Def. A point $\bar{x}$ is a fixed point of a function $g$ if $g(\bar{x}) = \bar{x}$.

Fixed point iteration is an algorithm for solving an equation of the form $g(x) = x$.

\[
\begin{align*}
x_0 & \quad \text{guess} \\
x_1 & = g(x_0) \\
x_2 & = g(x_1) \\
& \quad \vdots
\end{align*}
\]

Example. Set your calculator to radians, type in $1$, and keep hitting the $\cos$ button.

This converges to a number near $0.739$, which is the solution to $\cos x = x$.

It is obvious that if this converges, it will converge to a fixed point $\bar{x} = g(\bar{x})$. But, when does it converge?

Let \( e_n = x_n - \bar{x} \) then \( e_{n+1} = x_{n+1} - \bar{x} = g(x_n) - g(\bar{x}) = \left[ g(\bar{x}) + g'(\bar{x})(x_n - \bar{x}) + \frac{1}{2} g''(\bar{x})(x_n - \bar{x})^2 + \ldots \right] - g(\bar{x}) \)

If \( x_n - \bar{x} \) is small, then \[
|e_{n+1}| \approx |g'(\bar{x})| \cdot |e_n|
\]

Obviously, the error goes to $0$ if $|g'(\bar{x})| < 1$. 
To apply an implicit multistep method, we need to solve

\[ Y_{n+k} = h \beta_0 f(t_{n+k}, Y_{n+k}) + \text{known terms} \]

By discussion above, \( Y_{n+k} \) converges if

\[ |h \beta_0 \frac{df}{dy}| < 1. \]

We can always achieve that by taking \( h \) small enough.

The steps are

\[ \begin{align*}
P: & \quad \text{use an explicit method to estimate } Y_{n+k}^{(0)} \quad \text{(predict)} \\
E: & \quad f_{n+k} = f(t_{n+k}, Y_{n+k}^{(0)}) \quad \text{(evaluate)} \\
C: & \quad \text{do fixed point iteration on implicit method} \\
\text{repeat as needed} & \quad Y_{n+k}^{(1)} = Y_{n+k}^{(0)} + h \left[ \beta_0 f_{n+k} + \text{known terms} \right] \quad \text{(correct)}
\end{align*} \]

How often should you iterate?

There is no point in finding \( Y_{n+k} \) to very high accuracy. \( Y_{n+k} \) is only an approximation to \( y(t_{n+k}) \).

Usually, we only iterate once or twice. Keep in mind that the explicit method is not as accurate as the implicit method, but \( Y_{n+k}^{(0)} \) is still a fairly good guess.

Usually, you do a fixed number of steps.
absolute minimum: PEC
most popular: PECE
possible: PECECEC = P(EC)^2
            PECECE = P(EC)^2 e

Numerical Example \[ \begin{cases} y' = x + t \\ y(0) = 0 \end{cases} \] true solution \( y = e^t - t - 1 \)

predictor: 2-step AB \( y_{n+1} = y_n + h \left[ \frac{3}{2} f_n - \frac{1}{2} f_{n-1} \right] \)
corrector: 1-step AM \( y_{n+1} = y_n + h \left[ \frac{1}{2} f_{n+1} + \frac{1}{2} f_n \right] \)

Take \( h = 0.1 \), use \( y_{-1}, y_0 = \text{true values} \)

P: \( y_1^{(0)} = y_0 + \frac{h}{2} \left[ 3 f_0 - f_1 \right] = 0.004758 \) error \( \times 4 \cdot 10^{-4} \)
E: \( f_1^{(0)} = 0.104768 \)
C: \( y_1^{(1)} = y_0 + \frac{h}{2} \left[ f_1^{(0)} + f_0 \right] = 0.0052619 \) error \( \times 9 \cdot 10^{-5} \)
E: \( f_1^{(1)} = 0.105238 \)

true value: \( y(t_1) = 0.005170718 \) difference \( 9 \cdot 10^{-5} \)
true solution of corrector: \( y_1^{(0)} = 0.00526315 \)

Note
(1) implicit method has an error of \( \frac{1}{2} \) of explicit method
(2) one step of corrector gets us almost to true solution of the corrector
(3) the limit of corrector is not the true solution
Limitations on Step Size $h$

One thing you will find in any book on numerical ODEs is the statement that the behavior of a method can be examined by using it on the test equation: $y' = \lambda \cdot y$

solution: $y = c \cdot e^{\lambda t}$

$\lambda \in \mathbb{C}$

$\lambda \in \mathbb{R}$, $\lambda > 0$

$\lambda \in \mathbb{R}$, $\lambda < 0$

exponential growth

exponential decay
\[ \lambda = \alpha + i\beta, \alpha > 0 \]

My attempt at an explanation

Expand \( f \) in Taylor series around 0

\[ f(t, y) = f(0, 0) + \frac{\partial f}{\partial t}(0, 0) \cdot t + \frac{\partial f}{\partial y}(0, 0) \cdot y \]

This adds some fixed function to solution

Anyway:

What is important is the size of \( \frac{\partial f}{\partial y} \).

This is the Lipschitz constant \( L \), and it is also the \( \lambda \) in the test equation

\[ y' = \lambda y \]
In general, $\frac{df}{dy}$ is not constant.

Where it is large, $y$ changes rapidly
Where it is small, $y$ is smooth and slowly varying

If $y$ changes rapidly, we need to take small steps for accuracy.
If $y$ is smooth, we can take bigger steps.

For convergence of predictor-corrector, we need $h\lambda$ small.

Both conditions can be satisfied simultaneously by taking
$h\lambda$ small enough.