INVERSION OF k-PLANE TRANSFORMS AND APPLICATIONS IN COMPUTER TOMOGRAPHY*

FRITZ KEINERT†

Abstract. The mathematics behind Computerized Tomography (CT) is based on the study of the parallel beam transform P and the divergent beam transform D. Both of these map a function f in $\mathbb{R}^n$ into a function defined on the set of all lines in $\mathbb{R}^n$, by integrating f along these lines.

The parallel and divergent k-plane transforms are defined in a similar fashion by integration over k-planes (i.e., translates of k-dimensional subspaces) and are also denoted P and D, respectively. A related transform is the spherical k-plane transform S, which maps a function f on the sphere S^{n-1} into its integrals over k-dimensional great circles.

This paper discusses the properties of the k-plane transforms P, D, and S and their inverses, emphasizing relationships and similarities between the operators, and their relation to CT.

Some new results are included. Most notable are more general conditions under which the inversion formulas hold.

Key words. parallel beam transform, divergent beam transform, k-plane transform, Radon transform, x-ray transform, image reconstruction, tomography, local tomography

AMS(MOS) subject classifications. primary 44A15; secondary 47A05, 65R10

1. Introduction.

In 1917, Johann Radon published a paper ([12], reprinted in [5]), in which he posed and solved the problem of recovering a function f on $\mathbb{R}^2$ from its line integrals. This paper was then largely forgotten for decades.

Several decades later, John [8] used the decomposition of a function on $\mathbb{R}^n$ in terms of its hyperplane integrals (the so-called *Radon transform*) in the study of partial differential equations. Helgason [3]–[5] studied problems similar to Radon's in more general spaces (such as reconstructing a function in a space of constant curvature from its integral over geodesics.) All of these approaches were mostly of theoretical interest.

With the increasing availability of computers, problems similar to Radon's have become popular in applied sciences. Usually, a function f on $\mathbb{R}^2$ or $\mathbb{R}^3$ is determined from its integrals over lines or planes. Here f represents some internal property of an object which cannot be observed directly, while some integrals of f are observable. Let us consider the case of Computerized X-Ray Tomography (CT or CAT scans) as an example, ignoring some of the details.

When a monochromatic x-ray beam of intensity I_0 is sent through a uniform object of thickness Δs, the initial intensity I_0 and final intensity I of the beam are related by

$$I = I_0 e^{-\mu \Delta s}, \tag{1.1}$$

where μ is the *x-ray attenuation coefficient*, a material-specific constant. (For all practical purposes, $\mu = 0$ for air.) When several objects with attenuation coefficients μ_i and thicknesses Δs_i are stacked up, the relation becomes

$$I = I_0 e^{-\sum_i \mu_i \Delta s_i}. \tag{1.2}$$

* Received by the editors November 24, 1986; accepted for publication (in revised form) September 27, 1988

† Department of Mathematics, Iowa State University, Ames, Iowa 50011.

By the usual limiting arguments of calculus, we find

$$I = I_0 e^{-\int_L \mu(s)\,ds} \tag{1.3}$$

in the case when the beam is sent through a nonhomogeneous object along a line L. Thus we can measure the line integrals of the function μ by

$$\int_L \mu(s)\,ds = -\log(I/I_0). \tag{1.4}$$

In most applications line integrals are measured, as in CT. In some cases, such as Magnetic Resonance Imaging (MRI), it is possible to measure plane integrals directly. As mentioned above, hyperplane integrals have been used in theoretical research.

These examples motivate the study of the general problem of reconstructing a function f on $\mathbb{R}^n$ from its integrals over k-dimensional planes (called k-planes for short.) Here *plane* means a translate of a subspace.

In the early days of CT, line integrals were measured for a set of parallel lines (see Fig. 1), and these measurements repeated for a number of different directions. Modern CT uses lines emanating from a common source point (see Fig. 2), and measurements are made from a number of sources.

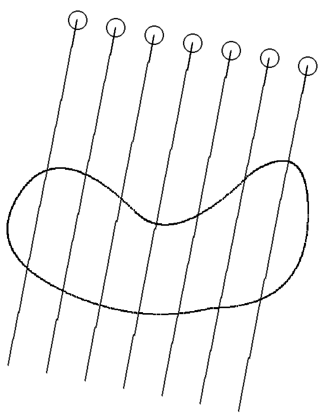

FIG. 1. *Parallel beam setup.*

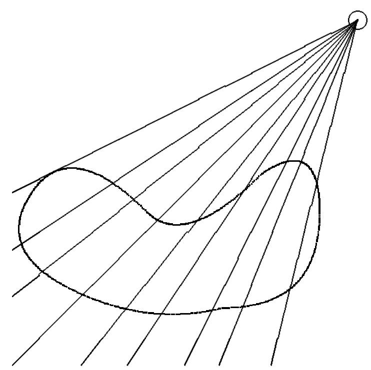

FIG. 2. *Divergent beam setup.*

For the mathematical description of these two geometries, we use two different transforms. The *parallel k-plane transform* Pf of a function f in $\mathbb{R}^n$ is given by

$$Pf(\pi, x'') = P_\pi f(x'') = \int_\pi f(x' + x'')\,dx', \tag{1.5}$$

where π is a k-dimensional subspace of $\mathbb{R}^n$. Both notations Pf and $P_\pi f$ are used, depending on whether we want to consider the measurements over one set of parallel planes or the totality of all measurements.

The *divergent k-plane transform* Df of f is given by

$$Df(\pi, a) = D_a f(\pi) = \int_\pi f(a + x')\,dx', \tag{1.6}$$

where a is the source point.

A transform closely related to D is the *spherical k-plane transform* S defined by

$$Sf(\pi) = \int_{S^{n-1}\cap\pi} f(\theta)\, d\theta, \tag{1.7}$$

where f is a function on S^{n-1}.

Treatments of the parallel k-plane transform can be found in Helgason [3], [5] and Solmon [20]. The spherical k-plane transform appears in Helgason [3], [5] and Strichartz [21]. The divergent line transform ($k = 1$) is discussed in Hamaker et al. [2]. The divergent k-plane transform for $k > 1$ has not been described before. Applications of these transforms to CT and MRI are discussed in detail in Herman [7] and Natterer [11].

One of the goals of this paper is to gather results from the literature, together with new results for the divergent k-plane transform, and arrange them in a coherent fashion. There are many similarities in the properties of the three transforms, and similar methods can be used to study them.

Another objective is to justify the formulas obtained for P, D, and their inverses in a rigorous way, under minimal conditions on the functions involved. Other authors have either used very smooth, rapidly decaying functions or functions of compact support, for which the formulas are not hard to prove. However, the functions actually used in many reconstruction algorithms do not satisfy these criteria, while they do meet the criteria established below.

Throughout the paper, the emphasis is on the inversion of the transforms, also called "reconstructing the original function." We think of Pf or Df as the measured data, from which we want to reconstruct the original function f.

The outline of the paper is as follows.

Sections 2, 3, and 4 introduce the transforms P, D, and S and describe some of their basic properties, such as domains of definition. Some auxiliary results needed later are also established there.

Section 5 contains uniqueness and nonuniqueness theorems. The purpose of these theorems is to decide how much information is necessary to uniquely determine the original functions, and how much we can infer about the original function from partial data. The theorems for P have been known for a while but never seem to have been stated in their full form in the literature.

In § 6, formal inversion formulas for D and P are derived, including both exact and approximate formulas. "Formal" means that we do not worry at this point whether the functions involved satisfy the proper smoothness or decay conditions that allow us to take Fourier transforms, interchange integrations, etc.

Section 7 is the longest and most technical, and states exact results about the conditions under which the results in § 6 are valid, extending earlier work in [17].

In § 8, approximate inversion formulas for the reconstruction of Λf rather than f are given, where the operator Λ is defined in terms of the Fourier transform $\tilde{}$ by

$$(\Lambda f)^{\tilde{}} = |\xi|\tilde{f}. \tag{1.8}$$

Thus $\Lambda^2 = -\Delta$, where Δ is the Laplace operator. While quantitative information is lost, Λ enhances the boundaries between regions of different densities and enables local reconstructions. This technique is known as *Lambda Tomography* or *Local Tomography* and is fairly new. Details and examples can be found in Faridani et al. [1].

2. The transforms P, D, S. As outlined in the Introduction, we would like to study the reconstruction of a function f in $\mathbb{R}^n$ from its integrals over k-planes, i.e., translates of k-dimensional subspaces of $\mathbb{R}^n$.

In many cases, including earlier versions of CT, integrals over a number of parallel lines in a given direction are measured (see Fig. 1.) The measurements are then repeated for many directions. This measuring geometry leads to the definition of the *parallel k-plane transform* P below. In the case $k = 1$ (integrals over lines) this transform is also known as the *parallel beam transform* or *x-ray transform.* In the case $k = (n-1)$ (integrals over hyperplanes) it is called the *Radon transform,* after Radon's early paper [12].

In modern CT, integrals are measured for lines emanating from a common source (see Fig. 2), which motivates the definition of the *divergent k-plane transform* D below. This transform has previously only been studied for the case $k = 1$, the so-called *divergent beam transform.*

In the study of D, especially, it is often useful to use polar coordinates. This leads in a natural way to the study of the *spherical k-plane transform* S*,* which maps a function f defined on the unit sphere S^{n-1} in $\mathbb{R}^n$ into its integrals over k-dimensional great circles on the sphere. The resulting formulas are interesting in their own right and are used in deriving formulas for P and D, but they do not have any direct application to imaging.

In the remainder of this section, we give exact definitions of the transforms P, D, and S and establish some identities needed in later sections.

Let f be a measurable function on $\mathbb{R}^n$. $G_{k,n}$ is the Grassmann manifold of (non-oriented) k-dimensional subspaces of $\mathbb{R}^n$. For $\pi \in G_{k,n}$, $\pi^\perp$ is the $(n-k)$-dimensional subspace perpendicular to π. A point x in $\mathbb{R}^n$ is often written $x = x' + x''$, where x', x'' are the components of x in π, $\pi^\perp$ respectively. In the case of line integrals ($k = 1$), it is customary to use the letter θ instead of π, where $\theta \in S^{n-1}$.

