

ASSIGNMENT 20—ASSIGNED 3/7, DUE 3/10

Definition: Let f be a real-valued function defined for all real numbers. We say that $\lim_{x \rightarrow a} f(x) = b$ provided that for every positive real number ε there exists a positive real number δ such that for all real numbers x , if $0 < |x - a| < \delta$ then $|f(x) - b| < \varepsilon$.

Remark 1: If f is only defined for a subset of the real numbers, the definition is slightly more complicated. In place of “for all real numbers x ” it becomes “for all real numbers x in the domain of f .”

Remark 2: The reason for $0 < |x - a|$ is so that the ε estimate does not have to hold when $x = a$. This allows a function to have a limit at a that is not equal to the value of the function at a . This is important when the function is discontinuous at a . We will have an example later.

Remark 3: Notice that we do not define the meaning of the mathematical expression $\lim_{x \rightarrow a} f(x)$. Instead we define the meaning of the statement $\lim_{x \rightarrow a} f(x) = b$. This statement is a three-place predicate $L(f, a, b)$ of a function and two real numbers. That is, we have

$$\begin{aligned} L(f, a, b) &\equiv \lim_{x \rightarrow a} f(x) = b \\ &\equiv (\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R}) 0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon. \end{aligned}$$

Notice that in the statement $\lim_{x \rightarrow a} f(x) = b$, the x is a bound variable (bound by the limit symbol), so x does not appear in $L(f, a, b)$. Similarly, in the second line above, ε , δ , and x are bound variables (bound by quantifiers), so these do not appear in $L(f, a, b)$.

Definition: We say that $\lim_{x \rightarrow a} f(x)$ *exists* if

$$(\exists b \in \mathbb{R}) \lim_{x \rightarrow a} f(x) = b,$$

or, in more detail,

$$(\exists b \in \mathbb{R})(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R}) 0 < |x - a| < \delta \implies |f(x) - b| < \varepsilon.$$

Definition: Let a_n be a real number for each $n \in \mathbb{N}$. (This is called a *sequence* of real numbers.) We say that $\lim_{n \rightarrow \infty} a_n = b$ if

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists k \in \mathbb{N})(\forall n \in \mathbb{N}) n > k \implies |a_n - b| < \varepsilon.$$

ASSIGNMENT

1. By finding the negation of the statement “ $\lim_{x \rightarrow a} f(x)$ exists,” find the meaning of the statement $\lim_{x \rightarrow a} f(x)$ does not exist.” Write this statement in logical symbols and in English. (Your statement should have four quantifiers.)
2. Prove that $\lim_{x \rightarrow 3} 2x + 5 = 11$. (The task of estimating $|f(x) - b|$ for this problem is extremely simple, because the exact value can already be put in the form $k|x - a|$, without the need for an assumption that $|x - a| < \ell$ for some ℓ . Therefore your definition of δ will also be very simple. It does not require taking the minimum of two values, as in the x^2 example.)
3. Prove that $\lim_{n \rightarrow \infty} 1/n = 0$. (Apply the definition, above, for the limit of a sequence, with $a_n = 1/n$. You can use the fact that for any real number, there is an integer that is larger. You will need to use this for the real number $1/\varepsilon$.)