nu)} \boldsymbol{z}_{j-s}^{(\nu)}\end{aligned} \tag{5.8}$$

for $j = \tilde{q}, \ldots, \tilde{p}-1$. These equations can again be solved successively for $\Upsilon_j^{(\nu)}$ and then $T_{k_0+j}^{(\nu)}$, $j = 0, \ldots, \tilde{p}-\tilde{q}$.

The modified algorithm is faster than the original one. However, it frequently results in longer new masks than the original algorithm. This is illustrated by the examples in section 7.

6. Algorithms. The following algorithms are implementations of the procedures outlined in the previous two sections. They can be used to find suitable lifting factors of any desired length, with free parameters explicitly identified.

Assume that $\mathcal{H}$, $\tilde{\mathcal{H}}$ are biorthogonal masks, with $\mathcal{H}$ satisfying condition E.

6.1. Algorithm 1. Given integers $\tilde{p} \geq 1$, $n \geq 1$, k_0 arbitrary, we want to find matrix trigonometric polynomials of length n with starting index k_0

$$L^{(\nu)}(\xi) = \sum_{k=k_0}^{k_0+n-1} L_k^{(\nu)} e^{-ik\xi} \tag{6.1}$$

so that the new dual mask $\tilde{\mathcal{H}}_{\text{new}}$ produced by the lifting process (3.3) has approximation order $\tilde{p}$.

Step 1. Compute the moments

$$M_j^{(\nu)} = \frac{1}{\sqrt{m}} \sum_k k^j h_k^{(\nu)} \tag{6.2}$$

for $\nu = 0, \ldots, m-1$ and $j = 0, \ldots, \tilde{p}-1$.

Step 2. Compute the vectors $\tilde{\boldsymbol{y}}^{(j)}$, $\boldsymbol{z}_j^{(\nu)}$ for $\nu = 1, \ldots, m-1$ and $j = 0, \ldots, \tilde{p}-1$ from

$$\begin{aligned} \tilde{\boldsymbol{y}}^{(0)} &= M_0^{(0)} \tilde{\boldsymbol{y}}^{(0)}, \\ \tilde{\boldsymbol{y}}^{(j)} &= \left(m^j I - M_0^{(0)} \right)^{-1} \sum_{s=0}^{j-1} \binom{j}{s} M_{j-s}^{(0)} \tilde{\boldsymbol{y}}^{(s)}, \\ \boldsymbol{z}_j^{(\nu)} &= m^{-j} \sum_{\ell=0}^{j} \binom{j}{\ell} M_\ell^{(\nu)} \tilde{\boldsymbol{y}}^{(j-\ell)}. \end{aligned} \tag{6.3}$$

Step 3. Form the matrices

$$A = \begin{pmatrix} k_0^0 I & k_0^1 I & \cdots & k_0^{\tilde{p}-1} I \\ (k_0+1)^0 I & (k_0+1)^1 I & \cdots & (k_0+1)^{\tilde{p}-1} I \\ \vdots & \vdots & \ddots & \vdots \\ (k_0+n-1)^0 I & (k_0+n-1)^1 I & \cdots & (k_0+n-1)^{\tilde{p}-1} I \end{pmatrix}, \tag{6.4}$$

$$Y = \begin{pmatrix} \binom{0}{0} \tilde{\boldsymbol{y}}^{(0)} & \binom{1}{0} \tilde{\boldsymbol{y}}^{(1)} & \binom{2}{0} \tilde{\boldsymbol{y}}^{(2)} & \cdots & \binom{\tilde{p}-1}{0} \tilde{\boldsymbol{y}}^{(\tilde{p}-1)} \\ & \binom{1}{1} \tilde{\boldsymbol{y}}^{(0)} & \binom{2}{1} \tilde{\boldsymbol{y}}^{(1)} & \cdots & \binom{\tilde{p}-1}{1} \tilde{\boldsymbol{y}}^{(\tilde{p}-2)} \\ & & \binom{2}{2} \tilde{\boldsymbol{y}}^{(0)} & \cdots & \binom{\tilde{p}-1}{2} \tilde{\boldsymbol{y}}^{(\tilde{p}-3)} \\ & & & \ddots & \vdots \\ & & & & \binom{\tilde{p}-1}{\tilde{p}-1} \tilde{\boldsymbol{y}}^{(0)} \end{pmatrix}, \tag{6.5}$$

and

$$Z = \begin{pmatrix} \boldsymbol{z}_0^{(1)} & \cdots & \boldsymbol{z}_{\tilde{p}-1}^{(1)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{z}_0^{(m-1)} & \cdots & \boldsymbol{z}_{\tilde{p}-1}^{(m-1)} \end{pmatrix}. \tag{6.6}$$

