Another example for stability analysis

Richardson's Method

\[ u_t = u_{xx} \]

\[ \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2} \]

\[ r = \frac{\Delta t}{(\Delta x)^2} \]

Again, set \( u_{i,j} = e^{i\mu x} \cdot e^{j\omega t} \), divide by \( u_{i,j} \):

\[ e^{\omega t} = e^{-\omega t} + 2r (e^{i\mu x} - 2 + e^{-i\mu x}) \]

\[ = e^{-\omega t} + 4r (\cos (\mu x) - 1) \]

Multiply by \( e^{\omega t} \), set \( z = e^{\omega t} \)

\[ z^2 - 4r (\cos (\mu x) - 1) z - 1 = 0 \]

We want \( |e^{\omega t}| = |z| \leq 1 \) for stability

\[ z_{\frac{1}{2}} = \frac{1}{2} \left( 4r (\cos (\mu x) - 1) \pm \sqrt{16r^2 (\cos (\mu x) - 1)^2 + 4} \right) \]

Note that both solutions are real.

Fact: For any monic polynomial (leading coefficient is 1)

\[ z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \]

the sum of the roots is \((-a_{n-1})\), and the product is \((-1)^n a_0\).

In our case: product of roots is \(-1 \Rightarrow \) one root is \(> 1 \)
in magnitude, one is smaller.
Now consider the standard sample problem
\[
\begin{align*}
U_{xx} &= U_{tt} \\
U(x, 0) &= f(x) \\
U_t(x, 0) &= g(x)
\end{align*}
\]

We will leave out boundary conditions.

**Explicit Method**
\[
\frac{U_{l+1,j} - 2U_{l,j} + U_{l-1,j}}{(\Delta x)^2} = \frac{U_{l+1,j} - 2U_{l,j} + U_{l-1,j}}{(\Delta t)^2}
\]

or
\[
U_{l,j+1} = m^2 \left( U_{l-1,j} + U_{l+1,j} \right) + 2(1-m^2)U_{l,j} - U_{l,j-1}
\]

where \( m = \frac{\Delta t}{\Delta x} \) is error is \( O \left( (\Delta x)^2 (\Delta t)^2 + (\Delta t)^4 \right) \)

Obviously, we need to use values from 3 rows to estimate \( U_{tt} \).

**Answer:** Just like before, introduce an artificial boundary at \( t_{-1} \).

**Row 0:**
\[
U_{l,0} = f_L \quad \text{(initial condition)}
\]

**Second IC:**
\[
\frac{U_{l,1} - U_{l,0}}{2\Delta t} = g_L \quad \Rightarrow \quad U_{l,-1} = U_{l,1} - 2\Delta t g_L
\]

**PDE for Row 0:**
\[
U_{l,1} = m^2 \left( U_{l-1,0} + U_{l+1,0} \right) + 2 \left( 1 - m^2 \right) U_{l,0} - U_{l,-1}
\]

\[
= m^2 \left( f_{L-1} + f_{L+1} \right) + \left( 1 - m^2 \right) f_L + \Delta t g_L
\]

\[
= m^2 \left( \frac{f_{L-1} + f_{L+1}}{2} + \frac{1}{2} \right) + (1 - m^2) f_L + \Delta t g_L
\]
Altogether:

\[ U_{n,0} = f_n \]
\[ U_{n,1} = \frac{m^2}{2} \left( f_{n-1} + f_{n+1} \right) + (1 - m^2) f_n \cdot s \cdot g_n \]
\[ j \geq 1 \]
\[ U_{n,j+1} = m^2 \left( U_{n,j+1} + U_{n,j-1} \right) + 2 (1 - m^2) U_{n,j} - U_{n,j-1} \]

Stability:

Again, let \( U_j = e^{i \mu j d x} \cdot e^{i \omega j t} \)

