Recall: If $\alpha, \beta \in \mathbb{R}$, then $e^{(\alpha + i\beta)t} = e^{\alpha t} \cos(\beta t) + ie^{\alpha t} \sin(\beta t)$

Test Equation: $y'(t) = \lambda y(t)$ \quad $\lambda \in \mathbb{C}$

solution: $y(t) = c \cdot e^{\lambda t}$

**Def:** A numerical ODE method is **stable** (for given $\lambda, h$) if the numerical solution of the test equation remains bounded as $t \to \infty$.

**Fact:** Stability depends only on the product $h\lambda$, not $h, \lambda$ separately.

**Def:** The region of stability of a method is $\{ h\lambda \in \mathbb{C} : \text{method is stable} \}$

**Examples:**

1. Euler's method

   $y_{n+1} = y_n + h \cdot f(t_n, y_n)$

   $= y_n + h \cdot \lambda \cdot y_n$

   $= (1 + h\lambda) y_n$

   obvious: $h\lambda$ is stable $\iff \ |1 + h\lambda| \leq 1$
This is a circle of radius 1 around (-1).

(2) Backward Euler

\[ y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) \]
\[ = y_n + h \lambda y_{n+1} \]

\[(1+h\lambda)y_{n+1} = y_n\]

\[ y_{n+1} = \frac{1}{1-h\lambda} y_n \]

\[ (1-h\lambda) \leq 1 \]

\[ |1-h\lambda| \geq 1 \]

This is the outside of a circle of radius 1 around 1.

What is the ideal region of stability?

We want the numerical solution to have the correct qualitative behavior (going to 0, going to \(\infty\)) as the true solution.

Since \( e^{\lambda t} \rightarrow \infty \) for \( Re(\lambda) > 0 \)
\[ \rightarrow 0 \] for \( Re(\lambda) < 0 \)

we want the left half plane.
Def: A method is $A$-stable if its region of stability includes the left half plane.

It is $A(\alpha)$-stable if it includes a wedge with opening angles $\pm \alpha$ in the left half plane.

Example: The backward Euler method is $A$-stable, Euler's method is not.

Discussion of Examples

For given $\lambda$ in the left half plane, $h\lambda$ will be stable if we take $h$ small enough.

If you take $h$ too large, the numerical solution will blow up.

Backward Euler

Perfectly stable for $\text{Re} \lambda < 0$.
However, this method is too stable.
For $\text{Re} \lambda > 0$, large $h$, the true solution blows up, numerical solution goes to 0.
Stability for One-Step Methods

Facts

1. When you apply any standard one-step method to the test equation, you find

   \[ y_{n+1} = \varphi(hz) \cdot y_n \]

   The region of stability is then \( \{ z : |\varphi(z)| \leq 1 \} \)

2. For any standard explicit method, \( \varphi(z) \) is a polynomial of rational function

\( (= \text{quotient of polynomials}) \)

Examples

(a) Euler \( \varphi(z) = 1 + z \)

(b) backward Euler \( \varphi(z) = \frac{1}{1 - z} \)

(c) 4-stage RK \( \varphi(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} \)

Consequence: For any explicit method, the region of stability is bounded, since a polynomial goes to \( \infty \) in all directions.

\( \Rightarrow \) all \( A \)-stable, \( A(\alpha) \)-stable methods are implicit.

Typical Region of Stability

Some blob to the left of \( 0 \).
(see separate page)
Fig. 3.1  Stability regions for Adams-Bashforth methods. Method of order $k$ is stable inside region indicated left of origin.

Region of absolute stability of fourth order explicit Runge-Kutta method.

Region of Stability for Backward Euler Method

Region of Stability for 1-Step Adams-Moulton
more facts  3. A method has order \( p \) \( \Longleftrightarrow \) \( \phi(z) = e^z + O(z^{p+1}) \)

Examples:
\[
e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots
\]

Euler \( \phi = 1 + z \) \( \text{order 1} \)

backward Euler \( \phi = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \) \( \text{order 1} \)

matches to have \( \phi = 1 + z + \frac{z^2}{2} \) \( \text{order 2} \)

4. How do you find the region of stability for given \( \phi \)?

Usually numerically:

pick some \( e^{i\theta} \)

evaluate \( \phi(h e^{i\theta}) \)

for increasing \( h \),

until \( |\phi(h e^{i\theta})| = 1 \)

Do this for many \( \theta \), and you get a polar coordinate representation of the boundary curve.

