MATH 504 FALL 2003 PRACTICE FINAL

(1) Prove or disprove the following statement for sets $A$, $B$, and $C$: 
   If $A \cup B$ is isomorphic to $A \cup C$, then $B$ is isomorphic to $C$.

(2) Express the poset $(2\{a,b,c\}, \subseteq)$ as an intersection of linear orders.

(3) Let $(X, G)$ be a transitive $G$-set for a group $G$. Prove that the following are equivalent:
   (a) $\forall x \in X, \forall g \in G, xg = x \Rightarrow g = 1$;
   (b) $\exists x \in X. \forall g \in G - \{1\}, xg \neq x$.

(4) Compute the size of each (group) conjugacy class of the group $A_5$.

(5) Let $e$ be an element of a group $(G, \cdot)$.
   (a) Show that the set $G$ forms a group under the multiplication
      $$*: G \times G \to G; (x, y) \mapsto xe^{-1}y.$$      
   (b) Show that the group $(G, *)$ with multiplication given in (a) is isomorphic with the original group structure $(G, \cdot)$ on $G$.

(6) Let $G$ be a 2-transitive permutation group on a set $X$ of size $n$. 
    Prove that $|G|$ is divisible by $n(n - 1)/2$.

(7) Let $A$, $B$, and $C$ be subgroups of a group $G$, with $A \subseteq C \subseteq AB$. 
    Show that $C = A(B \cap C)$.

(8) Show that a group $G$ of order 135 cannot be simple. [You may use Sylow’s Theorems, if you carefully quote which ones you use.]