

- 1 Extremal graph theory
- 2 Edit Distance Origins
- 3 Definitions related to Edit Distance
- 4 Hereditary Properties
- 5 Binary Chromatic Number
- 6 The Edit Distance Function
- 7 Weighted Turán lemma
- 8 Future Work
- 9 Thanks!

A classical extremal problem

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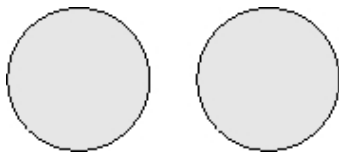
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Partition the vertices in half and delete edges inside each part.

Results on triangles

So, the maximum number of changes required to remove triangles from n -vertex graph G is

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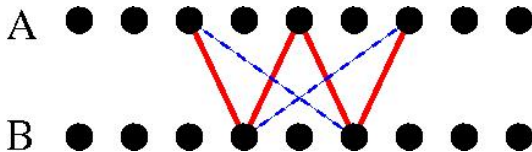
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How many edge-deletions plus edge-additions are necessary to ensure that G has no copy of “ W ” as an induced subgraph?

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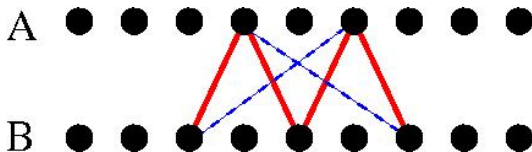
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How many edge-deletions plus edge-additions are necessary to ensure that G has no copy of “ W or M ” as an induced subgraph?

The question relates to consensus trees. Two trees are comparable if a corresponding bipartite graph has no induced W or M .

Szemerédi's Regularity Lemma gives:

Theorem (Axenovich-M. 2006)

Let H be any fixed bipartite graph that is neither empty nor complete. The number of edge-operations necessary to remove all induced copies of H from a random $N \times N$ bipartite graph is at least $(1/2)N^2 - o(N^2)$, with high probability.

Clearly, $(1/2)N^2$ edge operations suffices.

Bipartite graph editing settled asymptotically

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Clearly, $(1/2)N^2$ edge operations suffices.

The more general case is not so easy to settle, even asymptotically.

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A **HEREDITARY PROPERTY** is one that is preserved under vertex-deletion. Example: $\text{Forb}(K_{3,3})$, no induced copy of $K_{3,3}$.

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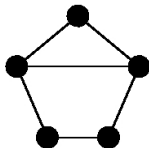
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A 5-cycle as a subgraph, but no **INDUCED** 5-cycle.

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For the rest of this talk, all of our hereditary properties are principal; i.e.,

$$\mathcal{H} = \text{Forb}(H), \text{ for some graph } H.$$

A useful parameter

Let $\mathcal{H} = \text{Forb}(H)$ and let $a, c \in \mathbb{N}$ have the property that $V(H)$ cannot be partitioned into a set of a independent sets and c cliques.

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Moreover, this holds with equality if H is self-complementary.

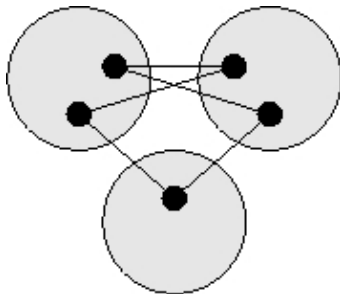
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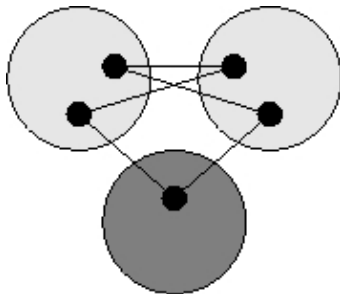


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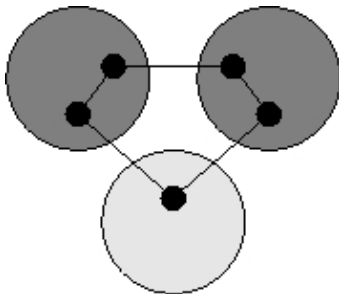


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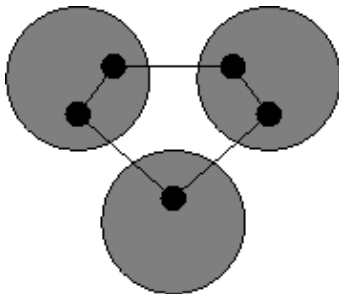


