Here listed are all interpolation problems.

1. Let \( g_1, \ldots, g_n \in C[a, b] \) be elements of a Chebyshev system, and let \( x_1, \ldots, x_n \) be pairwise distinct data points. For any two elements \( f, g \in C[a, b] \), let \( \langle f, g \rangle := \sum_{k=1}^{n} f(x_k)g(x_k) \). Show directly that if \( \tilde{f} \in \text{span}(g_1, \ldots, g_n) \) satisfies the normal equations for the best approximation of \( f \) with respect to \( \langle \cdot, \cdot \rangle \), then \( \tilde{f} \) interpolates \( f \) at \( x_1, \ldots, x_n \).

2. Let \( f \in C_1[a, b] \), and suppose that \( x_1 \cdots x_n \) are pairwise distinct points. Show that for every \( \epsilon > 0 \), there exists a polynomial \( p \) such that \( \| f - p \|_{\infty} < \epsilon \) and, simultaneously, satisfies the interpolation conditions \( p(x_k) = f(x_k), 1 \leq k \leq n \).

3. How small must the maximal distance between two neighboring data points be to insure that the polynomial interpolant \( \tilde{p} \in P_5 \) of the exponential function on \([-1, 1] \) satisfies \( \| f - \tilde{p} \|_{\infty} \leq 5 \cdot 10^{-8} \) and \( \| f' - \tilde{p}' \|_{\infty} \leq 5 \cdot 10^{-7} \) simultaneously?

4. Determine the interpolating polynomial of degree 2 in both the Lagrange and Newton forms for the functions \( f(x) := \frac{2}{1+x^2} \) and \( f(x) := \cos(\pi x) \) using the interpolation points \( x_1 = -1, x_2 = 0, x_3 = 1 \).

5. Compute the Peano representation of the remainders for linear interpolation at \( x_1 = a, x_2 = b \) under the hypothesis that \( f \in C_2[a, b] \) and that \( f \in C_1[a, b] \), respectively. Show that \( |R_1(f; x)| \leq \max_{x \in [a, b]} |f'(x)|(b - a) \).