1. Let $(A, +, 0)$ be an abelian group. On the direct product $\mathbb{Z} \times A$, define a componentwise group structure and a multiplication

$$(m, a) \cdot (n, b) = (mn, mb + na).$$

(a) Show that $(\mathbb{Z} \times A, +, \cdot)$ forms a commutative ring.
(b) Is $(\mathbb{Z} \times A, +, \cdot)$ unital? Justify your answer.

2. Suppose that $x^2 = x$ for each element $x$ of a ring $R$.
   (a) Show that $R$ has characteristic 2.
   (b) Show that $R$ is commutative.

3. Let $u$ be a non-zero element of a simple, commutative, unital ring $R$. Show that $u$ is a unit of the monoid $(R, \cdot, 1)$.

4. Let $G$ be the group of invertible matrices in the ring $(\mathbb{Z}/2)^2$ of $2 \times 2$ matrices over the ring of integers modulo 2. Determine $|G|$.

5. For natural numbers $a$ and $b$, show that $a\mathbb{Z} + b\mathbb{Z} = \text{gcd}(a, b)\mathbb{Z}$. 

Write clearly. Credit is given for the best three answers.