Occasionally, we also use the manifold $\mathcal{G}_{k,n}$ of all k-planes in $\mathbb{R}^n$,

$$\mathcal{G}_{k,n} = \{(\pi, x'') : \pi \in G_{k,n},\ x'' \in \pi^\perp\}.$$

If $\pi \in G_{k,n}$, $x'' \in \pi^\perp$, the *parallel k-plane transform* of f (in direction π) is defined by

$$Pf(\pi, x'') = P_\pi f(x'') = \int_\pi f(x' + x'')\, dx', \tag{2.1}$$

whenever the Lebesgue integral exists. Thus, $P_\pi f(x'')$ is the integral of f over the translate of the k-dimensional subspace π passing through x''.

From the definition it follows that if f is a function on $\mathbb{R}^n$, then $P_\pi f$ (for fixed π) is a function on $\pi^\perp$, and Pf is a function on $\mathcal{G}_{k,n}$. The notation $P_\pi f(x'')$ underscores the idea that one wants to consider various measurements in the same "direction" π.

If $a \in \mathbb{R}^n$ and $\pi \in G_{k,n}$, the *divergent k-plane transform* of f is defined by

$$Df(\pi, a) = D_a f(\pi) = \int_\pi f(a + x')\, dx', \tag{2.2}$$

whenever the Lebesgue integral exists. Here we are considering the integrals over k-planes passing through the common point a, called the *source*.

$D_a f$ (for fixed a) is defined on $G_{k,n}$, Df on $G_{k,n} \times \mathbb{R}^n$.

The transforms P and D are related by

$$Df(\pi, x) = Pf(\pi, E_{\pi^\perp} x), \tag{2.3}$$

where $E_{\pi^\perp} x$ is the orthogonal projection of x onto $\pi^\perp$.

If f is a measurable function on S^{n-1} (the unit sphere in $\mathbb{R}^n$), and $\pi \in G_{k,n}$, the *spherical k-plane transform* of f is defined by

$$Sf(\pi) = \int_{S^{n-1}\cap\pi} f(\theta)\, d\theta, \tag{2.4}$$

whenever the Lebesgue integral exists. Thus $Sf(\pi)$ is the integral of f over the k-dimensional great circle obtained by intersecting π with the sphere. If f is a function on S^{n-1}, Sf is defined on $G_{k,n}$.

While it is possible to derive most properties of these transforms directly by means of calculus, it is much easier and more elegant to consider the transform S as a special case of a general theory of Helgason, first mentioned in [3] (see [6] for a recent description.) One can then extend the results to P and D. Some similarities between S, P, and D also become more apparent this way. The important features of Helgason's theory are briefly summarized here.

Note. It is possible to fit S and P directly into the framework described below, but not D. The approach we are taking here is slightly different: we only treat S this way, then go to D from there and lastly to P, using relation (2.3).

The starting point for Helgason's generalization was the following observation: to calculate the parallel or divergent beam transform we need to integrate a function over all points on a line. The inversion formulas in both cases involve integration over all lines passing through a given point. This duality leads to the definition of *incidence* below and its application in the definition of a transform pair.

Let G be a Lie group with closed subgroups H_1, H_2, so that $X = G/H_1$, $\Xi = G/H_2$ are homogeneous spaces of G. X, Ξ are differentiable manifolds with G-invariant measures dx, $d\xi$, unique up to multiplication by a constant.

$x \in X$, $\xi \in \Xi$ are called *incident* if the corresponding cosets in G intersect. With the proper choice of G, H_1, H_2, the elements of X, Ξ can be identified with geometric objects, and incidence (in the sense just defined) is equivalent to geometric intersection.

Let

$$\begin{aligned} \check{x} &= \{\xi \in \Xi : x \text{ and } \xi \text{ are incident}\} \\ \hat{\xi} &= \{x \in X : x \text{ and } \xi \text{ are incident}\} \end{aligned} \tag{2.5}$$

Under rather general conditions (H_1, H_2 compact is sufficient), the sets $\check{x}$, $\hat{\xi}$ have canonical G-invariant measures, also called $d\xi$, dx in the following, so that dual transforms $\hat{}$, $\check{}$ can be defined by

$$\begin{aligned} \hat{f}(\xi) &= \int_{\hat{\xi}} f(x)\, dx \\ \check{\phi}(x) &= \int_{\check{x}} \phi(\xi)\, d\xi. \end{aligned} \tag{2.6}$$

Provided the measures are normalized correctly, the following theorem holds (reminiscent of the Plancherel theorem for the Fourier transform.)

Theorem 2.1 (Helgason). *Let f, g be nonnegative, measurable functions defined almost everywhere on X, Ξ, respectively. Then $\hat{f}$, $\check{g}$ are measurable, and*

$$\int_X f(x)\, \check{g}(x)\, dx = \int_\Xi \hat{f}(\xi)\, g(\xi)\, d\xi. \tag{2.7}$$

As the main example, consider the spherical k-plane transform. Here $G = O(n)$, $X = O(n)/O(n-1)$, $\Xi = O(n)/O(k)\times O(n-k)$, where $O(n)$ is the group of orthogonal transformations of $\mathbb{R}^n$. X can be identified with the unit sphere S^{n-1}, Ξ with the Grassmannian $G_{k,n}$ in such a way that the incidence relation has the usual geometric meaning. That is, a point θ on the sphere S^{n-1} is incident to a k-dimensional subspace $\pi \in G_{k,n}$ if and only if $\theta \in \pi$. The measure of $X = S^{n-1}$ is normalized to the usual

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{2.8}$$

and the measure of the Grassmannian is defined as

$$\begin{aligned} |G_{k,n}| &= \frac{|S^{n-1}|\,|S^{n-2}|\cdots|S^{n-k}|}{2\,|S^{k-1}|\,|S^{k-2}|\cdots|S^{1}|}, \qquad k \geq 2, \\ |G_{1,n}| &= |S^{n-1}|/2, \\ |G_{0,n}| &= 1 \end{aligned} \tag{2.9}$$

following Santalo [15].

For $\pi \in G_{k,n}$, the set $\hat{\pi}$ of points on S^{n-1} incident to π equals $\pi \cap S^{n-1}$ and can be identified with S^{k-1}, with its usual measure. Thus, S corresponds exactly to the $\hat{}$ transform above.

For $\theta \in S^{n-1}$, $\check{\theta}$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$ which contain the vector θ. We will write this set $\Gamma_{k,n}(\theta)$ from now on. $\Gamma_{k,n}(\theta)$ is in one-to-one correspondence with the $(k-1)$-dimensional subspaces of the hyperplane $\theta^\perp$ perpendicular to θ, and can therefore be identified with the Grassmannian $G_{k-1,n-1}$. This correspondence also gives the $O(n)$-invariant measure on $\check{\theta}$. The total measure of $\Gamma_{k,n}(\theta)$ will be denoted by $\gamma_{k,n}$, thus

$$\gamma_{k,n} = |G_{k-1,n-1}|. \tag{2.10}$$

With these normalizations, (2.7) reads

$$\int_{S^{n-1}} f(\theta) \int_{\Gamma_{k,n}(\theta)} g(\pi)\,d\pi\,d\theta = \int_{G_{k,n}} g(\pi) \int_{S^{n-1}\cap\pi} f(\theta)\,d\theta\,d\pi. \tag{2.11}$$

By setting $f \equiv 1$, $g \equiv 1$, we can check that the normalization is correct.

Corollary 2.2. *If g is nonnegative, measurable, and defined almost everywhere on $G_{k,n}$, then*

$$\int_{G_{k,n}} g(\pi)\,d\pi = \frac{1}{|S^{k-1}|} \int_{S^{n-1}} \int_{\Gamma_{k,n}(\theta)} g(\pi)\,d\pi\,d\theta. \tag{2.12}$$

Proof. Set $f \equiv 1$ in (2.11). □

Corollary 2.3. *If f is nonnegative, measurable, and defined almost everywhere on S^{n-1}, then*

$$\begin{aligned} \int_{S^{n-1}} f(\theta)\,d\theta &= \frac{1}{\gamma_{k,n}} \int_{G_{k,n}} \int_{S^{n-1}\cap\pi} f(\theta)\,d\theta\,d\pi \\ &= \frac{1}{\gamma_{n-k,n}} \int_{G_{k,n}} \int_{S^{n-1}\cap\pi^\perp} f(\theta)\,d\theta\,d\pi. \end{aligned} \tag{2.13}$$

Proof. The first part follows by setting $g \equiv 1$ in (2.11).

For the second part, write out (2.11) for the spherical $(n-k)$-transform and note that the correspondence $\pi \leftrightarrow \pi^\perp$ is one-to-one and measure preserving between $G_{k,n}$ and $G_{n-k,n}$. ∎

COROLLARY 2.4. *If f is nonnegative, measurable, and defined almost everywhere on $\mathbb{R}^n$, then*

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\,dx &= \frac{1}{\gamma_{k,n}} \int_{G_{k,n}} \int_\pi |x'|^{n-k} f(x')\,dx'\,d\pi \\ &= \frac{1}{\gamma_{n-k,n}} \int_{G_{k,n}} \int_{\pi^\perp} |x''|^k f(x'')\,dx''\,d\pi. \end{aligned} \tag{2.14}$$

Proof. Write the integral on the left-hand side in polar coordinates and use Corollary 2.3. ∎

3. Domains of definition. In the preceding section, we defined the transforms P, D, and S. The goal of this section is to establish conditions on f that will make Pf, Df, Sf well defined. This means that Pf, Df, Sf should be at least measurable and defined almost everywhere.

In most papers on the subject, one of two conditions has been imposed. For applications to imaging, we can assume that f has compact support and is square integrable. For theoretical purposes, where functions of unbounded support are needed, they are usually taken to be in the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions. In either case, it is easy to justify the formal calculations in later sections.

In this paper, we would like to impose as few conditions as possible on the functions involved, in order to determine a larger class of functions for which the inversion procedures of computed tomography are valid. This is mostly of theoretical interest. However, it should be noted that some algorithms in practical use in CT involve functions which satisfy neither of the usual requirements, but do satisfy the less stringent requirements established in § 7.

Measurability is never a problem and will be understood from now on. If f is measurable, the measurability of Sf follows from Helgason's general theory and can be extended to D and P from there. Alternatively, it could be established directly by standard measure theoretic arguments.

L^p is the space of complex-valued functions on $\mathbb{R}^n$ which satisfy

$$\|f\|_{L^p} = \left\{ \int_{\mathbb{R}^n} |f(x)|^p\,dx \right\}^{1/p} < \infty. \tag{3.1}$$

L^∞ is the space of essentially bounded functions on $\mathbb{R}^n$, L_0^p the space of functions in L^p of compact support. L^p spaces on domains other than $\mathbb{R}^n$ are written $L^p(S^{n-1})$, $L^p(G_{k,n})$, etc.