The equations (4.13) are equivalent to

$$LAY = -Z, \tag{6.7}$$

where L contains the desired coefficients of $L^{(\nu)}(\xi)$

$$L = \begin{pmatrix} L_{k_0}^{(1)} & \cdots & L_{k_0+\tilde{p}-1}^{(1)} \\ \vdots & \ddots & \vdots \\ L_{k_0}^{(m-1)} & \cdots & L_{k_0+\tilde{p}-1}^{(m-1)} \end{pmatrix}. \tag{6.8}$$

Step 4. Perform a singular value decomposition (SVD)

$$AY = U \Sigma V^*. \tag{6.9}$$

Here U is of size $nr \times nr$, Σ is of size $nr \times \tilde{p}$, and V is of size $\tilde{p} \times \tilde{p}$. Let s be the rank of Σ, then

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{pmatrix} \tag{6.10}$$

with Σ_{11} nonsingular and of size $s \times s$.

Substitute the SVD into (6.7), multiply by V on the right, and partition all matrices corresponding to the partitioning of Σ

$$((LU)_1, (LU)_2) \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{pmatrix} = -\left((ZV)_1, (ZV)_2\right), \tag{6.11}$$

or

$$\begin{aligned} (LU)_1 \Sigma_{11} &= -(ZV)_1, \\ 0 &= (ZV)_2. \end{aligned} \tag{6.12}$$

If $(ZV)_2 \neq 0$, the system is unsolvable. Go back to the start and increase n. Otherwise, the solution is

$$\begin{aligned} (LU)_1 &= -(ZV)_1 \Sigma_{11}^{-1}, \\ (LU)_2 &= \text{arbitrary}. \end{aligned} \tag{6.13}$$

The general solution is then

$$L = ((LU)_1, (LU)_2) U^*. \tag{6.14}$$

The free parameters are the elements of $(LU)_2$, of which there are $r(nr - s)(m - 1)$.

Step 5. Assemble the $L^{(\nu)}(\xi)$ and perform the lifting.

Step 6. If required, verify that $\tilde{\mathcal{H}}_{\text{new}}$ satisfies condition E or other properties. If necessary, use optimization on the free parameters to satisfy these conditions.

6.2. Algorithm 2. Given integers $\tilde{p} \geq 1$, $n \geq 1$, k_0 arbitrary, we want to find matrix trigonometric polynomials of length n and starting index k_0

$$T^{(\nu)}(\xi) = \sum_{k=k_0}^{k_0+n-1} T_k^{(\nu)} e^{-ik\xi} \tag{6.15}$$

so that the new dual mask $\tilde{\mathcal{H}}_{\text{new}}$ produced by the lifting process (3.3) performed with $L^{(\nu)}(\xi) = T^{(\nu)}(\xi) H^{(\nu)}(\xi)$ has approximation order $\tilde{p}$.

Steps 1 and 2 are the same as in Algorithm 1.

The existing approximation order $\tilde{q}$ of $\tilde{\mathcal{H}}$ is the largest number $\tilde{q}$ for which $\boldsymbol{z}_j^{(\nu)} = \mathbf{0}$ for all $\nu = 1, \ldots, m$ and $j = 0, \ldots, \tilde{q} - 1$.

Step 3. Form the matrices

$$A = \begin{pmatrix} k_0^0 I & k_0^1 I & \cdots & k_0^{\tilde{p}-\tilde{q}-1} I \\ (k_0+1)^0 I & (k_0+1)^1 I & \cdots & (k_0+1)^{\tilde{p}-\tilde{q}-1} I \\ \vdots & \vdots & \ddots & \vdots \\ (k_0+n-1)^0 I & (k_0+n-1)^1 I & \cdots & (k_0+n-1)^{\tilde{p}-\tilde{q}-1} I \end{pmatrix}, \tag{6.16}$$

$$Y^{(\nu)} = \begin{pmatrix} \binom{\tilde{q}}{0} m^{\tilde{q}} \tilde{\boldsymbol{z}}_{\tilde{q}}^{(\nu)} & \binom{\tilde{q}+1}{0} m^{\tilde{q}+1} \tilde{\boldsymbol{z}}_{\tilde{q}+1}^{(\nu)} & \cdots & \binom{\tilde{p}-1}{0} m^{\tilde{p}-1} \tilde{\boldsymbol{z}}_{\tilde{p}-1}^{(\nu)} \\ & \binom{\tilde{q}+1}{1} m^{\tilde{q}} \tilde{\boldsymbol{z}}_{\tilde{q}}^{(\nu)} & \cdots & \binom{\tilde{p}-1}{1} m^{\tilde{p}-2} \tilde{\boldsymbol{z}}_{\tilde{p}-2}^{(\nu)} \\ & & \ddots & \vdots \\ & & & \binom{\tilde{p}-1}{\tilde{p}-\tilde{q}-1} m^{\tilde{q}} \tilde{\boldsymbol{z}}_{\tilde{q}}^{(\nu)} \end{pmatrix}, \tag{6.17}$$

