

ASSIGNMENT 19 (ASSIGNED 3/5, DUE 3/7)

Sample problem: Use the estimate $|x^2 - 4| < 5|x - 2|$ when $|x - 2| < 1$ to prove the following statement:

for every positive real number ε (epsilon) there exists a positive real number δ (delta) such that for all real numbers x , if $|x - 2| < \delta$ then $|x^2 - 4| < \varepsilon$.

Discussion: The first quantifier of the statement to be proved is $\forall \varepsilon \in \mathbb{R}^+$. (\mathbb{R}^+ is the set of positive real numbers.) As with any statement that begins with “for all,” the proof will begin by supposing that ε is an element of the specified set \mathbb{R}^+ . Having written that in the proof, we can “cross out” the \forall and go on to the next quantifier, $\exists \delta \in \mathbb{R}^+$. As usual, we respond to the existence quantifier by posing a construction problem: Find a positive real number δ such that for all x , if $|x - 2| < \delta$ then $|x^2 - 4| < \varepsilon$. But from the given estimate, we can make $|x^2 - 4| < \varepsilon$ if we make $5|x - 2| < \varepsilon$ (and also make $|x - 2| < 1$, so that the estimate is valid). In other words we have to make $|x - 2|$ less than both $\varepsilon/5$ and 1. But the only control over $|x - 2|$ that we are allowed is to make it less than some number δ (which we are trying to find). So we have to choose δ to be less than both $\varepsilon/5$ and 1. The easiest way to do this is to choose $\delta = \min\{\varepsilon/5, 1\}$. Now we have solved the construction problem. It only remains to re-arrange the pieces into a proof. Notice that in the proof that follows, we respond to the first quantifier \forall in the first sentence, the second quantifier \exists in the second sentence, the third quantifier \forall in the third sentence, the hypothesis of the if-then statement in the fourth sentence, and the conclusion of the if-then statement in the rest of the proof.

Proof: Suppose that ε is a positive real number. Put $\delta = \min\{\varepsilon/5, 1\}$. Suppose that x is a real number. Suppose that $|x - 2| < \delta$. Then $|x - 2| < 1$ and also $|x - 2| < \varepsilon/5$. Since $|x - s| < 1$, we know (from the previously proved estimate) that $|x^2 - 4| < 5|x - 2|$. Since $|x - 2| < \varepsilon/5$, we have $|x^2 - 4| < 5(\varepsilon/5)$. Therefore $|x^2 - 4| < \varepsilon$, as claimed. \square

Here is a shorter form of the proof, which assumes that everything is understood to be a real number and that the reader understands the consequences of the definition of δ (in particular, that $\delta \leq 1$ and $\delta \leq \varepsilon/5$).

Proof. Suppose $\varepsilon > 0$, put $\delta = \min\{\varepsilon/5, 1\}$, and suppose x satisfies $|x - 2| < \delta$. Since $|x - 2| < 1$, the previously proved estimate gives

$$|x^2 - 4| < 5|x - 2| < 5\delta \leq 5\frac{\varepsilon}{5} = \varepsilon.$$

Therefore $|x^2 - 4| < \varepsilon$ as claimed. \square

Be sure that you understand why each $<$ sign and \leq sign is true, and why sometimes one and sometimes the other is used. Notice that in a “chain of inequalities” like this, as long as they all “go in the same direction,” if one inequality is strict then the inequality between the first and last elements is strict.

ASSIGNMENT:

1. Use the previously proved estimate for $x^3 - 8$ when x is near 2 to prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all x , if $|x - 2| < \delta$ then $|x^3 - 8| < \varepsilon$.
2. Let $a_n = (2n + 1)/(n - 3)$ for $n = 4, 5, 6, \dots$ (Here n takes only integer values, but a_n is a real number.) Then it can be shown that

$$(1) \quad (\forall n \in \mathbb{N})(n > 24) \implies |a_n - 2| < \frac{8}{n}.$$

(I am not assigning this as homework, but it would be a good idea to do it for your own practice.) Now consider the following problem: Use the estimate (1) to prove the statement

$$(2) \quad (\forall \varepsilon \in \mathbb{R}^+)(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n > k) \implies |a_n - 2| < \varepsilon.$$

Solve this problem by completing the following steps. Label your work with letters (a), (b), and so on. Except for part (d), your work should be written in complete sentences.

- a. Write (2) as a sentence in English.
- b. Write the first sentence of the proof, in response to the first quantifier in (1).
- c. Write the construction problem suggested by the second quantifier in (2).
- d. Solve the construction problem using (1). It may help to realize that k plays a role similar to δ in the first problem, but one has $n > k$ in place of $|x - 2| < 1$, so the inequality signs are reversed. In the first problem we think of x as a real number near 2 with δ telling how small the distance is in order to satisfy the ε inequality. In this problem we think of n as an integer “near infinity” (that is, large), and k tells and k tells how large n has to be in order to satisfy

the ε inequality. This part of your work does not need to be written in sentence form.

- e. Write the second sentence of your proof based on the result of part (d). This is your response to the second quantifier in (2).
- f. Write the third sentence of your proof based on the third quantifier in (2).
- g. Write the fourth sentence of your proof based on the hypothesis of the conditional statement in (2).
- h. Write the remainder of your proof showing that the conclusion of the conditional statement in (2) is true under the conditions that are already described.
- i. Rewrite the entire proof as it should appear if the problem were only to prove the theorem (and the previous steps were not laid out for you).