The inner product on L^2 is written

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx, \tag{3.2}$$

where the bar denotes complex conjugation.

By applying Fubini's theorem to the first formula in Corollary 2.3, we can easily establish the following theorem.

Theorem 3.1. *If f is integrable on S^{n-1}, then Sf is defined almost everywhere on $G_{k,n}$ and is integrable.*

The *Riesz kernel* R_k was introduced by M. Riesz in [14]. It is defined by

$$R_k(x) = c_{k,n}\,|x|^{k-n}, \qquad 0 < k < n, \tag{3.3}$$

where the constant

$$c_{k,n} = \frac{\Gamma((n-k)/2)}{2^k \pi^{n/2}\Gamma(k/2)} \tag{3.4}$$

is chosen so that

$$\tilde{R}_k(\xi) = (2\pi)^{-n/2}|\xi|^{-k}. \tag{3.5}$$

Here the Fourier transform $\tilde{f}$ of a function f is given by

$$\tilde{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\,dx. \tag{3.6}$$

Lemma 3.2 (Helgason [3]). *If f is nonnegative, measurable, and defined almost everywhere on $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$,*

$$\int_{G_{k,n}} D_x f(\pi)\,d\pi = \int_{G_{k,n}} P_\pi f(E_{\pi^\perp} x)\,d\pi = \frac{\gamma_{k,n}}{c_{k,n}} R_k * f(x), \tag{3.7}$$

where $*$ denotes convolution.

Proof. The first equality is relation (2.3). The second follows from

$$\begin{aligned} \int_{G_{k,n}} D_x f(\pi)\,d\pi &= \int_{G_{k,n}}\int_\pi f(x+x')\,dx' \\ &= \int_{G_{k,n}}\int_\pi |x'|^{n-k}|x'|^{k-n} f(x+x')\,dx' \\ &= \gamma_{k,n}\int_{\mathbb{R}^n} |y|^{k-n} f(x+y)\,dy \\ &= \frac{\gamma_{k,n}}{c_{k,n}} R_k * f(x), \end{aligned} \tag{3.8}$$

using Corollary 2.4. □

It should be noted that the integrals in the preceding lemma both represent the average of the k-plane integrals of f over all k-planes passing through the point x, that is, the $\check{}$-transform in Helgason's theory.

The existence of Pf and Df is thus tied to the existence of $R_k * f$. The following two lemmas were proved in [17].

Lemma 3.3. *If $\rho \in L^1$ and $(1+|x|)^n\rho$ is bounded, then*

$$|R_k * \rho(x)| \le c\,(1+|x|)^{k-n}(\|\rho\|_{L^1} + \|(1+|x|)^n\rho\|_{L^\infty}), \tag{3.9}$$

where the constant c depends only on n, k, and $\|(1+|x|)^n\rho\|_{L^\infty}$.

Lemma 3.4. *If $(1+|x|)^{k-n}f \in L^1$, then $R_k * f$ is defined almost everywhere and is locally integrable. Moreover, if $\rho \in L^1$ and $(1+|x|)^n\rho$ is bounded, then*

$$\begin{aligned} |\langle R_k * f, \rho\rangle| &= \left|\int_{\mathbb{R}^n} R_k * f(x)\bar{\rho}(x)\,dx\right| \\ &\le c\,\|(1+|x|)^{k-n}f\|_{L^1}(\|\rho\|_{L^1} + \|(1+|x|)^n\rho\|_{L^\infty}), \end{aligned} \tag{3.10}$$

where the constant c depends only on n, k, and $\|(1+|x|)^n\rho\|_{L^\infty}$.

If $(1+|x|)^{k-n}f \notin L^1$ *and* $f \geq 0$, *then* $R_k * f$ *is undefined everywhere.*

Lemmas 3.2 and 3.4, together with Fubini's theorem, yield the following theorem.

THEOREM 3.5. *If* $(1+|x|)^{k-n}f \in L^1$, *then*

(a) *For almost every* $x \in \mathbb{R}^n$, $D_x f$ *is defined for almost every* $\pi \in G_{k,n}$ *and is integrable on* $G_{k,n}$.

(b) Pf *is defined almost everywhere on* $\mathcal{G}_{k,n}$ *and is locally integrable. Thus, for almost every* $\pi \in G_{k,n}$, $P_\pi f$ *is defined almost everywhere on* $\pi^\perp$ *and is locally integrable.*

In summary, we have established that $L^1(S^{n-1})$ is a suitable domain for S, and $\{f : (1+|x|)^{k-n}f \in L^1\}$ is a suitable domain for both P and D.

4. The transforms as operators on L^2. In § 3 we established some conditions under which the transforms S, P, and D exist. As indicated in § 2, S and P are special cases of the $\hat{}$ operator from Helgason's general theory, and so both have a dual $\check{}$. These duals turn out to be exactly the adjoint operators. To get useful theorems about the adjoints, it is necessary to restrict the domains of the transforms somewhat, to L^2-type domains.

Unless otherwise stated, c denotes an unspecified constant, whose value may be different in each occurence.

THEOREM 4.1 (STRICHARTZ [21]). *S maps* $L^2(S^{n-1})$ *continuously into* $L^2(G_{k,n})$. *Its adjoint* S^* *is given by*

$$S^*g(\theta) = \int_{\Gamma_{k,n}(\theta)} g(\pi)\,d\pi \tag{4.1}$$

and maps $L^2(G_{k,n})$ *continuously into* $L^2(S^{n-1})$.

THEOREM 4.2 (SMITH, SOLMON, WAGNER [19]). *If* $f \in L^2_0(\Omega)$, *where* Ω *is a bounded open set in* $\mathbb{R}^n$, *then* $P_\pi f \in L^2_0(\pi^\perp)$ *for all* $\pi \in G_{k,n}$, $Pf \in L^2_0(\mathcal{G}_{k,n})$, *and*

$$\begin{aligned} \|P_\pi f\|_{L^2_0(\pi^\perp)} &\leq c\,\|f\|_{L^2_0(\Omega)}, \\ \|Pf\|_{L^2_0(\mathcal{G}_{k,n})} &\leq c\,\|f\|_{L^2_0(\Omega)}, \end{aligned} \tag{4.2}$$

where c is a constant depending only on n, k, and the diameter of Ω.

This shows that P_π, P can be regarded as unbounded operators on L^2 with domain L^2_0.

THEOREM 4.3 (SMITH, SOLMON, WAGNER [19]). *If P is considered as an unbounded operator on* L^2 *with domain* L^2_0, *its formal adjoint* $P^\#$ *is given by*

$$P^\# g(x) = \int_{G_{k,n}} g(\pi, E_{\pi^\perp}x)\,d\pi, \tag{4.3}$$

where $g \in L^2(\mathcal{G}_{k,n})$ *and* $E_{\pi^\perp}x$ *is the orthogonal projection of x onto* $\pi^\perp$. *For every* $g \in L^2(\mathcal{G}_{k,n})$, $P^\# g$ *is defined almost everywhere and is locally square integrable. Moreover, g is in the domain of the adjoint* P^* *of P if and only if* $P^\# g$ *is globally square integrable, in which case* $P^\# g = P^* g$.

By the remarks at the beginning of this section and following Lemma 3.2, another way to write the formula in Lemma 3.2 is

$$P^\# Pf(x) = \int_{G_{k,n}} P_\pi f(E_{\pi^\perp}x)\,d\pi = \frac{\gamma_{k,n}}{c_{k,n}} R_k * f(x). \tag{4.4}$$

Remark. Application of the operator $P^{\#}$ is often called *backprojection* in the literature. The reason for this is as follows: The operator P_π is thought of as a projection from $\mathbb{R}^n$ to $\pi^\perp$, by integrating in the "direction" π. The operator $P^{\#}$ can be thought of as first extending the projected function $P_\pi f$ from $\pi^\perp$ back to all of $\mathbb{R}^n$, by making it constant along the translates of π (the back-projection), and then averaging over all π.

THEOREM 4.4. *If $f \in L_0^2$, then Df is locally square integrable on $G_{k,n} \times \mathbb{R}^n$. In particular, $D_a f \in L^2(G_{k,n})$ for almost every $a \in \mathbb{R}^n$.*

Proof. If $f \in L_0^2$, let Ω be a bounded open set which contains the support of f, and let χ_Ω be the characteristic function of Ω.

$$\begin{aligned} |D_a f(\pi)|^2 &= |\int_\pi f(a+x')\,dx'|^2 \\ &\le \int_\pi \chi_\Omega(a+x')\,dx' \int_\pi |f(a+x')|^2\,dx' \\ &\le c\, D_a|f|^2(\pi). \end{aligned} \tag{4.5}$$

By this formula and Lemma 3.2,

$$\begin{aligned} \int_{G_{k,n}} |D_a f(\pi)|^2\,d\pi &\le c\int_{G_{k,n}} D_a|f|^2(\pi)\,d\pi \\ &\le c\frac{\gamma_{k,n}}{c_{k,n}} R_k * |f|^2(a), \end{aligned} \tag{4.6}$$

and the theorem follows from Lemma 3.4, since $(1+|x|)^{k-n}|f|^2 \in L^1$ if $f \in L_0^2$. $\square$

For later purposes, this theorem is not sufficient. $D_a f$ is known to exist for almost every $a \in \mathbb{R}^n$, but the inversion formula for D will involve integration over a sphere A in $\mathbb{R}^n$ (of n-dimensional measure zero.) The problem disappears if the domain of D is restricted to $L_0^2(\Omega)$, where Ω is a fixed bounded open set in $\mathbb{R}^n$, and a is chosen outside $\bar{\Omega}$, the closure of Ω.

