Problems listed below are related to topics
(a) Krylov Subspace methods
(b) The Conjugate Gradient method
(c) The GMRES method

1. Let $A$ be spd and let $\{x_k\}$ be the conjugate gradient iterates. The Krylov subspace is defined as

$$ K_k = \text{span}\{r_0, Ar_0, \cdots, A^{k-1}r_0\} = \text{span}\{r_0, r_1, \cdots, r_{k-1}\}. $$

Then

(i) $r_k^\top r_l = 0$ for all $0 \leq l < k$,

(ii) $p_k \in K_k$, $p_k^\top A\xi = 0$ for $\xi \in K_{k-1}$;

(iii) $\alpha_k = \|r_{k-1}\|^2_2 / (p_k^\top Ap_k)$;

(iv) $\beta_k = \|r_{k-1}\|^2_2 / \|r_{k-2}\|^2_2$.

2. Consider the system of equations $Ax = b$ for $N = 4, 8, 16, 32$, where $h := N^{-1}$, $k = N - 1$, $A \in R^{(k^2,k^2)}$, $B = R^{(k,k)}$, and $b \in R^{k^2}$ with

$$ A = \begin{pmatrix} B_k & -I_k & 0 \\ -I_k & B_k & -I_k \\ \vdots & \ddots & \ddots \\ 0 & \cdots & -I_k \\ -I_k & B_k \end{pmatrix}, \quad B_k = \begin{pmatrix} 4 & -1 & \cdots & 0 \\ -1 & 4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & -1 & 4 \end{pmatrix}. $$

Here $I_k$ is the identity matrix in $R^{(k,k)}$, and the vector $b$ is given by

$$ b = h^2 \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^k \end{pmatrix}, \quad f^\mu = \begin{pmatrix} f^\mu_1 \\ f^\mu_2 \\ \vdots \\ f^\mu_k \end{pmatrix} $$

and $f^j_i := 5 \pi^2 \sin(2 \pi i \cdot h) \sin(\pi j \cdot h)$, $1 \leq j, i \leq k$. Compute its solution by the Conjugate Gradient method and compare the result.
with those obtained by Jacobi method, SOR (with $\omega = 1, 1.6, 1.9$). Starting with the vector $x^{(0)} = 0$, how many iterations $k := k(N)$ are needed to get $\|Ax^k - b\|_\infty \leq 10^{-3}$?

3. Let $A$ be the $100 \times 100$ tridiagonal symmetric matrix with $1, 2, \cdots, 100$ on the diagonal and $1$ on the sub- and superdiagonals, and set $b = (1, 1, \cdots, 1)^T$. Write a program that takes 100 steps of the CG to approximately solve $Ax = b$. Produce a plot with two curves on it: the computed residual norms $\|r_n\|_2$ for CG, the actual residual norms $\|b - Ax_n\|_2$ for CG. Comment on your results.

4. Consider the centered difference discretization of

$$-u'' + u' + u = 1, \quad u(0) = u(1) = 0.$$ 

Solve this problem with GMRES. Try this for meshes with $n = 50, 100, 200, \cdots$ points and mesh width $h = 1/(1+n)$. How does the performance of the iteration depend on the mesh?