5. In the case of a system, the test equation is

\[
\frac{\vec{y}'}{\vec{y}} = A \vec{y}
\]

\( A = \) constant matrix.

The stability condition is then

"the method is stable if all eigenvalues of \( e^{hA} \) are \( \leq 1 \) in magnitude."

(3-2-6)
Stability For Multistep Methods

A general k-step method has the form
\[ y_{n+1} = \alpha_1 y_n + \alpha_2 y_{n-1} + \ldots + \alpha_k y_{n-k} + h \left[ \beta_0 f_{n+1} + \beta_1 f_n + \ldots + \beta_k f_{n-k} \right] \]

It is \underline{explicit} if \( \beta_0 = 0 \), \underline{implicit} if \( \beta_0 \neq 0 \)

Fact: If you apply a k-step method to the test equation, you get
\[ y_{n+1} = \phi_1(h\lambda) y_n + \ldots + \phi_k(h\lambda) y_{n-k} \]
where \( \phi_i(z) \) is a polynomial if the method is explicit, a rational function if the method is implicit.

\underline{Sideline Difference equations}

Recall the solution of \( y'' = y' + y \)
We guess \( y = e^{\lambda t} \), substitute that in, and get
\[ \lambda^2 = \lambda + 1 \]
Solutions are \( \lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2} \), and the general solution of the DE is
\[ y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \]

Now consider \( y_{n+1} = y_n + y_{n-1} \) (Fibonacci equation)
we guess \( y_n = r^n \), substitute that in, and get
\[ r^2 = r + 1 \]
The general solution is \( y_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \)

Obvious: \( y_n \) remains bounded if all \( r \) are \( \leq 1 \) in magnitude.
Example: Take the most accurate explicit 2-step method

\[ y_{n+1} = -4y_n + 5y_{n-1} + h \left[ 4f_n + 2f_{n-1} \right] \]

Apply this to the test equation:

\[ y_{n+1} = -4y_n + 5y_{n-1} + h \left[ 4\lambda y_n + 2\lambda y_{n-1} \right] \]

\[ = \left( -4 + 4h\lambda \right) y_n + \left( 5 + 2h\lambda \right) y_{n-1} \]

characteristic equation:

\[ r^2 = (-4 + 4h\lambda)r + (5 + 2h\lambda) \]

To test a particular \( h\lambda \):
- write out characteristic equation
- solve it for this \( h\lambda \)
- check if all roots are \( \leq 1 \), or not.

You can use this approach as before, to map out the boundary of the region of stability, but there is something you can do by hand:

Set \( h\lambda = 0 \). That produces the reduced characteristic equation.

Facts:
- The reduced char. eq. always has solution \( r = 1 \)
- When you go back to the full char. eq., this solution behaves like \( r = 1 + h\lambda \) \( \Rightarrow \) for \( \lambda \) in left half plane, \( h \) small enough, this will be \( < 1 \) in magnitude
- The roots of polynomials vary continuously with the coefficients. If all the other roots of the reduced char. eq. are \( < 1 \) in magnitude, they will remain \( < 1 \) for small \( h\lambda \).
Summary - To check the stability of a multistep method, find the roots of the reduced char. eq. If one of them is 0, the others < 1, the method is stable for small \( h \lambda \), Re(\( \lambda \)) < 0. Otherwise, the method is unstable.

- To find the region of stability, fix \( \lambda = e^{i\theta} \). Start at \( h = 0 \) and increase \( h \). For each \( h \lambda \), solve the full char. eq. If all roots are \( \leq 1 \), this \( h \lambda \) is inside the region of stability.

Examples - (most accurate 2-step, continued)

Reduced char. eq.

\[
\begin{align*}
\lambda^2 &= -4 \lambda + 5 \\
\lambda^2 + 4 \lambda - 5 &= 0 \\
(\lambda - 1)(\lambda + 5) &= 0
\end{align*}
\]

\[
y_n = c_1 \cdot 1^n + c_2 (-5)^n
\]

\(^A\) blows up

This method is unstable.

- \( AB/AM \)

\[
y_{n+1} = y_n + h [ \ldots \text{ various } f ]
\]

Reduced char. eq.

\[
\begin{align*}
\lambda &= \lambda^{-1} \\
solutions \quad \lambda_1 = 1, \lambda_2 = \ldots = \lambda_5 = 0
\end{align*}
\]

stable for small \( h \lambda \).

You sacrifice the \( A \)-coefficients to get good stability.