THEOREM 4.5. *If $f \in L_0^2(\Omega)$, then $D_a f \in L^2(G_{k,n})$ for all sources a outside $\bar{\Omega}$, and for such a,*

$$\| D_a f\|_{L^2(G_{k,n})} \le c\,[\mathrm{dist}(a,\Omega)]^{(k-n)/2}\, \|f\|_{L_0^2(\Omega)}, \tag{4.7}$$

where $\mathrm{dist}(a,\Omega)$ *is the distance from a to Ω, and c is a constant depending only on n, k, and the diameter of Ω.*

Proof. As in the proof of the previous theorem,

$$\begin{aligned} \| D_a f\|^2_{L^2(G_{k,n})} &\le c\,R_k * |f|^2(a) \\ &= c\int_{\mathbb{R}^n} |a-x|^{k-n}|f|^2(x)\,dx \\ &\le c\,[\mathrm{dist}(a,\Omega)]^{k-n}\,\|f\|^2_{L_0^2(\Omega)}. \end{aligned} \tag{4.8}$$

$\square$

THEOREM 4.6. *If D_a is considered as an operator on $L_0^2(\Omega)$, a outside $\bar{\Omega}$, its adjoint D_a^* is given by*

$$D_a^* g(x) = |x-a|^{k-n}\int_{\Gamma_k((x-a)/|x-a|)} g(\pi)\,d\pi, \qquad x \in \Omega, \tag{4.9}$$

where $g \in L^2(G_{k,n})$. $D_a^* g$ is defined almost everywhere on Ω and is in $L_0^2(\Omega)$.

Proof. Using Fubini's theorem, polar coordinates, and (2.11),

$$
\begin{aligned}
\int_{G_{k,n}} D_a f(\pi) g(\pi)\, d\pi &= \int_{G_{k,n}} g(\pi) \int_0^\infty \int_{S^{n-1}\cap\pi} t^{k-1} f(a+t\theta)\, d\theta\, dt\, d\pi \\
&= \int_0^\infty t^{k-1} \int_{G_{k,n}} g(\pi) \int_{S^{n-1}\cap\pi} f(a+t\theta)\, d\theta\, d\pi\, dt \\
&= \int_0^\infty t^{k-1} \int_{S^{n-1}} f(a+t\theta) \int_{\Gamma_{k,n}(\theta)} g(\pi)\, d\pi\, d\theta\, dt \qquad (4.10)\\
&= \int_{\mathbb{R}^n} |x|^{k-n} f(a+x) \int_{\Gamma_k(x/|x|)} g(\pi)\, d\pi\, dx \\
&= \int_{\mathbb{R}^n} f(x)\, |x-a|^{k-n} \int_{\Gamma_k((x-a)/|x-a|)} g(\pi)\, d\pi\, dx.
\end{aligned}
$$

The fact that $D_a^* g \in L_0^2(\Omega)$ follows from Theorem 4.1. ◻

5. Uniqueness and nonuniqueness. An important question is whether a function f is determined uniquely by its transform, that is, whether the operators P, D, S are one-to-one or not. Also, we would like to know whether knowing Sf, Df, or Pf on part of its domain is already sufficient, since this might allow reconstructions with fewer measurements.

As motivation, consider the case of two-dimensional divergent beam CT, which is used in modern CAT scanners. There is an inversion formula, derived later, which expresses f in terms of $D_a f$ for all sources a on a circle surrounding the support of f. This automatically implies that f is uniquely determined by Df.

But is it really necessary to have x-ray sources all around the object? Is it maybe sufficient to measure from sources only on an arc smaller than 360°?

The theorems in this section state that any infinite set of sources will do, whether or not they are spaced all around the object or just concentrated on an arc of 5° (or any other size.) Conversely, with only a finite set of measurements the object is essentially arbitrary, even if we know its outside shape.

As far as practical applications go, these statements are fairly academic. It is possible to reconstruct an object by measurements from angles less than 360° (indeed, most CT scanners skip the part of the full circle where the patient lies.) However, numerical conditioning becomes rapidly worse as the angle decreases, and the inevitable measuring errors make anything noticeably less than a full circle useless in practice.

On the other hand, it is being demonstrated every day that a finite set of measurements produces useful results. This is due to the fact that the functions in the nullspace are highly oscillatory and are eliminated during the reconstruction procedure (which involves truncating the Fourier transform, thus deleting the high-frequency components.) For a discussion of this subject, see Louis [10].

Thus, except for Theorem 5.1, the following theorems are mainly of theoretical interest.

The Fourier transforms of f and Pf are related by the *central slice theorem*, which is proved here for L^1 functions and will later be extended to more general functions. It states that the n-dimensional Fourier transform of f, restricted to the $(n-k)$-plane $\pi^\perp$, is equal, up to a constant, to the $(n-k)$-dimensional Fourier transform of $P_\pi f$ in $\pi^\perp$. This can be visualized as taking a cut or slice through the full transform.

THEOREM 5.1 (CENTRAL SLICE THEOREM). *If $f \in L^1$, then*

$$(P_\pi f)\tilde{}\,(\xi'') = (2\pi)^{k/2} \tilde{f}(\xi''), \qquad \xi'' \in \pi^\perp. \tag{5.1}$$

Proof. Let $\langle\,.\,,\,.\,\rangle$ denote the inner product on $\mathbb{R}^n$.

$$
\begin{aligned}
(P_\pi f)\tilde{}\,(\xi'') &= (2\pi)^{-(n-k)/2}\int_{\pi^\perp} e^{-i\langle x'',\xi''\rangle}\,P_\pi f(x'')\,dx'' \\
&= (2\pi)^{-(n-k)/2}\int_{\pi^\perp} e^{-i\langle x'',\xi''\rangle}\int_\pi f(x'+x'')\,dx'\,dx'' \\
&= (2\pi)^{-(n-k)/2}\int_{\mathbb{R}^n} e^{-i\langle x,\xi''\rangle}\,f(x)\,dx \\
&= (2\pi)^{k/2}\,\tilde f(\xi'').
\end{aligned} \tag{5.2}
$$

□

This relationship between the Fourier transform and the parallel k-plane transform will be exploited in various ways. It allows us to use the well-known properties of Fourier transforms to study the k-plane transforms.

The way it is used in this section is to observe that if all k-plane integrals in a particular direction π vanish, then the Fourier transform of f must vanish on $\pi^\perp$. If f has compact support, then $\hat f$ is analytic, which puts restrictions on its set of zeros.

These theorems are easy generalizations of the results in § 4 of Smith, Solmon, Wagner [19]. They must have been known to the authors of [19] and to others, but for some reason never seem to have been stated in the literature.

THEOREM 5.2. *Let $f \in L_0^1$ and let $\{\pi_1, \pi_2, \cdots\}$ be a collection of k-dimensional subspaces of $\mathbb{R}^n$. If the subspaces $\pi_j^\perp$ are not contained in a proper algebraic variety, then f is uniquely determined by $P_{\pi_j} f$.*

Proof. Since f has compact support, its Fourier transform $\tilde f$ extends to an entire function on $\mathbb{C}^n$ with a Taylor expansion

$$\tilde f(\xi) = \sum_{m=0}^{\infty} p_m(\xi), \tag{5.3}$$

where $p_m(\xi)$ is a homogeneous polynomial of degree m. By the linearity of P, it suffices to show that $f = 0$ whenever $P_{\pi_j} = 0$ for all j.

By Theorem 5.1, the central slice theorem, $\tilde f$ vanishes on $\pi_j^\perp$ whenever $P_{\pi_j} f = 0$. For any $\xi'' \in \pi_j^\perp$,

$$\tilde f(t\xi'') = \sum_{m=0}^{\infty} t^m p_m(\xi'') = 0 \qquad \text{for all } t, \tag{5.4}$$

from which it follows that $p_m = 0$ on $\pi_j^\perp$. Since by assumption no nonzero p_m can vanish on all the $\pi_j^\perp$, $P_{\pi_j} f = 0$ for all j implies $p_m = 0$ for all m, so $\tilde f = 0$ and finally $f = 0$. □

Remark. In the case $k = 1$, $\pi_j^\perp$ has dimension $(n-1)$. Since no nonzero polynomial on $\mathbb{R}^n$ can vanish on an infinite number of hyperplanes, any infinite set of directions determines f uniquely.

For $k \geq 2$, this is no longer the case. For example, let π be a fixed hyperplane through the origin in $\mathbb{R}^n$ and f a function supported in the unit ball, with values ± 1 on opposite sides of π. Then $\mathsf{P}_{\pi_j} f = 0$ whenever $\pi_j^\perp$ is contained in π, and for $k \geq 2$, there are infinitely many distinct π_j which satisfy this.

THEOREM 5.3. *Let $f_0 \in C_0^\infty(\mathbb{R}^n)$ (the set of infinitely differentiable functions with compact support), and let $\{\pi_1, \pi_2, \cdots\}$ be a collection of k-dimensional subspaces of $\mathbb{R}^n$.*

If all of the subspaces $\pi_j^\perp$ are contained in a proper algebraic variety on $\mathbb{R}^n$, if K is any compact set in the interior of the support of f_0, and if f_1 is any function in $C^\infty(\mathbb{R}^n)$, then there is a function $f \in C_0^\infty(\mathbb{R}^n)$ so that

$$\begin{array}{rcll} f &=& f_1 & \text{on } K, \\ P_{\pi_j} f &=& P_{\pi_j} f_0 & \text{for all } j, \\ \operatorname{supp} f &\subset& \operatorname{supp} f_0. & \end{array} \tag{5.5}$$

Proof. Let Q be a polynomial that vanishes on all $\pi_j^\perp$. The theorem of Ehrenpreis–Malgrange on the existence of solutions to constant coefficient partial differential equations guarantees the existence of functions u_0 and u_1 in $C^\infty(\mathbb{R}^n)$ so that

$$Q(-iD)u_m = f_m, \qquad m = 0, 1, \tag{5.6}$$

where $D = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$.

Choose $\rho \in C_0^\infty(\mathbb{R}^n)$ so that $\rho = 1$ in a neighborhood of K and ρ vanishes outside the support of f_0. Now let

$$v_m = Q(-iD)(\rho u_m), \qquad m = 0, 1. \tag{5.7}$$

The last two formulas show that $v_m = f_m$ in a neighborhood of K, and $v_m = 0$ outside the support of f_0.

By (5.1),

$$\begin{array}{rcl} (P_{\pi_j} v_m)\tilde{\ }(\xi'') &=& (2\pi)^{k/2} Q(\xi'')(\rho u_m)\tilde{\ }(\xi'') \\ &=& 0 \qquad \text{for } \xi'' \in \pi_j^\perp, \end{array} \tag{5.8}$$

thus

$$P_{\pi_j} v_m = 0 \qquad \text{for } m = 0, 1 \text{ and all } j. \tag{5.9}$$

Finally, define $f = f_0 - v_0 + v_1$. Then $f = f_1$ in a neighborhood of K, $\operatorname{supp} f \subset \operatorname{supp} f_0$ and $P_{\pi_j} f = P_{\pi_j} f_0$ for all j. □

Remark. The theorem automatically applies if $\{\pi_1, \cdots, \pi_N\}$ is a finite set (take $Q(\xi) = \langle \xi, \xi_1\rangle\langle \xi, \xi_2\rangle \cdots \langle \xi, \xi_N\rangle$, where $\xi_j \in \pi_j$, $\xi_j \neq 0$).

