**Assignment 23**

We are given that for every $\varepsilon_1 > 0$, there exists a $\delta_1 > 0$ such that for all $x$ satisfying $0 < |x - a| < \delta_1$,

$$|f(x) - b| < \varepsilon_1.$$

Similarly, for every $\varepsilon_2 > 0$, there exists a $\delta_2 > 0$ such that for all $x$ satisfying $0 < |x - a| < \delta_2$,

$$|f(x) - b| < \varepsilon_2.$$

We must prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x$ satisfying $0 < |x - a| < \delta$,

$$|f(x)g(x) - bc| < \varepsilon$$

Suppose that $\varepsilon$ is a fixed, but arbitrary, positive real number. We will now consider two cases, $b \neq 0$ and $b = 0$. First suppose $b \neq 0$. Applying Theorem 2 to $g$, choose $B > 0$ and $\delta_0 > 0$ such that if $0 < |x - a| < \delta_0$ then $|g(x)| < B$. Put $\varepsilon_1 = \frac{\varepsilon}{2B}$ and $\varepsilon_2 = \frac{\varepsilon}{2|b|}$. Let $\delta_1$ and $\delta_2$ be positive real numbers that meet the conditions described above for the values of $\varepsilon_1$ and $\varepsilon_2$ we have chosen. Put $\delta = \min\{\delta_1, \delta_2, \delta_0\}$, and suppose $x$ is a real number satisfying $0 < |x - a| < \delta$. Then $|f(x) - b| < \frac{\varepsilon}{2B}$ and $|g(x) - c| < \frac{\varepsilon}{2|b|}$ and $|g(x)| < B$. Therefore

$$\begin{aligned}
|f(x)g(x) - bc| &= |f(x)g(x) - bg(x) + bg(x) - bc| \\
&= |(f(x) - b)g(x) + b(g(x) - c)| \\
&\leq |f(x) - b||g(x)| + |b||g(x) - c| \\
&< \frac{\varepsilon}{2B}B + b\frac{\varepsilon}{2|b|} \\
&= \varepsilon.
\end{aligned}$$

Next suppose $b = 0$. Applying Theorem 2 to $g$, choose $B > 0$ and $\delta_0 > 0$ such that if $0 < |x - a| < \delta_0$ then $|g(x)| < B$. Put $\varepsilon_1 = \frac{\varepsilon}{B}$. Let $\delta_1$ be a positive real number that meets the conditions described above for the value of $\varepsilon_1$ we have chosen. Put $\delta = \min\{\delta_1, \delta_0\}$, and suppose $x$ is a real number satisfying $0 < |x - a| < \delta$. Then $|f(x) - b| < \frac{\varepsilon}{2B}$ and $|g(x)| < B$. Therefore

$$\begin{aligned} |f(x)g(x) - bc| &= |f(x)g(x) - bg(x) + bg(x) - bc| \\ &= |(f(x) - b)g(x) + b(g(x) - c)| \\ &\leq |f(x) - b||g(x)| + |b||g(x) - c| \\ &< \frac{\varepsilon}{B}B \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon$ was an arbitrary positive real number, we have proved this result for all $\varepsilon > 0$, as required. $\square$