and

$$Z^{(\nu)} = \left(\boldsymbol{z}_{\tilde{q}}^{(\nu)}, \ldots, \boldsymbol{z}_{\tilde{p}-1}^{(\nu)} \right). \tag{6.18}$$

The equations (5.8) are equivalent to the sequence of equations

$$T^{(\nu)} A Y^{(\nu)} = -Z^{(\nu)} \tag{6.19}$$

for $\nu = 1, \ldots, m-1$, where $T^{(\nu)}$ contains the desired coefficients of $T^{(\nu)}(\xi)$

$$T^{(\nu)} = \left(T_{k_0}^{(\nu)}, \ldots, T_{k_0+n-1}^{(\nu)} \right). \tag{6.20}$$

Step 4. This step is repeated for each $\nu = 1, \ldots, m-1$.
Perform the SVD

$$AY^{(\nu)} = U^{(\nu)} \Sigma^{(\nu)} V^{(\nu)*}. \tag{6.21}$$

and proceed as in Step 4 of Algorithm 1.

The free parameters are the elements of $(T^{(\nu)} U^{(\nu)})_2$. Their total number is

$$r \sum_{\nu} (nr - s^{(\nu)}) = nr^2(m-1) - r \sum_{\nu} s^{(\nu)}. \tag{6.22}$$

Steps 5 and 6 are the same as in Algorithm 1.

6.3. Further comments.

6.3.1. About the implementation. In Algorithm 1, it is possible in Step 3 to solve the equations for all ν simultaneously, since Y is independent of ν. This is not possible in Algorithm 2. As Example 2 in section 7 illustrates, the ranks $s^{(\nu)}$ may vary with ν.

In order to obtain dual approximation order $\tilde{p}$, we need to impose conditions on $\Lambda_j^{(\nu)}$, $j = 1, \ldots, \tilde{p}-1$. We can always find a suitable $L^{(\nu)}$ of length $\tilde{p}$, but we want $L^{(\nu)}$ of length n, with $n < \tilde{p}$ in general. Setting the higher moments to 0 does not result in shorter $L^{(\nu)}$. Instead, we incorporate the matrix A into the algorithms, and solve for $L_j^{(\nu)}$ directly.

6.3.2. Choosing $\boldsymbol{n}$, $\boldsymbol{k_0}$. Choosing n as small as possible results in the shortest possible new wavelets, which is usually desirable. Since each approximation order condition amounts to r scalar equations, and each $L_j^{(\nu)}$ contains r^2 coefficients, dual approximation order $\tilde{p}$ should require a smallest possible n of

$$n = \operatorname{ceil}(\tilde{p}/r). \tag{6.23}$$

This cannot be guaranteed (there are counterexamples), but (6.23) gives a good estimate. In particular, constant $L^{(\nu)}$, which does not increase the support lengths of the functions, will in general be able to achieve approximation order r already.

Larger n could be used if extra free parameters are desired.

The starting subscript k_0 of $L^{(\nu)}$ affects both the support length and the centering of the new multiwavelet (see formulas (3.6)). Numerical experiments indicate that the new dual wavelets tend to be smoother if k_0 is chosen so that the scaling function and new dual scaling function are approximately centered around the same point (see Example 1(c) in section 7).

The algorithms could easily be generalized to allow different n, k_0 for each ν.

6.3.3. Stability. We now address the question raised in section 2, regarding the stability properties of the new masks $\mathcal{H}_{\rm new}$, $\tilde{\mathcal{H}}_{\rm new}$ (see section 2). The complete answer is not known, but we offer some observations.

1. *Stability of $\boldsymbol{\phi}^{(0)}$ is preserved. Stability of $\tilde{\boldsymbol{\phi}}^{(0)}$ is not preserved in general, but we may be able to preserve it by choosing a suitable L (possibly of higher degree).*

The first part is obvious, since $\boldsymbol{\phi}^{(0)}_{\rm new} = \boldsymbol{\phi}^{(0)}$.