For the operator S, the uniqueness question has been partially answered by Strichartz [21], where it is shown that S is one-to-one on even functions in $L^2(S^{n-1})$. Obviously this is all that one can hope for, since all odd functions are in the nullspace of S.

In the case of the operator D, one can reduce the proof to a slight modification of the corresponding proof for the one-dimensional case, which has been settled before. The details are technical and not very illuminating, so we will just outline the connection and refer to the original papers.

For $m \geq 0$, the operator $\mathtt{D}_a^m$ is defined by

$$\mathtt{D}_a^m f(\theta) = \int_{-\infty}^{\infty} f(a + t\theta)\, |t|^m \, dt, \tag{5.10}$$

where $\theta \in S^{n-1}$, whenever the integral exists. $\mathtt{D}_a^m$ shares many of the properties of D_a for $k = 1$, in particular $\mathtt{D}_a^m$ is a bounded operator from $L_0^2(\Omega)$ to $L^2(G_{1,n})$ for a

outside the closure $\bar{\Omega}$. Identifying functions on $G_{1,n}$ with even functions on S^{n-1} and using polar coordinates on π, it is easy to show that for such a,

$$D_a f(\pi) = \frac{1}{2} S\, D_a^{k-1} f(\pi) \tag{5.11}$$

if $f \in L_0^2(\Omega)$.

THEOREM 5.4. *If $f \in L_0^2(\Omega)$ and A is an infinite set of points bounded away from $\hat{\Omega}$ (the closed convex hull of Ω), then f is determined uniquely by $D_a f(\pi)$ for $\pi \in G_{k,n}$ and $a \in A$.*

Proof. Since S is one-to-one on even functions in $L^2(S^{n-1})$, and $D_a^{k-1} f$ (for fixed a) is an even function on S^{n-1}, it suffices by (5.11) to prove the theorem for $\mathtt{D}_a^m$ instead of D_a. The proof of this is a slight modification of the proof of Theorem 5.1 in Hamaker et al. [2]. □

THEOREM 5.5. *Let $f_0 \in L_0^1(\Omega)$, let A be a finite set of sources outside $\bar{\Omega}$, let K be a compact set in the interior of the support of f_0, and let f_1 be any integrable function on K. Then there is a function $f \in L_0^1(\Omega)$ with the same shape as f_0, with $D_a f = D_a f_0$ for all $a \in A$, and $f = f_1$ on K.*

Remark. The shape of a function is the complement of the unbounded component of the complement of its support.

Proof. Theorem 6.15 in Leahy, Smith, Solmon [9] shows that there is an f so that $\mathtt{D}_a^m f(\theta) = \mathtt{D}_a^m f_0(\theta)$ for all $\theta \in S^{n-1}$ and for all $m \geq 0$, with $f = f_1$ on K. The theorem then follows immediately from (5.11). □

6. Formal inversion formulas. The object of this section is to find formulas to recover f exactly or approximately from Pf or Df. The word "formal" in the heading refers to the fact that we are not concerned at the moment about justifying the steps, such as taking Fourier transforms, interchanging the order of integration, etc. This will be done in the following section.

Remark. The inversion of S is harder than that of P and D. Helgason [3], [5] and Strichartz [21] consider a transform defined by integration over k-dimensional geodesic surfaces in a space of constant curvature. This approach is less general than the one outlined before, but covers both P and S. Inversion formulas were found for even k by Helgason and for all k by Strichartz. These formulas are very complicated, however, and not easy to derive, and we will not reproduce them here.

The operator Λ^k is defined by

$$(\Lambda^k f)\tilde{}(\xi) = |\xi|^k \tilde{f}(\xi). \tag{6.1}$$

If $k = 1$, we simply write Λ instead of Λ^1. Λ^k is well defined if f is a function whose (distribution) Fourier transform is a locally integrable function, and $|\xi|^k \tilde{f}$ is in L^2.

From the definitions of Λ^k and R_k and from Lemma 3.2 it follows that formally

$$\begin{aligned} f(x) &= \Lambda^k (R_k * f)(x) \\ &= \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k \int_{G_{k,n}} D_x f(\pi)\, d\pi \\ &= \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k \int_{G_{k,n}} P_\pi f(E_{\pi^\perp} x)\, d\pi \\ &= \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k P^{\#} P f(x). \end{aligned} \tag{6.2}$$

Λ^k commutes formally with $R_k = c\,P^{\#}\,P$ and with P. The first can be seen from (6.1) and (3.5), the latter from the central slice theorem: For each $\pi \in G_{k,n}$, $\xi'' \in \pi^{\perp}$,

$$
\begin{aligned}
(P_\pi \Lambda^k f)\tilde{}\,(\xi'') &= (2\pi)^{k/2}(\Lambda^k f)\tilde{}\,(\xi'') \\
&= (2\pi)^{k/2}|\xi''|^k \tilde{f}(\xi'') \\
&= |\xi''|^k (P_\pi f)\tilde{}\,(\xi'') \\
&= (\Lambda^k P_\pi f)\tilde{}\,(\xi'').
\end{aligned} \tag{6.3}
$$

We abbreviate this as

$$P\Lambda^k f = \Lambda^k P f. \tag{6.4}$$

Note that in this notation the operator Λ^k on the left acts on all of $\mathbb{R}^n$, the Λ^k on the right acts on each $\pi^{\perp}$. The notation $(Pf)\tilde{}$ is to be understood in a similar fashion.

Thus, in addition to (6.2), we can write the inversion formula for P as follows

$$f = \frac{c_{k,n}}{\gamma_{k,n}}\, P^{\#}\Lambda^k P f. \tag{6.5}$$

Formula (6.5) contains the so-called convolution-backprojection algorithm used in modern CT-scanners. For $n = 2$, $k = 1$ it reads

$$f = \frac{1}{2\pi}\, P^{\#}\Lambda\, P f, \tag{6.6}$$

which is interpreted as follows.

Pf is the measured data. For each fixed direction θ, $P_\theta f$ is a function defined on the line $\theta^{\perp}$ perpendicular to θ. For each θ, we apply Λ by taking a one-dimensional Fourier transform, multiplying by $|\xi|$ and transforming back. We then backproject these new functions (i.e., apply $P^{\#}$.)

The original formula (6.2) represents the so-called ρ-filtered layergram method (Herman [7].)

Equation (6.2) also contains the inversion formula for D on the second line, but it is not suitable in this form, since it requires the values of $D_x f(\pi)$ for all $x \in \mathbb{R}^n$. However, a little thought shows that if $D_a f(\pi)$ is known for all a on a surface enclosing the object, we know the k-plane integrals of f over all k-planes intersecting the object. Therefore, we should be able to reparameterize the integral. The following lemma and theorem accomplish this.

Lemma 6.1 (Leahy, Smith, Solmon [9]). *Let A be a sphere of radius r, x a point not on A, and let g be nonnegative and measurable on S^{n-1}. Then*

$$\int_{S^{n-1}} g(\theta)\,d\theta = \frac{1}{r}\int_A g\left(\frac{a-x}{|a-x|}\right)|a-x|^{-n}|\langle a, a-x\rangle|\,da. \tag{6.7}$$

Theorem 6.2. *If $f \in L_0^2(\Omega)$ and A is a sphere of radius r surrounding $\hat{\Omega}$, then for almost every $x \in \Omega$,*

$$f(x) = \frac{c_{k,n}}{r|S^{k-1}|\gamma_{k,n}}\,\Lambda^k \int_A D_a^* D_a f(x)|a-x|^{-k}|\langle a, a-x\rangle|\,da. \tag{6.8}$$

Proof. By Corollary 2.2, Lemma 6.1, and Theorem 4.6,

$$
\begin{aligned}
\int_{G_{k,n}} D_x f(\pi)\, d\pi &= \frac{1}{|S^{k-1}|} \int_{S^{n-1}} \int_{\Gamma_{k,n}(\theta)} D_x f(\pi)\, d\pi\, d\theta \\
&= \frac{1}{r|S^{k-1}|} \int_A \int_{\Gamma_{k,n}((a-x)/|a-x|)} D_x f(\pi)\, d\pi \quad |a-x|^{-n}\, |\langle a, a-x\rangle|\, da \\
&= \frac{1}{r|S^{k-1}|} \int_A \int_{\Gamma_{k,n}((a-x)/|a-x|)} D_a f(\pi)\, d\pi \quad |a-x|^{-n}\, |\langle a, a-x\rangle|\, da \\
&= \frac{1}{r|S^{k-1}|} \int_A D_a^* D_a f(x)\, |a-x|^{-k}\, |\langle a, a-x\rangle|\, da.
\end{aligned}
\tag{6.9}
$$

(Note that $D_x f(\pi) = D_a f(\pi)$, if $\pi \in \Gamma_{k,n}((a-x)/|a-x|)$.) ∎

Special case. For $k = 1$,

$$
D_a^* D_a f(x) = |a-x|^{1-n} D_a f\left(\frac{a-x}{|a-x|}\right). \tag{6.10}
$$

Equation (6.8) reduces to the formula in Smith [16]:

$$
f(x) = \frac{c_{1,n}}{2r}\, \Lambda \int_A D_a f\left(\frac{a-x}{|a-x|}\right) |a-x|^{-n} |\langle a, a-x\rangle|\, da. \tag{6.11}
$$

From a theoretical point of view we are finished now. However, the formulas are not suitable yet for numerical evaluation.

All of the above formulas require the evaluation of Λ. In principle, we should do this by taking the Fourier transform, multiplying by $|\xi|$, and transforming back. Numerically, this is not possible: the function $|\xi|$ must be cut off somewhere.

Engineers like to visualize this truncation process in Fourier space, in terms of "cutting off high-frequency components" and thereby controlling the noise level.

Another way to think about it and understand the effect on the reconstruction is the following. Pick an approximate δ-function e, called the *point-spread function* (the name is explained below.)