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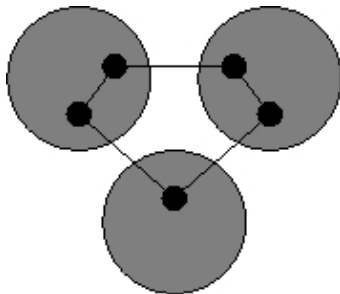


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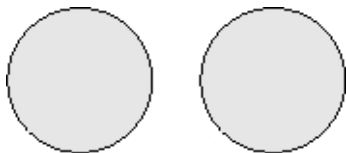
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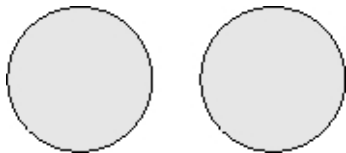


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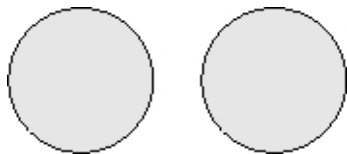
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Edit distance for C_5

Since $\chi_B(C_5) = 3$, and C_5 is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(H)) = \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$

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$$\lim_{n \rightarrow \infty} \text{Dist}(n, \text{Forb}(H)) \binom{n}{2}^{-1} = \frac{1}{4} .$$

We denote

$$d^*(\mathcal{H}) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \text{Dist}(n, \mathcal{H}) \binom{n}{2}^{-1} .$$

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For every hereditary property, \mathcal{H} , there exists a $p^ = p^*(\mathcal{H}) \in [0, 1]$ such that*

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(The expression $\text{Dist}(G_{n,p^*}, \mathcal{H})$ is tightly concentrated about the mean, so the right-hand side is well-defined.)

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Roughly, the hardest density- p graph to edit is the random graph.

The Edit Distance Function

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Theorem (Balogh-M., 2008)

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- There is an irrational $q \in [0, 1]$ and an H , such that $p^*(\text{Forb}(H)) = q$.

Theorem (Balogh-M., 2008)

$$p^*(\text{Forb}(K_{3,3})) = \sqrt{2} - 1 \quad d^*(\text{Forb}(K_{3,3})) = 3 - 2\sqrt{2}.$$

Weighted Turán lemma

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Lemma (Balogh-M., 2008)

Let $a \geq 2$ and K , $|V(K)| = k$ be a graph with edges colored **BLACK**, **WHITE** and **GRAY**, with the property that any set A of a vertices has at least one of the following conditions:

- 1 A contains at least one **WHITE** edge;
- 2 A contains a spanning subgraph of **BLACK** edges.

With $E^W(K)$ denoting the white edges and $E^B(K)$ the black edges,

$$(a - 1)|E^W(K)| + |E^B(K)| \geq \left\lceil \frac{k}{2}(k - a + 1) \right\rceil.$$

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With $\text{EW}(K)$ denoting the white edges and $\text{EB}(K)$ the black edges,

$$(a - 1)|\text{EW}(K)| + |\text{EB}(K)| \geq \left\lceil \frac{k}{2}(k - a + 1) \right\rceil.$$

If we only apply (1) and not (2), then, with a fixed, Turán's theorem is, asymptotically, a consequence.

Future Work

- $g_{\mathcal{H}}(p)$ is a function of \mathcal{H} and is derived from the random graphs $G_{n,p}$. What other functions of \mathcal{H} have similar properties? The following function has been studied and shares some of the properties of $g_{\mathcal{H}}(p)$:

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What are the axioms that will define such metrics as being “good”?
- Graph limits are new but well-studied. The limits of graphs are functions of a measure space on $[0, 1]^2$. Balázs Szegedy reports that limits of hereditary properties can be identified. Graph limits and graph metrics may share a common theoretical bond.

Conjecture

Fix $p_0 \in [0, 1]$ and let $H \sim G(n_0, p_0)$ with $\mathcal{H} = \text{Forb}(H)$. Then,

$$g_{\mathcal{H}}(p) \sim \frac{2 \log_2 n_0}{n_0} \min \left\{ \frac{p}{\log_2 \frac{1}{1-p_0}}, \frac{1-p}{\log_2 \frac{1}{p_0}} \right\}.$$

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Counterintuitive because $\frac{\log(1-p_0)}{\log(p_0(1-p_0))} = p_0$ if and only if $p_0 \in \{0, 1/2, 1\}$.