Math 502 HW#7

Problems listed below are related to topics
(a) Iterative Methods for nonlinear problems
(b) Newton’s Methods
(c) Quasi-Newton Methods.

1. If we determine an improved approximation \( x^{(k+1)} \) of a zero from the approximation \( x^{(k)} \) by using not only the linear part, but also the terms of second order in the Taylor expansion of \( f \) about \( x^{(k)} \), the resulting method is called a Newton method of order 2. Develop this iterative method.

2. Consider the two-step Newton method

\[
y^{(k)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, \quad x^{(k+1)} = y^{(k)} - \frac{y^{(k)}}{f'(y^{(k)})},
\]

and show:

a) If the method converges, then

\[
\lim_{k \to \infty} \frac{x^{(k+1)} - \xi}{y^{(k)} - \xi} = \frac{f''(\xi)}{f'(\xi)}.
\]

b) The convergence is cubic:

\[
\lim_{k \to \infty} \frac{x^{(k+1)} - \xi}{(x^{(k)} - \xi)^3} = \frac{1}{2} \left( \frac{f''(\xi)}{f'(\xi)} \right)^2.
\]

3. Consider applying Newton’s method to

\[
F(x) = \begin{bmatrix} x_1 \\ x_2^2 + x_2 \\ e^{x_3} - 1 \end{bmatrix}
\]

(a) What is \( J(x) \) at the root \( x^* = (0, 0, 0)^\top \)?
(b) What is a Lipschitz constant on \( J(x) \) in an interval of radius \( a \) around \( x^* \)?
(c) What region of convergence for Newton method on \( F(x) \) is predicated?
Hint: the region is

\[ N(x^*, \epsilon) := \{ x \mid \|x - x^*\| \leq \epsilon \}, \]

where \( \epsilon = \min\{a, \frac{1}{2n} \} \) with \( \|J(x^*)^{-1}\| \leq \beta \) and \( J \in Lip_\gamma(N(x^*, a)) \).

4. Let

\[ F(x) = \begin{bmatrix} x_1^2 + x_2^2 - 2 \\ e^{x_1-1} + x_2^3 - 2 \end{bmatrix}, \]

which has a root \( x^* = (1, 1)^T \). With initial guess \( x_0 = (1.5, 2)^T \) apply both Newton’s method and Broyden’s method to find the above root and compare their performance.