Problems listed below are related to methods for solving 2nd or higher order equations

(C1) Compute the Cauchy problem

\[ u_t = u_{xx}, \quad u_0(x) = f(x) \]

where

\[ f(x) := 1 + \sin(x) + \sin(10x). \]

Using (1) the Euler method: \[ v_j^{n+1} = (I + kD_+D_-)v_j^n; \]
(2) the leap-frog scheme \[ v_j^{n+1} = 2kD_+D_-v_j^n + v_j^{n-1}; \]
(3) The DuFort-Frankel method:

\[ v_j^{n+1} = \frac{1}{1 + 2\sigma}(2\sigma(v_{j+1}^n + v_{j-1}^n) + (1 - 2\sigma)v_j^{n-1}), \quad \sigma = k/h^2. \]


Take say \( \sigma = 0.5, N = 100 \) and plot the solution at \( t = 0, 10^{-2}, 1 \) followed by comments on your numerical results.

(C2) Consider the problem \( u_t + au_x = \eta u_{xx} \) subject to initial data in (C1). Write a program based on the explicit method

\[ v_j^{n+1} = v_j^n + k(\eta D_+D_- - aD_0)v_j^n, \quad j = 0, \cdots N, \]

for \( N = 10, 20, 40 \cdots \). Choose the time step such that

(i) \( \alpha = \frac{2\eta}{k^2} \) is a constant with \( \alpha \leq 1; \)
(ii) \( \lambda = \frac{4k}{\eta} \) is a constant with \( |\lambda| \leq 1. \)

Compare the solutions and explain the difference in their behavior.

(C3) What method could be used for the Schrödinger type equation

\[ u_t = iu_{xx}? \]

Write a program testing the equation with the initial data \( f(x) = 1 + \sin(x) + \sin(10x). \)
(C4) Define the Crank-Nicholson approximation for the Korteweg de Vries type equation

\[ u_t = u_{xxx} + au_x. \]

Prove unconditional stability and numerically test the case with \( a = 1 \) and initial data \( f(x) = 1 + \sin(x) + \sin(10x) \).

(C5) Testing your codes designed for (C1)-(C4) with discontinuous initial data

\[ u(x, 0) = \begin{cases} 
0, & 0 \leq x < \frac{2\pi}{3}, \\
1, & \frac{2\pi}{3} \leq x \leq \frac{4\pi}{3}, \\
0, & \frac{4\pi}{3} < x \leq 2\pi.
\end{cases} \]