Problems listed below are related to topics
(a) Approximation in Pre-Hilbert spaces
(b) The Method of Least Squares

1. Let $f \in C[-1, 1]$, $f(x) = e^x$. Find the best approximations of $f$ from $P_k$, $0 \leq k \leq 2$, with respect to the norm $\| \cdot \|_2$.
   a) using the normal equations;
   b) by extension of $f$ in Legendre polynomials.

2. Let $f \in C(-\infty, +\infty)$ be periodic with $f(x) := x^2$ for $x \in [-\pi, +\pi]$.
   a) Find the Fourier series expansion of $f$ in terms of trigonometric functions, and sketch the best approximations of $f$ from $span(g_1, g_2, g_3)$ and from $span(g_1, \cdots, g_5)$.
   b) How can one use this expansion to compute the value of $\pi$, and how many terms are needed to get $\pi$ to an accuracy of $5 \cdot 10^{-k}$?

3. Using the method of least squares, find all best approximations from $span(1, e^x)$, from $P_2$ and from $P_3$ for the following data:
   
   $\{(x_i, y_i)\}_{i=1}^5 = \{(-1, 0), (-1, 1), (0, 1), (1, 2), (1, 3)\}$.

4. Consider the set $\{1/\sqrt{2} T_0, T_1, \cdots, T_{n-1}, 1/\sqrt{2} T_n\}$ of Chebyshev polynomials of the first kind. Show that they form an ONS with respect to the discrete inner product
   
   $< f, g > := \frac{1}{n} [f(x_0)g(x_0) + 2 \sum_{k=1}^{n-1} f(x_k)g(x_k) + f(x_n)g(x_n)]$

   with $x_k := \cos \left( \frac{k\pi}{n} \right)$, $0 \leq k \leq n$. 