Hyperbolic PDEs

Recall: A second order ODE \( y'' = f(t, y, y') \)
can be converted to a first-order system \( \dot{z} = (y')' = (y' f(t, z, z')) \)

We can do the same for PDEs.

Example: \( w_{xx} - w_{yy} = 0 \)

Let \( U = w_x, V = w_y \), then \( U_y = W_{xy} = W_{yx} = V_x \)

we get \( \begin{cases} U_x - V_y = 0 \quad \text{(original PDE)} \\ V_y - V_x = 0 \quad \text{(mixed derivative)} \end{cases} \)

Consider a general linear equation in \( \mathbb{R}^2 \):
\[
\begin{cases}
\alpha(x, y) U_x(x, y) + b(x, y) U_y(x, y) = f(x, y) \\
U(x_0, y_0) = u_0
\end{cases}
\]

Suppose I have a parametric curve
\[
\vec{s}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \vec{s}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

with \( x'(t) = \alpha(\vec{s}(t)), \ y'(t) = b(\vec{s}(t)) \),

then by the chain rule
\[
\frac{d}{dt} U(\vec{s}(t)) = \frac{d}{dt} U(x(t), y(t)) = U_x \cdot x' + U_y \cdot y' = \alpha U_x + b U_y = f
\]

So \( U(\vec{s}(t)) = u_0 + \int_0^t f(\vec{s}(\tau)) d\tau \).

We know the solution along \( \vec{s} \).

\( \vec{s} \) is called a characteristic.
value at \((x_0, y_0)\) determines the value of \(u\) along \(\gamma(t)\).

We can repeat this for other \((x_0, y_0)\).

If \(u\) is given along some curve \(\Gamma\) which intersects the characteristics, this determines the solution in a whole region.

Notes:
1. If there is a jump in the initial conditions along \(\Gamma\), this discontinuity will propagate along the characteristics.
2. You can try to find the characteristics numerically. This leads to the method of characteristics. We will not cover that.
Example:
\[
\begin{cases}
 x' = x \\
 y' = -y \\
 x(0) = c, e^t \\
 y(t) = c_2 e^{-t}
\end{cases}
\]
so we want \( x' = x \) so that \( x(t) = c, e^t \)
\( y(t) = c_2 e^{-t} \)
We want \( x(0) = 1, y(0) = 1 \) \( \Rightarrow \)
\( c_1 = c_2 = 1 \)
\[
\vec{y}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}
\]
\( f(\vec{y}(t)) = x(t) \cdot y(t) = e^t \cdot e^{-t} = 1 \)
\[
\Rightarrow \quad U(\vec{y}(t)) = U(e^t, e^{-t}) = 1 + \int_0^t e^t d\tau = 1 + t
\]
\( \text{or} \quad U(x, \frac{1}{x}) = 1 + \ln x \)
\( \text{determines} \ U \text{ along hyperbola} \)

Now consider two functions \( U(x, y), V(x, y) \),
\[
\begin{cases}
 a, \quad bx + by + c_1 vx + d_1 vy = f_1 \\
 a_2 bx + b_2 by + c_2 vx + d_2 vy = f_2
\end{cases}
\]
\( U(x_0, y_0) \) given
\( V(x_0, y_0) \) given

For simplicity, assume \( a_1, a_2, b, \) etc. are constant.
Again, let \( \vec{x}(t) = (x(t), y(t)) \), then
\[
\frac{d}{dt} \left[ U + \mu V \right] = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \mu u_x x' + \mu v_y y'
\]
Let \( \vec{n}_1 = (a_1, b_1), \vec{n}_2 = (c_1, d_1) \) and assume \( \vec{n}_1 \cdot \vec{n}_2 = y' \)
for some \( \mu \).
Then we can do the same as before, and get values of \( U + \mu V \) along \( \vec{x} \).
Of course, in general \( \vec{n}_1 \neq \mu \vec{n}_2 \), but maybe we can use linear combinations of the two equations.
Take $\lambda_1$ (first equation) + $\lambda_2$ (second equation) we get:

$$(\lambda_1 a_1 + \lambda_2 a_2) u_x + \ldots = \lambda_1 f_1 + \lambda_2 f_2$$

$$\vec{v}_1 = (\lambda_1 a_1 + \lambda_2 a_2) = (a_1, a_2) \cdot (\lambda_1)$$
$$A \cdot \lambda$$

$$\vec{v}_2 = (C_1, C_2) \cdot \lambda$$
$$B$$

We want $A \vec{\lambda} = \mu \cdot B \vec{\lambda} \iff (A - \mu B) \cdot \vec{\lambda} = 0$
This is a sort of eigenvalue problem. A nontrivial solution $\vec{\lambda}$ exists $\iff \det(A - \mu B) = 0$