By writing out the integrals involved, we can calculate that formally

$$
P_\pi(e * f) = P_\pi e * P_\pi f \tag{6.12}
$$

(again, the convolution on the left is in $\mathbb{R}^n$, the convolution on the right in $\pi^\perp$) and that

$$
\Lambda^k (P_\pi e * P_\pi f) = (\Lambda^k P_\pi e) * P_\pi f. \tag{6.13}
$$

Thus, (6.5) gives

$$
e * f = \frac{c_{k,n}}{\gamma_{k,n}} P^{\#} \Lambda^k P(e * f) = P^{\#}(K * Pf) \tag{6.14}
$$

where the kernel K is defined by

$$
K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k Pe. \tag{6.15}
$$

Conversely, if K is known, then

$$e = P^{\#} K. \tag{6.16}$$

Thus, we approximate Λ numerically by a convolution with K. This is usually done by calculating the Fourier transforms of f and K, multiplying them together, and transforming back. The Fourier transform of K looks like $|\xi|$ for small ξ. The effect of approximating Λ in this way is that instead of a reconstruction of f we get a reconstruction of $e * f$, where e is an approximate δ-function.

For $k = 1$, this leads to the convolution-backprojection algorithm used in most modern CAT scanners, first suggested by Ramachandran and Lakshminarayanan in [13].

Remark. Putting mathematical rigor aside for a moment, consider what happens when f is a δ-function. The reconstruction of f is $f * e = e$. The reconstruction process "spreads out" f from a point to the function e, which is the reason for calling e the point-spread function.

To derive a corresponding theorem for D, another integration theorem is needed.

Let $B(r)$, $S(r)$ be the ball and sphere of radius r around the origin in $\mathbb{R}^n$.

LEMMA 6.3. *Let π be a fixed k-dimensional subspace of $\mathbb{R}^n$. If g is nonnegative and measurable on $\pi^\perp$ and has support in $B(r) \cap \pi^\perp$, then*

$$\int_{S(r)} g(E_{\pi^\perp}\theta)\, d\theta = r\, |S^{k-1}| \int_{\pi^\perp} g(x'')(r^2 - |x''|^2)^{(k-2)/2} dx''. \tag{6.17}$$

Proof. Fix an arbitrary $(k-1)$-dimensional subspace π' of π. By first rewriting the integral over $S(r)$ as one over $\pi' + \pi^\perp$ and then applying Fubini's theorem,

$$\begin{aligned} &\int_{S(r)} g(E_{\pi^\perp}\theta)\, d\theta \\ &= 2r \int_{B(r)\cap(\pi'+\pi^\perp)} g(E_{\pi^\perp}x)(r^2 - |x|^2)^{-1/2} dx \\ &= 2r \int_{\pi^\perp \cap B(r)} g(x'') \int_{\pi' \cap B(\sqrt{r^2 - |x''|^2})} (r^2 - |x'|^2 - |x''|^2)^{-1/2} dx'\, dx''. \end{aligned} \tag{6.18}$$

By setting $x' = t\theta$, the inner integral in polar coordinates becomes

$$|S^{k-2}| \int_0^{\sqrt{r^2-|x''|^2}} t^{k-2}(r^2 - |x''|^2 - t^2)^{-1/2} dt. \tag{6.19}$$

With the substitutions $s = (r^2 - |x''|^2)^{-1/2}\, t = \cos v$ this equals

$$\begin{aligned} &(r^2 - |x''|^2)^{(k-2)/2}\, |S^{k-2}| \int_0^1 s^{k-2}(1 - s^2)^{-1/2}\, ds \\ &= (r^2 - |x''|^2)^{(k-2)/2}\, |S^{k-2}| \int_0^{\pi/2} \cos^{k-2} v\, dv \\ &= \frac{\Gamma((k-1)/2)\Gamma(1/2)}{2\Gamma(k/2)} |S^{k-2}|\, (r^2 - |x''|^2)^{(k-2)/2} \\ &= \frac{|S^{k-1}|}{2} (r^2 - |x''|^2)^{(k-2)/2}. \end{aligned} \tag{6.20}$$

∎

COROLLARY 6.4. *If f, g are nonnegative and measurable functions on $\mathbb{R}^n$ and the support of f is contained in $B(r)$, then for almost every $\pi \in G_{k,n}$,*

$$\begin{aligned}\int_{\pi^\perp} & P_\pi f(x'') g(x'')\, dx'' \\ &= \frac{1}{r|S^{k-1}|} \int_{S^{n-1}(r)} D_a f(\pi) g(E_{\pi^\perp} a)(r^2 - |E_{\pi^\perp} a|^2)^{(2-k)/2}\, da.\end{aligned} \tag{6.21}$$

Proof. By (2.3) and Lemma 6.3,

$$\begin{aligned}\int_{\pi^\perp} & P_\pi f(x'') g(x'')\, dx'' \\ &= \int_{\pi^\perp} Df(\pi, x'') g(x'')(r^2 - |x''|^2)^{(k-2)/2}(r^2 - |x''|^2)^{(2-k)/2} dx'' \\ &= \frac{1}{r\,|S^{k-1}|} \int_{S^{n-1}(r)} Df(\pi, E_{\pi^\perp} a) g(E_{\pi^\perp} a)(r^2 - |E_{\pi^\perp} a|^2)^{(2-k)/2} da.\end{aligned} \tag{6.22}$$

Then note that $Df(\pi, E_{\pi^\perp} a) = Df(\pi, a)$. ∎

THEOREM 6.5. *If e and f are functions on $\mathbb{R}^n$ with the support of f contained in the interior of $B(r)$, then formally*

$$\begin{aligned}e * f(x) = \frac{1}{r|S^{k-1}|} \int_{G_{k,n}} \int_{S^{n-1}(r)} & D_a f(\pi) K(E_{\pi^\perp}(x - a)) \\ & \times (r^2 - |E_{\pi^\perp} a|^2)^{(2-k)/2} da\, d\pi,\end{aligned} \tag{6.23}$$

where the kernel is given by

$$K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k P_\pi e. \tag{6.24}$$

Proof. By (6.14),

$$\begin{aligned}e * f(x) &= \int_{G_{k,n}} K * P_\pi f(E_{\pi^\perp} x)\, d\pi \\ &= \int_{G_{k,n}} \int_{\pi^\perp} K(E_{\pi^\perp} x - x'')\, P_\pi f(x'')\, dx''\, d\pi \\ &= \frac{1}{r|S^{k-1}|} \int_{G_{k,n}} \int_{S^{n-1}(r)} D_a f(\pi) K(E_{\pi^\perp}(x - a)) \\ &\qquad \times (r^2 - |E_{\pi^\perp} a|^2)^{(2-k)/2} da\, d\pi.\end{aligned} \tag{6.25}$$

∎

Special case. For $k = 1$, with $\pi = \theta \in S^{n-1}$, $E_{\pi^\perp} = E_{\theta^\perp}$ we obtain

$$e * f(x) = \frac{1}{4r} \int_{S(r)} \int_{S^{n-1}} D_a f(\theta) K(E_{\theta^\perp}(x - a)) |\langle a, \theta\rangle|\, d\theta\, da, \tag{6.26}$$

which is formula (6.2) in Smith [16]. (The factor of 2 by which the formulas differ is hidden in the kernel.)

7. Exact conditions. All calculations in § 6 were purely formal, that is they are valid if the functions involved satisfy certain unspecified conditions. It remains to be shown what these conditions are. It suffices to do this for P (6.2), (6.15), since the corresponding formulas for D (see (6.8), (6.23)) follow by a change of variable in the integration, which does not affect convergence or divergence of the integrals.

The derivations in this section are fairly technical, but there is no way to avoid this. The main reason is the following.

The convolution theorem of Fourier transform theory is well known:

$$(f * g)\tilde{} = (2\pi)^{-n/2} \tilde{f}\tilde{g}. \tag{7.1}$$

If we could apply it to $R_k * f$, we could justify all formulas in § 6 and be done. It is possible to do that if we assume that f is a rapidly decreasing C^∞ function or an L^2 function with compact support, which is why most authors make one of these two assumptions. However, a careful study of the theory of Fourier transforms and convolutions will reveal that there is a variety of conditions on f and g which make the convolution theorem valid, but that our milder assumptions on f are not covered by any of the standard cases. Thus, (7.1) needs to be proved from scratch, using the Calderon–Zygmund theory of singular integrals. This is where the technicalities come in.

The formulas that will be considered are

$$f = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k P^\# Pf \tag{7.2}$$

and

$$e * f = P^\#(K * Pf), \tag{7.3}$$

with

$$K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k Pe. \tag{7.4}$$

Note that by Lemma 3.2, formula (7.2) is equivalent to

$$f = \Lambda^k (R_k * f). \tag{7.5}$$

If $f \in L^2$, then by the definition of Λ^k it suffices to show that $(R_k * f)\tilde{} = |\xi|^{-k} \tilde{f}$ and that this is a locally integrable function.

The domain $\mathcal{D}_k$ is defined as

$$\mathcal{D}_k = \{f \in L^2 : (1 + |x|)^{k-n} f \in L^1\}, \tag{7.6}$$

with

$$\|f\|_{\mathcal{D}_k} = \|f\|_{L^2} + \|(1 + |x|)^{k-n} f\|_{L^1}. \tag{7.7}$$

Note that if $k < n/2$, then by an application of the Cauchy–Schwarz theorem, $\mathcal{D}_k = L^2$.

The following lemma is proved in [17].

LEMMA 7.1. *Fix $e \in L_0^\infty$, and set $e_r(x) = r^{-n} e(x/r)$.*

(a) *If $\hat{u} \in L^1 + L^2$, then $\langle u, e_r \rangle \to 0$ as $r \to \infty$.*

(b) *If $g \in \mathcal{D}_k$, then $\langle R_k * g, e_r \rangle \to 0$ as $r \to \infty$.*

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, let $|\alpha| = \sum \alpha_i$, $\alpha! = \alpha_1! \cdots \alpha_k!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha/\partial x^\alpha = (\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \cdots (\partial^{\alpha_n}/\partial x_n^{\alpha_n})$, etc. The classical derivative of a function f is written $(\partial^\alpha/\partial x^\alpha)f$, the distribution derivative $D^\alpha f$. If $|\alpha| = k$, then $(\partial^\alpha/\partial x^\alpha)R_k$ is of the form $p(x)|x|^{-k-n}$, where $p(x)$ is a homogeneous polynomial of degree k whose integral over the unit sphere vanishes.

