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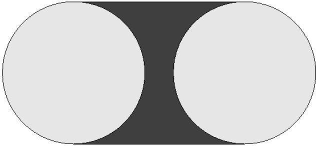

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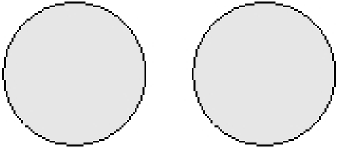

Partition the vertices in half and delete edges inside each part.

Results on triangles

So, the maximum number of changes required to remove triangles from n-vertex graph G is

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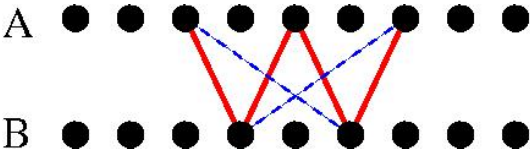
How many edge-deletions plus edge-additions are necessary to ensure that G has no copy of "W" as an induced subgraph?

An induced "W" (edges in red, nonedges in blue):



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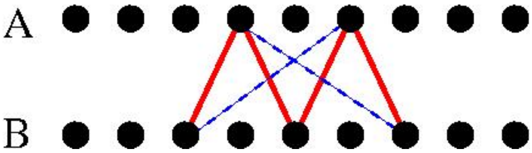

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The question relates to consensus trees. Two trees are comparable if a corresponding bipartite graph has no induced W or M.

Bipartite graph editing settled asymptotically

Szemerédi's Regularity Lemma gives:

Theorem (Axenovich-M. 2006)

Let H be any fixed bipartite graph that is neither empty nor complete. The number of edge-operations necessary to remove all induced copies of H from a random $N \times N$ bipartite graph is at least $(1/2)N^2 - o(N^2)$, with high probability.

Clearly, $(1/2)N^2$ edge operations suffices.

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The more general case is not so easy to settle, even asymptotically.

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A HEREDITARY PROPERTY is one that is preserved under vertex-deletion. Example: $\mathrm{Forb}(K_{3,3})$, no induced copy of $K_{3,3}$.

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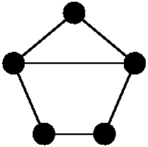

A 5-cycle as a subgraph, but no INDUCED 5-cycle.

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For the rest of this talk, all of our hereditary properties are principal; i.e.,

$$\mathcal{H} = \mathrm{Forb}(H), \text{ for some graph } H.$$

A useful parameter

Let $\mathcal{H} = \mathrm{Forb}(H)$ and let $a, c \in \mathbb{N}$ have the property that $V(H)$ cannot be partitioned into a set of a independent sets and c cliques.

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Moreover, this holds with equality if H is self-complementary.

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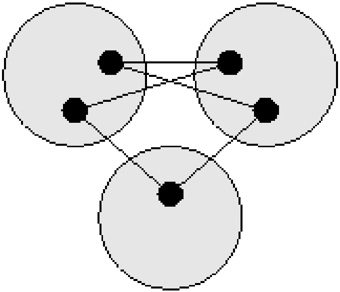

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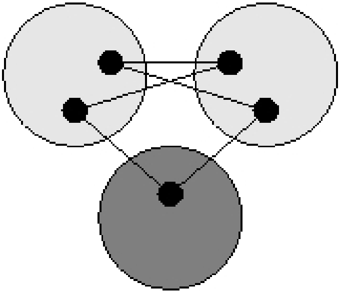

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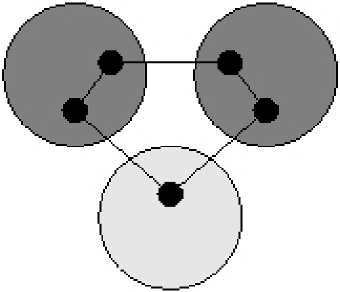

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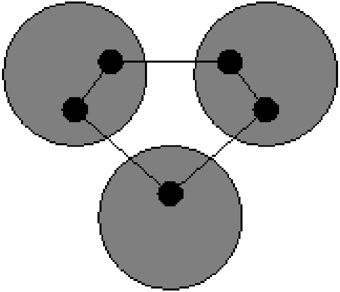

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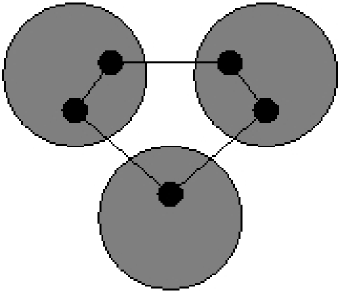

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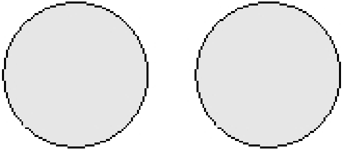