**Example**

$w_{xx} - w_{yy} = 0 \iff \begin{cases} u_x - v_y = 0 \\ v_y - u_x = 0 \end{cases}$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = I, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$A - \mu B = \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix}$

$\det(A - \mu B) = 1 - \mu^2 = 0 \Rightarrow \mu = \pm 1$

$\mu = 1$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{\lambda} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\lambda} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

This means we want to subtract equations:

$u_x - u_y + v_x - v_y = 0$

$\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, want $x' = 1$, $y' = -1$

so $\vec{\gamma}(t) = \begin{pmatrix} t \\ -t \end{pmatrix} + \text{const}$

These characteristics are the lines $x + y = \text{const}$. 
along these characteristics,
\[ \frac{d}{dt} (u + v) = 0 \implies u + v = \text{const} \]

\[ \mu = -1 \]

Likewise characteristics are \( x - y = \text{const} \),
along them \( u - v = \text{const} \).

\[ u + v \text{ known} \]

\[ u - v \text{ known} \]

at \( \Theta \) we have \( u + v = u_0 + v_0 \) \( u - v = u_1 - v_1 \), \( \implies \) we can figure out \( u, v \)

General situation: (in \( \mathbb{R}^2 \))

\[ \det (A - \mu B) = \text{quadratic polynomial in } \mu \]

if it has 2 real solutions: \underline{hyperbolic}

1 double real solution: \underline{parabolic}

complex solutions: \underline{elliptic}

9-1-5
hyperbolic
characteristic for $\mu_1$, $v_0 + \mu_1v_0$ known
characteristic for $\mu_2$, $v_1 + \mu_2v_1$ known

$\Gamma$, where $v, \nu$ are given

One last look at example

\[
\begin{cases}
  u_{xx} = u_{tt} \\
  u(x,0) = f(x) \\
  u_t(x,0) = g(x)
\end{cases}
\]

Translating the previous calculations:

\[
u(x,t) = \frac{1}{2} \left[ f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\xi) \, d\xi \right]
\]

interval of dependence
$v(x,t)$ depends on IC in $[x-t, x+t]$
not anything else

domain of influence
IC at $(x_0, 0)$ influences the solution in this region
In preparation for studying the stability of finite difference methods for hyperbolic equations, we need to build up some background first.

**Complex vectors**

- $\vec{v}, \vec{w}$ complex, then
  \[
  \vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle = \sum_j v_j \overline{w}_j
  \]

  \(\overline{w}_j = \) complex conjugate

**Def:** \{\(\vec{f}_j\)\} are **orthonormal** if

\[
\langle \vec{f}_j, \vec{f}_k \rangle = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & \text{otherwise} \end{cases}
\]

(Kronecker delta)

**Recall** If $\vec{f}$ is a unit vector, then

\[
\langle \vec{v}, \vec{f} \rangle \vec{f} = \text{component of } \vec{v} \text{ in direction } \vec{f}
\]

**Fact** An orthonormal basis of $\mathbb{R}^N$ is given by

\[
\vec{f}_j = \frac{1}{\sqrt{N}} \left( e^{2\pi ijk/N} \right)_{k=0..N-1}
\]

\[
= \frac{1}{\sqrt{N}} (w_{jk})_{k=0..N-1}
\]

(DFT = discrete Fourier transform)
Here $w = e^{\frac{2\pi i}{N}} = N^{th}$ root of unity

Example: $N = 3$

$w = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

$w^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i$

\[
\tilde{f}_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} w^0 & 0 & 1 \\ w^0 & 1 & 0 \\ w^0 & 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\tilde{f}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} w^1 & 0 & 1 \\ w^1 & 1 & 0 \\ w^1 & 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ w^2 \\ w^0 \end{pmatrix}
\]

\[
\tilde{f}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} w^2 & 0 & 1 \\ w^2 & 1 & 0 \\ w^2 & 0 & 2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ w^0 \\ w^2 \end{pmatrix}
\]

