MATH 301A FALL 2013 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your five best answers.

(1) Prove or disprove the following statement:
A function \( f : X \to Y \) is surjective if and only if there
is a function \( s : Y \to X \) such that \( f \circ s = \text{id}_Y \).

(2) For a positive integer \( d \), define \( \varphi(d) = |(\mathbb{Z}/d, \cdot, 1)^*| \).
   (a) Show that \( \varphi(p) = p - 1 \) if \( p \) is prime.
   (b) Show that \( \varphi(mn) = \varphi(m)\varphi(n) \) if \( m \) and \( n \) are coprime.

(3) Define
\[
M = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in (\mathbb{Z}/13)^2 \mid a_{12} = 0 \right\}.
\]
   (a) Show that \( M \) forms a submonoid of \( (\mathbb{Z}/13)^2, \cdot, I_2) \).
   (b) Determine the order \( |M^*| \) of the group \( M^* \) of units of \( M \).

(4) Suppose that \( H \) and \( K \) are subgroups of a group \( G \).
   (a) Let \( L \) be the intersection of all the subgroups of \( G \) that
       contain both \( H \) and \( K \). Show that \( L \) is a subgroup of \( G \).
   (b) Give an example to show that \( H \cup K \) need not be a sub-
       group of \( G \).

(5) Let \( x \) and \( y \) be elements of a group \( G \). Let \( x \) have finite order
    \( a \), and \( y \) have finite order \( b \), with \( \gcd(a, b) = 1 \).
    (a) Show that \( xy \) has order \( ab \) if \( xy = yx \).
    (b) Give an example where \( xy \neq yx \) and \( xy \) does not have
        order \( ab \).

(6) Prove the identity
\[
\sum_{r=0}^{n/2} \binom{n}{2r} = \sum_{r=1}^{n/2} \binom{n}{2r-1}
\]
for even natural numbers \( n \).