Math 503 Final work

1. Let \( s(t) = s_i(t), \quad t \in [t_i, t_{i+1}], i = 0, \ldots, n - 1 \) be a cubic spline interpolation of the points \( \{(t_i, y_i) : i = 0 \cdots n\} \) with \( s''(t_0) = s''(t_n) = 0 \). Show that
\[
\int_{t_0}^{t_n} |s''(t)|^2 dt \leq \int_{t_0}^{t_n} |f''(t)|^2 dt
\]
for any twice continuously differentiable function \( f \) such that \( y_i = f(t_i), \quad i = 0 \cdots n \).

2. Let \( a \leq x_0 < x_1 < \cdots < x_n \leq b \) be an arbitrary partition of the interval \([a, b]\). Show that there exist unique numbers \( \gamma_0, \cdots \gamma_n \) such that
\[
\sum_{i=1}^{n} \gamma_i P(x_i) = \int_{a}^{b} P(x)dx
\]
for all polynomials \( P \) with degree(\( P \)) \( \leq n \).

3. Consider the family of semi-implicit Runge-Kutta methods
\[
\begin{align*}
k_1 &= f(y_n + \beta hk_1) \\
k_2 &= f(y_n + hk_1 + \beta hk_2) \\
y_{n+1} &= y_n + h \left[ \left( \frac{1}{2} + \beta \right) k_1 + \left( \frac{1}{2} - \beta \right) k_2 \right].
\end{align*}
\]
a) Apply this method to the problem \( \dot{y} = \lambda y, \quad y(0) = 1 \), where \( \lambda \) is a complex constant. Obtain an expression of the form \( y_{n+1} = L(h\lambda, \beta)y_n \), where \( L(h\lambda, \beta) \) is a rational function of \( h\lambda \). By comparing \( y_{n+1} \) to the exact solution \( y(t_{n+1}) = \exp(h\lambda t)y(t_n) \), under the localizing assumption \( y_n = y(t_n) \), determine the order and the principal part of the local truncation error.
b) Show that if \( \beta > 1/2 \) then the negative real axis \( \{z : Im(z) = 0, \quad Re(z) < 0\} \), is contained in the region of absolute stability of the method.