2.3 Problems. 1) Show that by applying a finite number of Jacobi rotations, we can transform any matrix \( A \in \mathbb{R}^{(n,n)} \) into a Hessenberg matrix (or if \( A \) is symmetric, to a tridiagonal matrix).

2) Show that the cyclic Jacobi method converges as long as in each step of the process, we operate on an element whose absolute value exceeds \( N(A)/2n^2 \).

3) Write a computer program for the classical Jacobi method, and use it to find the eigenvalues of the matrix

\[
\begin{pmatrix}
  n & n-1 & n-2 & \cdots & 2 & 1 \\
n-1 & n-1 & n-2 & \cdots & 2 & 1 \\
n-2 & n-2 & n-2 & \cdots & 2 & 1 \\
\vdots & & & & & \\
2 & 2 & 2 & \cdots & 2 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}
\]

for \( n = 12 \).

Hint: The eigenvalues are \( \lambda \mu = \frac{1}{2} (1 - \cos \frac{2\mu - 1}{2n+1} \pi) \).

4) Estimate the eigenvalues of the matrix

\[
\begin{pmatrix}
  4.2 & 0.65 & 3.2 \\
0.65 & 6.4 & 1.6 \\
3.2 & 1.6 & 4.8
\end{pmatrix}
\]

using the method of Gerschgorin.

5) Prove the following assertion: For every eigenvalue \( \lambda \mu \) of a matrix \( A \in \mathbb{C}^{(n,n)} \) and for every matrix \( B \in \mathbb{C}^{(n,n)} \), either \( \det(\lambda \mu I - B) = 0 \), or \( \lambda \mu \in \mathbb{T} := \{ \lambda \in \mathbb{C} : \| (\lambda I - B)^{-1} \cdot (A - B) \| \geq 1 \} \).

6) Apply Problem 5 to prove the Gerschgorin Theorem. Hint: The matrix \( B \) must be chosen appropriately.

7) Prove that if \( A = (a_{\mu \nu}) \) is Hermitian, then corresponding to every diagonal element \( a_{\mu \mu} \), there exists an eigenvalue \( \lambda \) of the matrix \( A \) satisfying

\[
|\lambda - a_{\mu \mu}| \leq \left( \sum_{\nu = 1}^{n} |a_{\mu \nu}|^2 \right)^{1/2}.
\]