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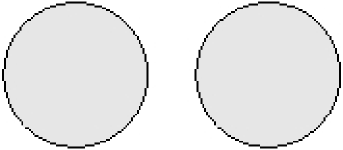

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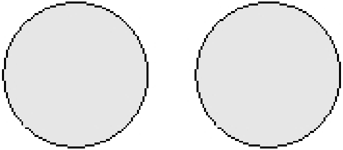

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Since $\chi_{\mathrm{B}}(C_5) = 3$, and C_5 is self-complementary, the theorem gives

$$\mathrm{Dist}(n, \mathrm{Forb}(H)) = \frac{1}{2(\chi_{\mathrm{B}}(H) - 1)}\binom{n}{2} - o(n^2).$$

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We denote

$$d^*(\mathcal{H}) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \mathrm{Dist}(n, \mathcal{H})\binom{n}{2}^{-1}.$$

Understanding d^*

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For every hereditary property, $\mathcal{H}$, there exists a $p^* = p^*(\mathcal{H}) \in [0,1]$ such that

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(The expression $\mathrm{Dist}\left(G_{n,p^*}, \mathcal{H}\right)$ is tightly concentrated about the mean, so the right-hand side is well-defined.)

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Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0,1]$, if

$$g_{\mathcal{H}}(p) = \lim_{n\to\infty} \mathrm{Dist}\left(G_{n,p}, \mathcal{H}\right) \binom{n}{2}^{-1}$$

then it is also true that

$$g_{\mathcal{H}}(p) = \lim_{n\to\infty} \max\left\{\mathrm{Dist}\left(G, \mathcal{H}\right) : |V(G)| = n, |E(G)| = p\binom{n}{2}\right\} \binom{n}{2}^{-1}.$$

Understanding d^*

Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0,1]$, if

$$g_{\mathcal{H}}(p) = \lim_{n\to\infty} \mathrm{Dist}\left(G_{n,p},\mathcal{H}\right)\binom{n}{2}^{-1}$$

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Roughly, the hardest density-p graph to edit is the random graph.

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- For any rational $r \in [0,1]$, there is an H, such that $p^*(\mathrm{Forb}(H)) = r$.

Theorem (Balogh-M., 2008)

$$p^*(\mathrm{Forb}(K_a + E_b)) = \frac{a-1}{a+b-1} \qquad d^*(\mathrm{Forb}(K_a + E_b)) = \frac{1}{a+b-1}.$$

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- There is an irrational $q \in [0,1]$ and an H, such that $p^*(\mathrm{Forb}(H)) = q$.

Theorem (Balogh-M., 2008)

$$p^*(\mathrm{Forb}(K_{3,3})) = \sqrt{2}-1 \qquad d^*(\mathrm{Forb}(K_{3,3})) = 3-2\sqrt{2}.$$

Weighted Turán lemma

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Lemma (Balogh-M., 2008)

Let $a \geq 2$ and K, $|V(K)| = k$ be a graph with edges colored **BLACK**, WHITE *and GRAY, with the property that any set A of a vertices has at least one of the following conditions:*

1. *A contains at least one* WHITE *edge;*
2. *A contains a spanning subgraph of* **BLACK** *edges.*

With $\mathrm{E}\mathbf{W}(K)$ denoting the white edges and $\mathrm{E}\mathbf{B}(K)$ the black edges,

$$(a-1)|\mathrm{E}\mathbf{W}(K)| + |\mathrm{E}\mathbf{B}(K)| \geq \left\lceil \frac{k}{2}(k-a+1) \right\rceil .$$

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If we only apply (1) and not (2), then, with a fixed, Turán's theorem is, asymptotically, a consequence.

Future Work

- $g_{\mathcal{H}}(p)$ is a function of $\mathcal{H}$ and is derived from the random graphs $G_{n,p}$. What other functions of $\mathcal{H}$ have similar properties? The following function has been studied and shares some of the properties of $g_{\mathcal{H}}(p)$:

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What are the axioms that will define such metrics as being "good"?

- Graph limits are new but well-studied. The limits of graphs are functions of a measure space on $[0,1]^2$. Balázs Szegedy reports that limits of hereditary properties can be identified. Graph limits and graph metrics may share a common theoretical bond.

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Conjecture

Fix $p_0 \in [0,1]$ and let $H \sim G(n_0, p_0)$ with $\mathcal{H} = \mathrm{Forb}(H)$. Then,

$$g_{\mathcal{H}}(p) \sim \frac{2\log_2 n_0}{n_0} \min\left\{ \frac{p}{\log_2 \frac{1}{1-p_0}}, \frac{1-p}{\log_2 \frac{1}{p_0}} \right\}.$$

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Counterintuitive because $\frac{\log(1-p_0)}{\log(p_0(1-p_0))} = p_0$ if and only if $p_0 \in \{0, 1/2, 1\}$.