MATH 510 SPRING 2003 FINAL

No more than 5 answers will be graded: each worth 6 points.

(1) Let A be a diagonalizable $n \times n$ matrix with complex coefficients. Suppose that A has rank r. Show that there are 2^r distinct matrices B such that $B^2 = A$.

(2) Let A be a real $n \times n$ matrix, and let A have n real eigenvalues. Show that such A is normal if and only if it is symmetric.

(3) Show that a normal matrix A in $\mathrm{M}_n(\mathbb{C})$ is unitary if and only if all its eigenvalues are of the form $e^{i\theta}$ for a real number θ.

(4) Show that two Hermitian matrices are similar if and only if they are unitarily similar.

(5) Let A be a rectangular $m \times n$ complex matrix. Show that AA^* is positive semidefinite.

(6) Show that each invertible matrix A in $\mathrm{M}_n(\mathbb{C})$ may be expressed in the form $A = Pe^{iQ}$ with positive definite matrices P and Q. [*Hint*: recall that for $B = W\Phi W^*$ with unitary W and $\Phi = \mathrm{diag}(p_1, \ldots, p_n)$, the matrix $e^B = W\mathrm{diag}(e^{p_1}, \ldots, e^{p_n})W^*$.]

(7) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Is there a 3×3 unitary matrix U such that $AU = B$? If your answer is "yes," exhibit such a matrix U. If your answer is "no," explain clearly why no such matrix exists. [*Hint*: consider polar decompositions.]

(8) Let A be a 3×3 complex matrix with $A^2 = A$. Determine all possible Jordan canonical forms for A.

(9) Let A and B be 3×3 complex matrices, of equal rank, with $A^3 = 0$ and $B^2 = 0$. Show that A and B are similar.