For $|\alpha| = k$, the formula

$$\langle \text{v.p.} \frac{\partial^\alpha}{\partial x^\alpha} R_k, \phi \rangle = \lim_{\substack{\epsilon \to 0 \\ r \to \infty}} \int_{\epsilon < |x| < r} \frac{\partial^\alpha}{\partial x^\alpha} R_k(x) \, \overline{\phi(x)} \, dx \tag{7.8}$$

defines a tempered distribution called the *principal value* of $(\partial^\alpha/\partial x^\alpha)R_k$, which can be used as a convolution operator

$$(\text{v.p.} \frac{\partial^\alpha}{\partial x^\alpha} R_k) * f(x) = \lim_{\substack{\epsilon \to 0 \\ r \to \infty}} \int_{\epsilon < |x-y| < r} \frac{\partial^\alpha}{\partial x^\alpha} R_k(x-y) \, f(y) \, dy. \tag{7.9}$$

From the Calderon–Zygmund theory of singular integrals it follows that $D^\alpha R_k = \text{v.p.}(\partial^\alpha/\partial x^\alpha)R_k$ and that for $g \in \mathcal{D}_k$, $D^\alpha(R_k * g) = (D^\alpha R_k) * g$ and $(D^\alpha(R_k * g))\tilde{} = (i\xi)^\alpha |\xi|^{-k} \tilde{g}$.

$\mathcal{S}$ is the space of rapidly decreasing C^∞ functions.

THEOREM 7.2. *If $g \in \mathcal{D}_k$ and $\phi \in \mathcal{S}$ with $(\partial^\alpha/\partial x^\alpha)\phi(0) = 0$ for all $0 \le |\alpha| < k$, then*

$$\langle (R_k * g)\tilde{}, \phi \rangle = \int |\xi|^{-k} \tilde{g}(\xi) \, \overline{\phi(\xi)} \, d\xi. \tag{7.10}$$

If $|\xi|^{-k}\tilde{g}$ is locally integrable, then $(R_k * g)\tilde{} = |\xi|^{-k}\tilde{g}$.

Proof. It follows from (3.4) that $R_k * g$ is a tempered distribution. If $\phi \in \mathcal{S}$ with $(\partial^\alpha/\partial x^\alpha)\phi(0) = 0$ for all $|\alpha| < k$, then

$$\phi(\xi) = \sum_{|\alpha| = k} \xi^\alpha \phi_\alpha(\xi), \tag{7.11}$$

where

$$\phi_\alpha(\xi) = (-1)^{|\alpha|} \frac{k!}{\alpha!} \int_1^\infty \cdots \int_1^\infty \frac{\partial^\alpha}{\partial \xi^\alpha} \phi(t_1 t_2 \cdots t_k \xi) \, dt_1 \cdots dt_k. \tag{7.12}$$

It is easy to show that $\phi_\alpha \in \mathcal{S}$. Then

$$\begin{aligned}
\langle (R_k * g)\tilde{}\,, \phi\rangle &= \sum_{|\alpha|=k} \langle (R_k * g)\tilde{}\,, \xi^\alpha \phi_\alpha\rangle \\
&= \sum_{|\alpha|=k} \langle R_k * g, (-iD)^\alpha \check{\tilde{\phi}}_\alpha\rangle \\
&= \sum_{|\alpha|=k} \langle (-iD)^\alpha (R_k * g), \check{\tilde{\phi}}_\alpha\rangle \\
&= \sum_{|\alpha|=k} \langle ((-iD)^\alpha (R_k * g))\tilde{}\,, \phi_\alpha\rangle \qquad (7.13)\\
&= \sum_{|\alpha|=k} \langle \xi^\alpha |\xi|^{-k} \tilde{g}, \phi_\alpha\rangle \\
&= \sum_{|\alpha|=k} \int_{\mathbb{R}^n} \xi^\alpha |\xi|^{-k} \tilde{g}(\xi) \overline{\phi_\alpha(\xi)}\, d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^{-k} \tilde{g}(\xi) \overline{\phi(\xi)}\, d\xi,
\end{aligned}$$

where $\check{\phi}(x) = \phi(-x)$.

If $|\xi|^{-k}\tilde{g}$ is locally integrable, it is a tempered distribution, and $\langle (R_k * g)\tilde{}\,, \phi\rangle = \langle |\xi|^{-k}\tilde{g}, \phi\rangle$ for all ϕ with $(\partial^\alpha/\partial x^\alpha)\phi(0) = 0$, $0 \le |\alpha| < k$. Thus

$$(R_k * g)\tilde{} = |\xi|^{-k}\tilde{g} + \sum_{|\alpha|=0}^{k-1} c_\alpha D^\alpha \delta. \qquad (7.14)$$

$|\xi|^{-k}\tilde{g}$ is in $L^1 + L^2$, so its inverse Fourier transform u satisfies $\tilde{u} \in L^1 + L^2$, and $R_k * g = u + \sum_{|\alpha|=0}^{k-1} i^{-|\alpha|} c_\alpha x^\alpha$.

If $e \in L_0^\infty$, $e_r(x) = r^{-n} e(x/r)$, then by Lemma 7.1,

$$\begin{aligned}
\langle R_k * g, e_r\rangle &\to 0 \\
\langle u, e_r\rangle &\to 0
\end{aligned} \qquad (7.15)$$

Take e to be the characteristic function of $[0, t_1] \times [0, t_2] \times \cdots \times [0, t_n]$, then

$$\left\langle \sum_{|\alpha|=0}^{k-1} i^{-|\alpha|} c_\alpha x^\alpha, e_r \right\rangle = \sum_{j=0}^{k-1} r^{-n} p_j(rt), \qquad (7.16)$$

where

$$p_j(t) = \sum_{|\alpha|=j} \frac{i^{-|\alpha|} c_\alpha}{(\alpha_1 + 1)\cdots(\alpha_n + 1)} t_1^{\alpha_1+1} \cdots t_n^{\alpha_n+1} \qquad (7.17)$$

is a homogeneous polynomial of degree $j + n$.

For each fixed t, $\sum r^{-n} p_j(rt)$ tends to zero as $r \to \infty$. This implies that all the p_j and therefore all the c_α must be zero. □

Corollary 7.3. *If $f \in \mathcal{D}_k$, then (7.2) is valid, that is, for almost every $x \in \mathbb{R}^n$,*

$$f(x) = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k P^\# P f(x). \qquad (7.18)$$

To establish the validity of (7.3) it must first be determined under what conditions on e the kernel K is welldefined.

THEOREM 7.4. *If* $(1+|x|)^{k-n} f \in L^1$, $\phi \in L^1$ *and* $(1+|x|)^n \phi \in L^\infty$, *then*

$$\int_{G_{k,n}} \langle P_\pi f,\, P_\pi \phi\rangle \, d\pi = \frac{\gamma_{k,n}}{c_{k,n}} \langle R_k * f, \phi\rangle \tag{7.19}$$

and

$$\Big| \int_{G_{k,n}} \langle P_\pi f,\, P_\pi \phi\rangle \, d\pi \,\Big| \le c\, \|(1+|x|)^{k-n} f\|_{L^1} (\|\phi\|_{L^1} + \|(1+|x|)^n \phi\|_{L^\infty}). \tag{7.20}$$

Proof. (7.19) follows from

$$\begin{aligned} \int_{G_{k,n}} \langle P_\pi f,\, P_\pi \phi\rangle \, d\pi &= \int_{G_{k,n}} \int_{\pi^\perp} P_\pi f(x'')\, \overline{P_\pi \phi(x'')}\, dx''\, d\pi \\ &= \int_{G_{k,n}} \int_{\pi^\perp} P_\pi f(x'') \int_\pi \overline{\phi(x'+x'')}\, dx'\, dx''\, d\pi \\ &= \int_{G_{k,n}} \int_{\mathbb{R}^n} P_\pi f(E_{\pi^\perp} x) \overline{\phi(x)}\, dx\, d\pi \\ &= \frac{\gamma_{k,n}}{c_{k,n}} \langle R_k * f, \phi\rangle \end{aligned} \tag{7.21}$$

by Lemma 3.2. Inequality 7.20 follows from Lemma 3.4 and justifies the application of Fubini's theorem a posteriori. □

THEOREM 7.5. *If* $f \in \mathcal{D}_k$, *then for almost every* $\pi \in G_{k,n}$, *the following holds:*

(a) $|\xi''|^{k/2} \tilde f \in L^2(\pi^\perp)$.

(b) $P_\pi f$ *is a tempered distribution on* $\pi^\perp$.

(c) *If* $\phi \in \mathcal{S}(\pi^\perp)$, $D^\alpha \phi(0) = 0$ *for* $0 \le |\alpha| < k$, *then*

$$\langle (P_\pi f)\tilde{}\,, \phi\rangle = (2\pi)^{k/2} \int_{\pi^\perp} \tilde f(\xi'') \overline{\phi(\xi'')}\, d\xi''. \tag{7.22}$$

(d) (Central slice theorem) *If* $|\xi|^{-k} \tilde f$ *is locally integrable on* $\mathbb{R}^n$, *then* $\tilde f$ *is locally integrable on* $\pi^\perp$ *and* $(P_\pi f)\tilde{} = (2\pi)^{k/2} \tilde f$ *on* $\pi^\perp$.

Proof. Throughout the proof, c stands for an unspecified constant (which need not be the same in each case.)

(a) By Corollary 2.4,

$$\int_{\mathbb{R}^n} |\tilde f(\xi)|^2 \, d\xi = \frac{1}{\gamma_{n-k,n}} \int_{G_{k,n}} \int_{\pi^\perp} |\xi''|^k |f(\xi'')|^2 \, d\xi''\, d\pi \tag{7.23}$$

and (a) holds whenever the inner integral on the right-hand side is finite.

(b) In Theorem 7.4, take $\phi(x) = (1+|x|^2)^{(-n-1)/2}$, so that $P_\pi \phi(x'') = c(1+|x''|^2)^{(k-n-1)/2}$, to get

$$\int_{G_{k,n}} \int_{\pi^\perp} P_\pi f(x'') (1+|x''|^2)^{(k-n-1)/2} \, dx''\, d\pi \le c \|f\|_{\mathcal{D}_k}, \tag{7.24}$$

thus

$$\int_{\pi^\perp} P_\pi f(x'')(1+|x''|^2)^{(k-n-1)/2}\,dx'' < \infty \tag{7.25}$$

for almost every π, and for any such π, (b) holds.

