MATH 301B SPRING 2010 PRACTICE TEST #2

Write clearly, on separate paper. All questions carry equal weight. You will receive credit for your three best answers.

(1) Set

\[ G = \left\{ \begin{bmatrix} p & q \\ r & s \end{bmatrix} \mid p, q, s \in \mathbb{Z}, \ r \in 3\mathbb{Z}, \ ps - qr = 1 \right\}. \]

Show that \( G \) is a subgroup of the group of invertible \( 2 \times 2 \) real matrices under (the usual) matrix multiplication.

(2) Let \((G, \cdot, e)\) be a finite group, and let \( p \) be an odd prime number. Consider the equation

\[ x^p = e \]

for an element \( x \) of \( G \). Show that the number of solutions \( x \) in \( G \) is odd.

(3) Let \( M \) and \( N \) be normal subgroups of a group \( G \). Show that \( MN \) is a normal subgroup of \( G \).

(4) Let \( d_k d_{k-1} \ldots d_1 d_0 \) be the decimal expansion of a positive integer \( n \), so that

\[ n = \sum_{j=0}^{k} d_j 10^j \]

with \( 0 \leq d_j < 10 \). Show that 9 divides \( n \) if and only if

\[ d_k + d_{k-1} + \ldots + d_1 + d_0 \equiv 0 \mod 9. \]