Substitute that, divide by \( U_j \):

\[ e^{i \omega t} = \frac{m^2}{2} \left( e^{-i \mu j d x} + e^{i \mu j d x} \right) + 2 (1 - m^2) - e^{-i \omega t} \]

\[ \cos(\mu j d x) = 2 - 4 \sin^2 \frac{\mu j d x}{2} \]

Set \( z = e^{i \omega t} \), multiply by \( z \):

\[ z^2 - 2 \left( 1 - 2 m^2 \sin^2 \frac{\mu j d x}{2} \right) z + 1 = 0 \]

Result:

\[ z = \frac{1}{2} \left[ 2 \left( 1 - 2 m^2 \sin^2 \frac{\mu j d x}{2} \right) \pm \sqrt{4 \left( 1 - 2 m^2 \sin^2 \frac{\mu j d x}{2} \right)^2 - 4} \right] \]

As before: if the solutions are real, one of them is bigger than 1

\[ \Rightarrow \text{unstable} \]

For stability, we need

\[ 4 \left( \cdot \right)^2 - 4 \leq 0 \quad \text{automatic} \]

\[ \Leftrightarrow \quad -1 \leq 1 - 2 m^2 \sin^2 \frac{\mu j d x}{2} \leq 1 \]

\[ \Rightarrow \quad m^2 \leq \frac{1}{\sin^2 \frac{\mu j d x}{2}} \quad \text{satisfied if } |m| \leq 1 \]

Remark: If the solutions are complex, they are complex conjugates \( \Rightarrow \) same magnitude \( \Rightarrow \) \( |z| = 1 \)
The condition $m \leq 1$ means $\Delta t = \Delta x^2$, which is a lot better than $\Delta t = \frac{1}{2} (\Delta x)^2$ in the parabolic case. We can still do better with implicit methods.

Sideline: interpreting the condition $m \leq 1$ in terms of characteristics.

The true value $u(x,t)$ depends on IC in $[x-t, x+t]$.

The numerical value $u_{ij}$ depends on IC from $i-j$ to $i+j$.

If $m > 1$, the numerical solution does not use some of the initial conditions.
That can't be good.
Implicit Method

\[
\frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{(dt)^2} = \lambda \cdot \text{(difference formula at level } j+1) \\
+ (1-2\lambda) \cdot (\ldots - \ldots j \ldots) \\
+ \lambda \cdot (\ldots - \ldots j-1 \ldots)
\]

\(\lambda = 0\) is explicit case

\[e^{\lambda dt} - 2 + e^{-\lambda dt} = \mu^2 \left[ \lambda \cdot (e^{i\mu dx} - 2 + e^{-i\mu dx}) \cdot e^{\mu dt} \\
+ (1-2\lambda) \cdot (e^{i\mu dx} - 2 + e^{-i\mu dx}) \cdot 1 \\
+ \lambda \cdot (e^{i\mu dx} - 2 + e^{-i\mu dx}) \cdot e^{-\mu dt} \right] \]

The analysis gets very messy, so here is the result:

\[
\begin{cases}
\text{if } 0 \leq \lambda < \frac{1}{4}, \text{ we need } \mu \leq \frac{1}{\sqrt{1-4\lambda}} \\
\text{if } \lambda \geq \frac{1}{4}, \text{ there is no restriction}
\end{cases}
\]

Different Approach: \(v_{xx} = u_{tt}\), let \(v = u_t, w = u_x\) to get
\[
\begin{cases}
v_t = w_x \\
v_x = w_t
\end{cases}
\]

discretize that:

\[
\begin{align*}
\frac{v_{i,j+1} - v_{i,j}}{dt} &= \frac{w_{i,j+1} - w_{i,j-1}}{2Ax} \\
\frac{w_{i,j+1} - w_{i,j}}{dt} &= \frac{v_{i+1,j} - v_{i-1,j}}{2Ax}
\end{align*}
\]

This particular scheme is unstable, but there are variations on this that work quite well.