It is shown in [7] (for scalar wavelets) that any polyphase matrix can be completely factored into lifting steps. Since lifting steps are reversible, this means that any polyphase matrix can be converted into any other by multiple lifting and dual lifting steps. Obviously, stability can get lost in the process.

However, we can always preserve condition E for $\tilde{\boldsymbol{\phi}}^{(0)}$, which is a prerequisite for stability. By (3.5), line 3,

$$\tilde{M}^{(0)}_{{\rm new},0} = \tilde{M}^{(0)}_0 - \sum_{\nu=1}^{m-1} \Lambda^{(\nu)*}_0 \tilde{M}^{(\nu)}_0. \tag{6.24}$$

If we choose $\Lambda^{(\nu)}(0) = 0$, then $\tilde{M}^{(0)}_0 = \tilde{M}^{(0)}_{{\rm new},0}$, and condition E remains valid. This approach may require increasing n.

In most of the numerical examples we tried, condition E was preserved automatically. In the remaining cases, a simple change in the free parameters was sufficient, with no increase in n needed.

We conjecture that stability of $\tilde{\boldsymbol{\phi}}^{(0)}$ can also be preserved, but the necessary additional conditions on L are not known.

2. *Stability of $\boldsymbol{\phi}$ is preserved.*

If L has finite degree, and $\{\boldsymbol{\phi}^{(\nu)}\}$ form a stable completion of $\boldsymbol{\phi}^{(0)}$, so do $\{\boldsymbol{\phi}^{(\nu)}_{\rm new}\}$. This follows from Proposition 3.1 in [2].

3. *It is not known whether stability of $\tilde{\boldsymbol{\phi}}$ or stability over all levels for $\boldsymbol{\phi}$ and $\tilde{\boldsymbol{\phi}}$ can be preserved.*

6.3.4. Free parameters. Free parameters that occur during the lifting process can be used for numerical optimization. One defines a function that takes the free parameters as input, calculates the new masks produced by a lifting step with these parameters, and then calculates some objective function which is to be maximized or minimized.

In Example 1(c) in section 7, we used the Sobolev smoothness estimate from [18] as the objective function, to produce the smoothest possible new dual scaling functions for given n and k_0.

7. Examples. We illustrate our algorithms with some numerical examples.

Example 1. This example has dilation factor $m = 2$, multiplicity $r = 2$. We start with cubic Hermite splines as the original scaling function [31]. A basic completion to a biorthogonal pair of masks has the symbols

$$\begin{aligned} H(z) &= \frac{1}{16}\begin{pmatrix} 4+8z+4z^2 & 6-6z^2 \\ -1+z^2 & -1+4z-z^2 \\ 8 & 0 \\ 0 & 8 \end{pmatrix}, \\ \tilde{H}(z) &= \frac{1}{4z}\begin{pmatrix} 4z^2 & 0 \\ 0 & 8z^2 \\ -2+4-2z^2 & -1+z^2 \\ 3-3z^2 & 1+4z+z^2 \end{pmatrix}, \end{aligned} \tag{7.1}$$

where $z = \exp(-i\xi)$. The original dual approximation order is 0. $\mathcal{H}$ satisfies condition E, $\tilde{\mathcal{H}}$ does not.

(a) Raise the dual approximation order from 0 to 2.

Algorithm 1 can achieve this with $n = 1$, and no free parameters. For $k_0 = 0$, the result is

$$L(z) = \frac{1}{4} \begin{pmatrix} -2 & 15 \\ 0 & -1 \end{pmatrix}, \tag{7.2}$$

$$\begin{aligned} H_{\text{new}}(z) &= \frac{1}{64} \begin{pmatrix} 16 + 32z + 16z^2 & 24 - 24z^2 \\ -4 + 4z^2 & -4 + 16z - 4z^2 \\ 9 - 16z + 7z^2 & -27 + 60z - 3z^2 \\ 1 - z^2 & 33 - 4z + z^2 \end{pmatrix}, \\ \tilde{H}_{\text{new}}(z) &= \frac{1}{16z} \begin{pmatrix} -4 + 8z + 12z^2 & -2 + 2z^2 \\ 33 - 60z + 27z^2 & 16 + 4z + 18z^2 \\ -8 + 16z - 8z^2 & -4 + 4z^2 \\ 12 - 12z^2 & 4 + 16z + 4z^2 \end{pmatrix}. \end{aligned} \tag{7.3}$$

The new masks have length 3. Any other choice of k_0 results in longer masks.

Algorithm 2 gives identical results, since $H^{(1)}(z)$ is a multiple of the identity.

(b) Raise the dual approximation order from 2 to 4, starting with the H_{new} from (a).

