Simulated Flow

Water waves

free surface

$h$

$\vec{v}$

$\vec{g}$

solid ground

$x$

$z$
Shallow water equation

We will concentrate on the time evolution of the flow model from an initial state that consists of two semi-infinite uniform zones which are separated by a discontinuity. The objective is to determine the resulting induced wave motion as a function of the initial state. This problem is geometrically one-dimensional in that the solution only depends on one space coordinate normal to the diaphragm. In this project we will first restrict our simulation to one-dimensional motion, and the flat-bottom case without bottom friction.

\[ \partial_t u + \partial_x f(u) = 0 \]

\[ u = \begin{pmatrix} h \\ v h \end{pmatrix}, \quad f(u) = \begin{pmatrix} v h \\ v^2 h + gh^2 / 2 \end{pmatrix} \]
Eigen-structure

• The Jacobian matrix of equations is

\[ J = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 1 \\ -v^2 + gh & 2v \end{pmatrix} \]

has two distinct and real eigenvalues and complete eigenvectors

\[ \lambda_1 = v - c \quad \lambda_2 = v + c \quad c = \sqrt{gh} \]

\[ R = \begin{pmatrix} 1 & 1 \\ v - c & v + c \end{pmatrix} \quad R^{-1} = \frac{1}{2c} \begin{pmatrix} c + v & -1 \\ c - v & 1 \end{pmatrix} \]
Numerical Test 1

\[(h_0, v_0)(x) = (1 - a(x), 1)\]

\[a(x) := \begin{cases} 
0.3(\cos(\pi(x - 1)/2))^{30} & |x - 1| \leq 1 \\
1 & \text{otherwise}
\end{cases}\]

• The computation domain is \([-1, 3]\);

• The step size \(\delta x = 1/50, 1/100, 1/200\);
The computation domain is $[-10, 10]$ and the left boundary data is $(1, 1)$.
Roe’s Method

• Roe’s approach is quite pragmatic and successful. The exact solution to the linearized Riemann problem is constructed.

\[ \partial_t \hat{u} + \hat{A}(u_l, u_r) \partial_x \hat{u} = 0 \]

• The Roe’s flux function is given by

\[ F^{Roe}(u_l, u_r) = \frac{1}{2} [f(u_l) + f(u_r)] - |\hat{A}(u_l, u_r)|(u_r - u_l) \]

\[ \hat{A}(u_l, u_r) = R(u_l, u_r) \cdot |\hat{A}(u_l, u_r)| \cdot R^{-1}(u_l, u_r) \]

and \( R, R^{-1} \) are the diagonalization matrices and \( \text{Lambda} \) the diagonal eigenvalue matrix.
Test by Roe’s scheme

• Find the Roe’s matrix;

• Write a program based on the Roe’s method;

• Discuss a possible sonic entropy fix for capturing the entropy solution;

• Do the numerical test 1 & 2.
Test by LxF Scheme

• The first-order Lax-Friedrichs’ scheme

\[ u_{j}^{n+1} = \frac{u_{j+1}^{n} + u_{j-1}^{n}}{2} - \frac{\lambda}{2} [f(u_{j+1}^{n}) - f(u_{j-1}^{n})]. \]

Here \( \lambda = k/h \) is the fixed mesh ratio.

The LxF scheme has the advantage of simplicity, since no Riemann solvers are involved in its construction.