(1) Suppose that a function \( f : A \to B \) has non-empty domain. Show that there is a function \( g : B \to A \) with \( fgf = f \).

(2) Let \( A \) be a set. Prove that the direct power \( A^n \) is isomorphic to \( \text{Set}(\{1, 2, \ldots, n\}, A) \) for each positive integer \( n \).

(3) Let \( \alpha \) be a reflexive and transitive relation on a set \( A \). Define a relation \( \beta \) on \( A \) by
\[ x \beta y \iff (x \alpha y \text{ and } y \alpha x). \]
(a) Prove that \( \beta \) is an equivalence relation on \( A \).
(b) Prove that \( x \beta y \Rightarrow x \alpha y \) yields a well-defined order relation on the quotient \( A^\beta \).

(4) An element \( x \) of a monoid \( M \) is invertible if and only if its image \( R_x : M \to M \) under the right regular representation \( R : M \to M^M \) is an invertible function.
(a) Show that the set \( M^* \) of invertible elements of \( M \) forms a submonoid of \( M \).
(b) Prove \( x \in M^* \Rightarrow \exists y \in M. xy = 1_M \).

(5) If \( A \) and \( B \) are finite subgroups of a group \( G \), prove that
\[ |AB| \cdot |A \cap B| = |A| \cdot |B|. \]