Math 481 HOMEWORK 4 due Thursday, April 23, 2009
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For each problem that uses Matlab, you should hand in a printout of the relevant script or function file(s), or a transcript of your interactive session (use the `diary` feature), plus whatever outputs or plots are requested. Put the problems in the proper order, and label all printouts clearly. The final output should have full accuracy (`format long`); intermediate results can be shorter, if you want.

1. Estimate the second derivative of $f(x) = \sqrt{x}$ at $x = 2$ by using the formula

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

for $h = 1$, 0.5, 0.25, and extrapolate. Remember that the error behaves like

$$\text{error} = c_2 h^2 + c_4 h^4 + c_6 h^6 + \cdots$$

Your homework should show the extrapolation table

h	numerical derivative		
1.0	******		
		******	
0.5	******		*******
		******	
0.25	******		

Hint: the number at the tip of the triangle should be fairly accurate. (10)

2. The Bessel function of order 0 is defined by the equation

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta)\, d\theta.$$

Calculate $J_0(1)$ to accuracy 10^{-10} by evaluating this integral numerically.
Hint: you can check your result by using the Matlab built-in Bessel function. (10)

3. Numerically evaluate the two-dimensional integral

$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} (2y \sin x + \cos^2 x)\, dy\, dx$$

Look at the Matlab demo from class for an example. You need to set up a function that evaluates the inner integral, and then integrate that again.
Hint: The Matlab symbolic toolbox can do this integral in closed form. You can use that to check your answer. (10)

4. Derive a quadrature rule of the form

$$\int_0^1 \sqrt{x} f(x)\, dx \approx a f(0) + b f(1/2) + c f(1).$$

Try out the formula for $f(x) = e^x$.
Hint: Set the two sides equal to each other for $f(x) = x^0$, x^1, x^2. (10)

5. The Gauss rule with n points on the interval $[-1, 1]$ is derived as follows: Let x_i, $i = 1, \ldots, n$ be the zeros of the Legendre polynomial $P_n(x)$. Find weights w_i so that the formula

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

is accurate for polynomials up to order $n-1$ (see previous problem). The rule will then in fact be accurate up to order $2n-1$. See section 4.5 in the book.

Derive the Gauss rule with 4 points. The relevant Legendre polynomial is

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

Verify that the formula is in fact correct for all powers of x up to x^7, but not for x^8.

Hint: I would recommend doing this using the Matlab symbolic toolbox. The points and weights can all be found in closed form, but the formulas are too complicated for hand calculations. (10)