1. Introduction

Von Neumann analysis is a widely used method to study how an initial wave is propagated with certain numerical schemes for a linear wave equation or heat equation. It is the most straight-forward analysis method although only limited to linear model equations.

With the new development of compact higher order method such as discontinuous Galerkin method, spectral volume method, spectral difference method, etc. To analyze the properties of this type of schemes, for 1D case or 2D quadrilateral or triangular meshes, it is necessary to extend this traditional analysis method to these higher order schemes.

The current presentation focuses on linear wave equation, starting with the simplest case, the 1D finite difference upwind scheme, then extending to 1D higher order scheme analysis, 2D quadrilateral mesh and finally 2D triangular mesh. All the analysis here still deals with uniform mesh.

2. Basic information about Von Neumann analysis – 1D upwind scheme

1D wave equation,

\[ u_t + au_x = 0 \]

The exact solution is \( u(t, x) = f(x - at) \), with \( u(0, x) = f(x) \).

If the initial condition is a harmonic wave \( f(x) = e^{ikx} \), then the exact solution is \( u(t, x) = e^{ik(x-at)} \).

For wave propagation properties, we need to look at the derivative of the solution, and the derivative of the exact solution is \( \frac{\partial u}{\partial t} = -ikau \).

Now replace the spatial derivative with upwind scheme,

\[ \frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \Rightarrow u_t = -a \frac{u_i - u_{i+1}}{\Delta x} \]

Assuming \( a > 0 \), and substitute in the initial wave,

\[ \Rightarrow u_t = -a \frac{1-e^{-ik\Delta x}}{\Delta x} u_i = -i \frac{a}{\Delta x} \left[ \sin (k\Delta x) + i \left( \cos (k\Delta x) - 1 \right) \right] u_i \]

Then define “non-dimensional wave number” \( K = k\Delta x \)

\[ \Rightarrow u_t = -i \frac{a}{\Delta x} \left[ \sin K + i \left( \cos K - 1 \right) \right] u_i \]

Define the expression inside the square parenthesis as “Fourier footprint” \( r \) of the upwind scheme, i.e.
\[ r_{\text{upw}}(K) = \sin K + i(\cos K - 1) \]

And compare with the exact derivative at i-th grid point,
\[
\frac{\partial u_i}{\partial t} = -ikau_i = -i \frac{a}{\Delta x} K u_i
\]

Easy to see, \( r_{\text{exact}}(K) = K \)

So what kind of information about the scheme can we get from Fourier footprint?

A. The order of accuracy.

If for a certain scheme, the Fourier footprint is \( r(K) \), and \( r(K) - K = O(r^{p+1}) \), the scheme is \( p \)-th order accurate.

E.g. for the upwind scheme, \( r_{\text{upw}}(K) - K = O(K^2) \), then upwind scheme is first-order accurate.

What if the expression of \( r(K) \) is complicated and hard to do the Taylor expansion?

Choose \( K = \tilde{K} \), which is small enough, evaluate \( r(\tilde{K}) - \tilde{K} \) and \( r\left(\frac{\tilde{K}}{2}\right) - \frac{\tilde{K}}{2} \), then
\[
\frac{r(\tilde{K}) - \tilde{K}}{r\left(\frac{\tilde{K}}{2}\right) - \frac{\tilde{K}}{2}} \approx 2^{p+1} \Rightarrow p \approx \log_2 \left( \frac{r(\tilde{K}) - \tilde{K}}{r\left(\frac{\tilde{K}}{2}\right) - \frac{\tilde{K}}{2}} \right) - 1
\]

B. Dispersion property

The real part of the Fourier footprint represents the phase the wave propagated, with the exact propagation speed \( K \).

In this upwind case, there is a phase lag between the wave propagation and the exact value.

C. Dissipation Property

The imaginary part of the Fourier footprint denotes the dissipation property of the wave propagated, if the negative part is positive, it will cause the amplitude of the wave to grow exponentially, and finally make the solution blow up, and therefore, for the stability of the scheme, the imaginary part of the Fourier footprint should normally be negative. For the upwind scheme, it is easy to see it is stable.
D. CFL limit

Plot the number $-ir(K)$ in a complex coordinate, we will have a closed loop, and in the upwind case, it is just a circle with the center at (-1, 0) and the radius 1.
First, for the scheme to be stable, the loop should normally be completely on the left hand side of the complex plane. Although weak instability may be consolidated by certain time integration scheme, the CFL limit for this to happen is often very stringent, and therefore, it is desirable to have the whole loop on the left hand side of the plane, i.e., we always have negative dissipation.

Secondly, since $-ir(K)CFL$ must lie inside the stability region of the time integration scheme used (The stability region can be derived from absolute stability theory of ODEs). It is desirable to have a smaller-sized loop, which can provide higher CFL limit for the same time integration scheme.

3. Extension to 1D compact higher order schemes

For compact higher order methods, such DG, SV and SD, a common feature is that each cell carries more than one piece of information, and these pieces of information carried by each cell is called degrees of freedom (DOFs). For DG, the DOFs can be chosen in different ways, resulting a family of methods, for SV, the DOFs are chosen as average solutions at sub cells, and for SD, the DOFs are chosen as the solutions at certain points inside the cell.