(c) Let $f_n \to f$ in $\mathcal{D}_k$, $f_n \in L^1$ (for instance, $f_n = f$ restricted to a ball of radius n.) By (7.24),

$$\int_{G_{k,n}}\int_{\pi^\perp} |\,P_\pi f(x'') - \,P_\pi f_n(x'')|(1+|x''|^2)^{(k-n-1)/2}\,dx''\,d\pi \to 0. \tag{7.26}$$

For some subsequence (still called f_n),

$$\int_{\pi^\perp} |\,P_\pi f(x'') - \,P_\pi f_n(x'')|(1+|x''|^2)^{(k-n-1)/2}\,dx'' \to 0 \tag{7.27}$$

for almost every π. If (7.25), (7.27) hold, then $P_\pi f_n$ converges to $P_\pi f$ in $\mathcal{S}'(\pi^\perp)$, so that

$$(2\pi)^{k/2}\int_{\pi^\perp} \tilde f_n(\xi)\overline{\phi(\xi)}\,d\xi = \langle (\,P_\pi f_n)\tilde{}\,, \phi\rangle \to \langle (\,P_\pi f)\tilde{}\,, \phi\rangle, \tag{7.28}$$

since Theorem 5.1 (the Central slice theorem) is valid for f_n. By (7.23), for some subsequence (still called f_n),

$$\|\,|\xi''|^{k/2}(\tilde f - \tilde f_n)\|_{L^2(\pi^\perp)} \to 0 \tag{7.29}$$

for almost every π. For such π and $\phi \in \mathcal{S}(\pi^\perp)$,

$$\begin{aligned}&\int_{|\xi''|\ge 1} |\tilde f(\xi'') - \tilde f_n(\xi'')|\,|\phi(\xi'')|\,d\xi'' \\ &\qquad \le \|\phi\|_{L^2(\pi^\perp)}\|\,|\xi''|^{k/2}(\tilde f - \tilde f_n)\|_{L^2(\pi^\perp)}\end{aligned} \tag{7.30}$$

also goes to zero.

If $D^\alpha\phi(0) = 0$, $0 \le |\alpha| < k$, then $|\phi(\xi'')| \le c|\xi''|^k$, so

$$\begin{aligned}&\int_{|\xi''|\le 1} |\tilde f(\xi'') - \tilde f_n(\xi'')|\,|\phi(\xi'')|\,d\xi'' \\ &\qquad \le c\,\Big(\int_{|\xi''|\le 1} |\xi''|^k\,d\xi''\Big)^{1/2}\,\|\,|\xi''|^{k/2}(\tilde f - \tilde f_n)\|_{L^2(\pi^\perp)},\end{aligned} \tag{7.31}$$

which also goes to zero, thus proving (c).

(d) By Corollary 2.4,

$$\int_{G_{k,n}}\int_{|\xi''|\le 1} |\tilde f(\xi'')|\,d\xi''\,d\pi = \gamma_{n-k,n}\int_{|\xi|\le 1} |\xi|^{-k}|\tilde f(\xi)|\,d\xi, \tag{7.32}$$

so $\tilde f$ is locally integrable on $\pi^\perp$ for almost every π. Fix π so that this holds and that (a), (b), (c) hold. This implies that $\tilde f$ is a tempered distribution on $\pi^\perp$, and

$$(\,P_\pi f)\tilde{}\; = (2\pi)^{k/2}\tilde f + \sum_{|\alpha|=0}^{k-1} c_\alpha D^\alpha \delta. \tag{7.33}$$

From Lemma 7.1 and Theorem 7.4,

$$\int_{G_{k,n}} \langle P_\pi |f|,\, P_\pi e_r \rangle \, d\pi = \frac{\gamma_{k,n}}{c_{k,n}} \langle R_k * |f|, e_r \rangle \to 0, \tag{7.34}$$

so for some sequence r_j, $\langle P_\pi f,\, P_\pi e_{r_j} \rangle \to 0$ for almost every π. Since $(P_\pi e)_r = P_\pi e_r$, it follows as in the proof of Theorem 7.2 that $c_\alpha = 0$ for all α. ∎

The Sobolev space $\mathcal{H}^m$ is defined by

$$\mathcal{H}^m = \{ f \in L^2 : |\xi|^m \tilde{f} \in L^2 \} \tag{7.35}$$

with the norm

$$\|f\|^2_{\mathcal{H}^m} = \|f\|^2_{L^2} + \|\, |\xi|^m \tilde{f} \|^2_{L^2}. \tag{7.36}$$

COROLLARY 7.6. *If $e \in \mathcal{D}_k \cap \mathcal{H}^{k/2}$ and $|\xi|^{-k} \tilde{e}$ is locally integrable, then for almost every α, $|\xi''|^k \tilde{e} \in L^2(\pi^\perp)$, and*

$$(\Lambda^k P_\pi e)^{\tilde{}}(\xi'') = (2\pi)^{k/2} |\xi''|^k \tilde{e}, \qquad \xi'' \in \pi^\perp. \tag{7.37}$$

Proof. By Theorem 7.5, it suffices to show that $|\xi''|^k \tilde{e} \in L^2(\pi^\perp)$ for almost every π. This follows from

$$\int_{G_{k,n}} \int_{\pi^\perp} |\xi''|^{2k} |\tilde{e}|^2(\xi'') \, d\xi'' = \gamma_{n-k,n} \int_{\mathbb{R}^n} |\xi|^k |\tilde{e}|^2(\xi) \, d\xi < \infty, \tag{7.38}$$

since $e \in \mathcal{H}^{k/2}$. ∎

THEOREM 7.7. *If $e \in \mathcal{D}_k \cap \mathcal{H}^{k/2}$, if $|\xi|^{-k} \tilde{e}$ is locally integrable and if $f \in L^2_0$, then*

$$\int_{G_{k,n}} \langle \Lambda^k P_\pi e,\, P_\pi f \rangle \, d\pi = \frac{\gamma_{k,n}}{c_{k,n}} \langle e, f \rangle. \tag{7.39}$$

Proof. Recall that $f \in L^2_0$ implies that $P_\pi f \in L^2_0(\pi^\perp)$ for all $\pi \in G_{k,n}$. Using Corollary 2.4, Theorems 4.2 and 7.5, and Corollary 7.6,

$$\begin{aligned} \int_{G_{k,n}} \langle \Lambda^k P_\pi e,\, P_\pi f \rangle \, d\pi &= \int_{G_{k,n}} \langle (\Lambda^k P_\pi e)^{\tilde{}}, (P_\pi f)^{\tilde{}} \rangle \, d\pi \\ &= (2\pi)^k \int_{G_{k,n}} \langle |\xi''|^k \tilde{e}, \tilde{f} \rangle \, d\pi \\ &= (2\pi)^k \gamma_{n-k,n} \langle \tilde{e}, \tilde{f} \rangle \\ &= (2\pi)^k \gamma_{n-k,n} \langle e, f \rangle. \end{aligned} \tag{7.40}$$

The relation $(2\pi)^k \gamma_{n-k,n} = \gamma_{k,n}/c_{k,n}$ can be checked by calculation. ∎

The last formula contains the desired approximate inversion formula for P.

THEOREM 7.8. *If $e \in \mathcal{D}_k \cap \mathcal{H}^{k/2}$, $|\xi|^{-k} \tilde{e}$ is locally integrable, and $f \in L^2_0$, then for almost every $x \in \mathbb{R}^n$,*

$$e * f(x) = \int_{G_{k,n}} K * P_\pi f(E_{\pi^\perp} x) \, d\pi, \qquad K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^k P_\pi e. \tag{7.41}$$

Proof. Replace f by $\overline{f(y-x)}$ in Theorem 7.7. ∎

8. Local reconstruction. The main goal in x-ray tomography is to produce a sharp and accurate picture of a cross section of an object. This means that the reconstructed function $e * f$ should be close to the original f, which can be achieved by using a point spread function e with small support.

Another objective is to keep measurements (and therefore radiation dose) to a minimum. If only the region around the spine is of interest, it should not be necessary to x-ray the rest of the body as well. This requirement can be met by using a kernel K with small support.

When the dimension k of the subspaces is even, then $\Lambda^k = (-\Delta)^{k/2}$, where Δ is the Laplacian. This means that if e (and therefore Pe) have compact support, so does K, and both goals can be met at the same time. If k is odd (which includes the usual case $k = 1$), this is impossible.

In practice, this means that even if only a small part of the picture is needed, measurements must be taken for the entire cross section of the object. It is, however, possible to get a good local reconstruction of Λf:

$$e * \Lambda f = \Lambda e * f = P^{\#}(K * Pf), \tag{8.1}$$

where

$$K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^{k+1} Pe. \tag{8.2}$$

Conditions for validity of the Λ reconstruction formula are easy to find.

Corollary 8.1. *If $e \in \mathcal{D}_k \cap \mathcal{H}^{(k+2)/2}$, $|\xi|^{-k}\tilde{e}$ is locally integrable, and $f \in L^2_0$, then for almost every $x \in \mathbb{R}^n$,*

$$e * f(x) = \int_{G_{k,n}} K * P_\pi f(E_{\pi^\perp} x)\, d\pi, \tag{8.3}$$

where

$$K = \frac{c_{k,n}}{\gamma_{k,n}} \Lambda^{k+1} P_\pi e. \tag{8.4}$$

Proof. As in the proof of Theorem 7.8. □

This process is known as *Lambda Tomography* or *Local Tomography* and can be easily adapted to divergent beam reconstructions, as well. The numerical algorithm is exactly the same as in the case of regular tomography, the only thing that changes is the kernel K. It is also possible to do a reconstruction first of f, then of Λf with the same data.

The obvious question is: How does Λf relate to f? In applications in x-ray tomography, f is the x-ray density function of a cross section of the body. The support of f is divided into regions, corresponding to organs, bones, tumors, etc., with f approximately constant on each region. The density differences between adjoining regions are often quite small, especially the difference between a tumor and surrounding tissue.

Since $\Lambda = (-\Delta)^{1/2}$, one expects Λ to act in a manner similar to a differentiation operator. For the functions encountered in medical tomography, Λ is expected to enhance the boundaries between regions, while the density information for each region is lost.

Possible applications include the detection of very slight density differences between adjacent tissues and reconstructions of small regions from a reduced set of measurements.

This new method has only been tested in a few cases, but results look promising. Recent results and examples can be found in Faridani et al. [1].

Acknowledgements. I would like to thank Professor Kennan T. Smith for his help and guidance, and Professors D. Finch and S. Helgason for valuable discussions. I would also like to thank the editor for many helpful suggestions.

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