So, any vector $\tilde{v}$ in $\mathbb{R}^N$ can be represented by

\[
\tilde{v} = \sum_j <\tilde{v}, \tilde{f}_j> \tilde{f}_j
\]

where

\[
<\tilde{v}, \tilde{f}_j> = \frac{1}{N} \sum_k v_k e^{-2\pi i jk/N}
\]
Now apply this to stability.

\[ t \]
\[ U = 0 \]
\[ V_j \]
\[ U_0 \text{ given} \]
\[ U = 0 \text{ at ends} \]
\[ A \overrightarrow{V_j} = B \overrightarrow{V_{j-1}} \]

we want to actually solve this in closed form.

\[ \int dt \]

Take \( \overrightarrow{U_0} \) and expand it in DFT:

\[ \overrightarrow{U_0} = \sum_k c_k \overrightarrow{f_k} \]

For each fixed \( k = 0, N-1 \), look for solutions of the form

\[ \overrightarrow{W_j} = \overrightarrow{f_k} \cdot e^{\alpha_k j \Delta t} \]

so \( \overrightarrow{W_0} = \overrightarrow{f_k} \)

If we succeed, the total solution will be

\[ \overrightarrow{V_j} = \sum_k c_k \overrightarrow{f_k} \cdot e^{\alpha_k j \Delta t} \]

Now \( f_{kl} = l^{th} \) entry in \( \overrightarrow{f_k} = e^{2\pi i l k / N} = e^{im\Delta x} \)

with \( m = 2\pi \), \( \Delta x = \frac{1}{N} \)

but we set this up for general \( m \):

Look for solutions \( \overrightarrow{W_j} \) whose value at \((lx, jdt)\) is

\[ W_{lj} = e^{imlx} \cdot e^{jdt} \]
Example: try this for the original finite difference method for parabolic PDE:

\[
\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} = \frac{U_{i-1,j}^n - 2U_{i,j}^n + U_{i+1,j}^n}{(\Delta x)^2}
\]

or

\[
U_{i,j}^{n+1} = \Gamma U_{i-1,j}^n + (1-2\Gamma) U_{i,j}^n + \Gamma U_{i+1,j}^n \quad \Gamma = \frac{\Delta t}{(\Delta x)^2}
\]

Substitute \( \times \)

\[
e^{i\mu L \Delta x} e^{\alpha_j(s+1) \Delta t} = \Gamma e^{i\mu (L-1) \Delta x} e^{\alpha_j \Delta t}
+ (1-2\Gamma) e^{i\mu L \Delta x} e^{\alpha_j \Delta t}
+ \Gamma e^{i\mu (L+1) \Delta x} e^{\alpha_j \Delta t}
\]

divide by \( e^{i\mu L \Delta x} e^{\alpha_j(s+1) \Delta t} \)

\[
e^{\alpha_j \Delta t} = \Gamma e^{-i\mu \Delta x} + (1-2\Gamma) + \Gamma e^{i\mu \Delta x}
= 1 - 2\Gamma + 2\Gamma \cos(\mu \Delta x)
= 1 - 4\Gamma \sin^2 \frac{\mu \Delta x}{2}
\]

so, for any \( \mu \) we can find some \( \alpha \) that works.

For stability, we want \(|e^{\alpha j \Delta t}| \leq 1\)

which means

\[ -1 \leq 1 - 4\Gamma \sin^2 \frac{\mu \Delta x}{2} \leq 1 \]

or

\[ \Gamma \leq \frac{1}{2} \sin^2 \frac{\mu \Delta x}{2} \]

This is always satisfied if \( \Gamma \leq \frac{1}{2} \) (same result as before)
How does this relate to error propagation?

Recall we had \( A \vec{e}_j = B \vec{e}_{j-1} + \vec{c}_j \).

Look at it like this:

\[ \text{error at level } J = \text{error at level } 0, \text{ propagated to level } J \]
\[ (= 0, \text{ actually}) \]
\[ + \vec{c}_1, \text{ propagated to level } J \]
\[ + \vec{c}_2, \text{ propagated to level } J \]
\[ \vdots \]

Each \( \vec{c}_j \) produces a setup like we had above

(\( \vec{c}_j \) corresponds to \( \vec{u}_0 \))

So, the analysis from above also applies to error propagation.

**Advantage:** this method also works for more complicated methods, of the form

\[ A \vec{u}_j = B \vec{u}_{j-1} + C \vec{u}_{j-2} + \vec{d}_j \]

for example.