To analyze these schemes, all the DOFs of a certain cell form a vector,

$$\tilde{u}_i = (u_1 \  u_2 \ldots \ u_n)^T$$

Then we still plug in the spatial discretization scheme to substitute the spatial derivative, and after some simplification, it will be in the form,
\[ \frac{\partial \bar{u}_i}{\partial t} = -\frac{a}{\Delta x} (A_{i-1}\bar{u}_{i-1} + A_{i}\bar{u}_i + A_{i+1}\bar{u}_{i+1}) \]

The initial condition is still given as a harmonic wave \( f(x) = e^{\pm ikx} \), and then,

\[ G(K) = \frac{A_{i-1}e^{-ik\Delta x} + A_{i} + A_{i+1}e^{ik\Delta x}}{i} = \frac{A_{i-1}e^{-ik} + A_{i} + A_{i+1}e^{ik}}{i} \]

With,

\[ \frac{\partial \bar{u}_i}{\partial t} = -\frac{a}{\Delta x} (A_{i-1}e^{-ik\Delta x} + A_{i} + A_{i+1}e^{ik\Delta x}) \bar{u}_i = -i \frac{a}{\Delta x} G(K) \bar{u}_i \]

And then the eigenvalues of matrix \( G(K) \) is the Fourier footprint of the scheme in question.

Clearly, there are multiple eigenvalues, but only one of them denotes the principal mode, the other modes, are substantially damped, which means in analysis, they can be ignored. Moreover, it is obvious that \( G(K) \) and \( G(K + 2\pi n) \) has the same eigenvalue, so actually all the modes can be obtained by translating the principal mode by \( 2\pi n \). This means in terms of stability, if the principal mode is stable, there is no need to worry about the stability of other modes.

In practice, only principal mode is studied. To pick out the principal mode, it is only needed to look for the eigenvalue that approximates the exact value when \( K \) is small, i.e. \( r_p(K) - K \rightarrow 0 \), as \( K \rightarrow 0 \).

Once the principal eigenvalue is chosen, one can get the same kind of properties: The order of accuracy, dispersion, dissipation properties and CFL limit, the same way as is done for the upwind case. Below are examples for several 4th order schemes.
4. Extension to 2D quadrilateral mesh
2D wave equation: \( u_t + a(\cos \alpha, \sin \alpha) \nabla u = 0 \), and the initial condition is given as a plane wave:
\[ u(0, x, y) = e^{i k(x \cos \theta + y \sin \theta)} \]. Generally, the initial wave direction and the wave propagating direction is not the same, but for the simplicity of analysis, we always let \( \theta = \alpha \).

Then again, we have the property: \( u_{t, exact} = -i \frac{a}{\Delta x} Ku \).

For 2D quad meshes, the stencil include five cells, so after plug in the spatial scheme, it should be in the form,
\[
\frac{\partial \tilde{u}_{i,j}}{\partial t} = - \frac{a}{\Delta x} \left( A_{i,j-1,j} \tilde{u}_{i,j-1,j} + A_{i,j-1,j-1} \tilde{u}_{i,j-1,j-1} + A_{i+1,j,j} \tilde{u}_{i+1,j,j} + A_{i,j+1,j} \tilde{u}_{i,j+1,j} + A_{i,j+1,j+1} \tilde{u}_{i,j+1,j+1} \right)
\]

Plug in the initial condition, it can also be written in the form,
\[
\frac{\partial \tilde{u}_{i,j}}{\partial t} = -i \frac{a}{\Delta x} G(K) \tilde{u}_{i,j}
\]

The next step is just to obtain the eigenvalue of \( G(K) \), choose the principal mode and analyze the all the properties of the scheme just as the 1D case.

5. 2D Triangular mesh analysis

The most difficult obstacle in analyzing triangular mesh is that it is hard to plug in the initial solution and relate all the DOFs in the whole stencil to the DOFs of the particular cell. Therefore, to do this, the basic building block contains 2 triangles, such that this block can be repeated and form the whole computational domain.
Then, the solution vector also contains the DOFs of the two triangles, as in,

$$\vec{u}_i = \begin{pmatrix} u_{i,1} & u_{i,2} & \cdots & u_{i,n} & u_{i,1} & \cdots & u_{i,n} \end{pmatrix}^T$$

Then plug in the spatial scheme,

$$\frac{\partial \vec{u}_{i,j}}{\partial t} = -\frac{a}{\Delta x}(A_{i-1,j} \vec{u}_{i-1,j} + A_{i,j} \vec{u}_{i-1,j} + A_{i,j} \vec{u}_{i,j} + A_{i+1,j} \vec{u}_{i+1,j} + A_{i,j} \vec{u}_{i,j+1})$$

And the expression can be manipulated to be like after plug in initial wave,

$$\frac{\partial \vec{u}_{i,j}}{\partial t} = -i \frac{a}{\Delta x} G(K) \vec{u}_{i,j}$$

After this, everything is the same as the first order case.

The following are examples of resulted dispersion and dissipation plot of 3rd order spectral volume schemes with different partition.
**Acknowledgement**

The material of this presentation mainly comes from the lecture by *Kris Van den Abeele*, during his visit to ISU in Jan – Mar 2007, more details and published papers can be found at his website: [http://mech.vub.ac.be/thermodynamics/members/Kris%20Van%20den%20Abeele.htm](http://mech.vub.ac.be/thermodynamics/members/Kris%20Van%20den%20Abeele.htm).