Algorithm 1 can achieve this with $n = 2$, and no free parameters. The shortest new masks have length 5, for $k_0 = -1$, and are generated by lifting using

$$L(z) = \frac{1}{48z} \begin{pmatrix} -12 + 12z & -63 - 117z \\ 2 - 2z & 9 + 21z \end{pmatrix}. \tag{7.4}$$

Algorithm 2 requires only $n = 1$, with no free parameters, but the shortest new masks (also for $k_0 = -1$) have length 7. The lifting factor is

$$L(z) = \frac{1}{13824z} \begin{pmatrix} -729 + 1152z - 423z^2 & -729 - 3996z + 135z^2 \\ 81 - 160z + 79z^2 & -567 + 636z - 39z^2 \end{pmatrix}, \tag{7.5}$$

which is produced from

$$T(z) = \frac{1}{216z} \begin{pmatrix} -72 & -81 \\ 10 & -9 \end{pmatrix}. \tag{7.6}$$

(c) Raise the dual approximation order from 0 to 2 (starting again with the original $\mathcal{H}$, $\tilde{\mathcal{H}}$) with free parameters, and optimize for smoothness.

The algorithm described in [18] can be used to determine a lower bound on the Sobolev exponent s of a multiscaling function. The shortest dual scaling function with approximation order 2 derived in (a) is in the Sobolev space $W^{-1.2294}$, so it is not even an L^2-function.

If we apply Algorithm 1 with $n = 2$, there are 4 free parameters. The shortest new scaling function symbols have length 5 for $k_0 = -1$ or $k_0 = 0$.

For $k_0 = 0$, the coefficients of H_{new} and $\tilde{H}_{\text{new}}$ are centered at 2 and -1, respectively. Numerical optimization of the Sobolev exponent yields a smoothest $\tilde{H}_{\text{new}}$ in $W^{-0.7877}$.

For $k_0 = -1$, the centers are 0 and 1, which is a better fit. The smoothest $\tilde{H}_{\text{new}}$ is in $W^{0.8289}$, which matches the result of [30], [32]. This $\tilde{H}_{\text{new}}$ satisfies condition E, and could be used as the basis for a further dual lifting step.

Example 2. This example has dilation factor $m = 3$, multiplicity $r = 1$. We take $\phi^{(0)}$ to be the characteristic function of $[0, 1]$, i.e., the Haar scaling function, with approximation order 1. A completion with dual approximation order 1 is

$$\begin{aligned} H(z) &= \frac{1}{9}\begin{pmatrix} 3 + 3z + 3z^2 \\ \sqrt{3}\left(-1 + 2z - z^2\right) \\ \sqrt{3}\left(-1 - z + 2z^2\right) \end{pmatrix}, \\ \tilde{H}(z) &= \frac{1}{3}\begin{pmatrix} 1 + z + z^2 \\ \sqrt{3}\,(-1 + z) \\ \sqrt{3}\left(-1 + z^2\right) \end{pmatrix}. \end{aligned} \tag{7.7}$$

Condition E is satisfied by the original masks. Since $r = 1$, it is automatically preserved.

We want to raise the dual approximation order to 3.

Algorithm 1 requires $n = 3$, with no free parameters. The choice $k_0 = -1$ produces the shortest new masks of length 9

$$\begin{aligned} H_{\text{new}}(z) &= \frac{1}{243z^3}\begin{pmatrix} 81z^3 + 81z^4 + 81z^5 \\ \sqrt{3}\left(1 + z + z^2 - 29z^3 + 52z^4 - 29z^5 + z^6 + z^7 + z^8\right) \\ \sqrt{3}\left(4 + 4z + 4z^2 - 26z^3 - 26z^4 + 55z^5 - 5z^6 - 5z^7 - 5z^8\right) \end{pmatrix}, \\ \tilde{H}_{\text{new}}(z) &= \frac{1}{81z^3}\begin{pmatrix} -4 - z + 5z^2 + 26z^3 + 29z^4 + 26z^5 + 5z^6 - z^7 - 4z^8 \\ \sqrt{3}\left(-27z^3 + 27z^4\right) \\ \sqrt{3}\left(-27z^3 + 27z^5\right) \end{pmatrix}, \end{aligned} \tag{7.8}$$

using

$$L(z) = \frac{1}{27\sqrt{3}z}\begin{pmatrix} 1 - 2z + z^2 \\ 4 + z - 5z^2 \end{pmatrix}. \tag{7.9}$$

Algorithm 2 requires at least $n = 2$, and produces new masks of length 12. There is one free parameter, in $T^{(1)}$ only.

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