Chapter 2. Linear Systems of Equations

After completing all elimination steps, the components of the solution vector $x$ can be simply computed as $x_\mu = b^{(n)}_\mu/a^{(n)}_{\mu\mu}$, $\mu = 1, 2, \ldots, n$. The complexity of the Gauss-Jordan method is $T_A^R(n) = O(n^3)$ as $n \to \infty$. Although its complexity is no better than that of Gauss elimination, the Gauss-Jordan method has some advantages when the computation is to be carried out on a computer capable of parallel operations on several processors simultaneously. We do not have space here, however, to discuss this point further.

1.7 Problems. 1) Find the LR-decomposition of the matrix

$$A := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \\ 1 & 16 & 81 & 256 \end{pmatrix},$$

and use this decomposition to solve the system of equations $Ax = b$ with right-hand side $b := (3, 1, -15, -107)^T$.

2a) Let $\{a^1, a^2, \ldots, a^n\}$ be a basis for $\mathbb{R}^n$, and let $\{a^1, \ldots, a^k, \ldots, a^n\}$ be another basis which differs from the first only in that the vector $a^k$ is replaced by the vector $\tilde{a}^k$. How can one find the coordinates of a prescribed vector with respect to the second basis, given the coordinates of $\tilde{a}^k$ with respect to the first basis?

b) Consider the following situation: Suppose we want to solve a linear system of equations, but after having computed the LR-decomposition, we discover that one column in the original matrix $A$ is wrong. How can the decomposition nevertheless be used to find the correct solution? Formulate a corresponding algorithm, and apply it to the system of equations in Problem 1, where the first column of $A$ is to be replaced by $(0, 0, 6, 36)^T$.

3a) Let $a, b, c \in \mathbb{R}^n$ with $|a_\nu| \geq \sum_{\mu \neq \nu} |a_\mu|$, $a_\nu \neq 0$, $|b_\kappa| \geq \sum_{\mu \neq \kappa} |b_\mu|$ and $\nu \neq \kappa$. Suppose the vector $c$ is defined by $c_\mu := b_\mu - a_\nu a_\mu$, $1 \leq \mu \leq n$. Show that $|c_\kappa| \geq \sum_{\mu \neq \kappa} |c_\mu|$.

b) If $|a_{\mu\nu}| \geq \sum_{\nu \neq \mu} |a_{\mu\nu}|$, then the matrix $A = (a_{\mu\nu})$ is called weakly diagonally dominant. Prove that if $A$ is a weakly diagonally dominant nonsingular matrix, then Gauss elimination without pivoting can be used to compute a decomposition of the form $L \cdot R = A$.

4) In general, is the inverse of a nonsingular band matrix a band matrix? 5) Write a computer program for Gauss elimination with complete pivoting. Test your program on the example

$$a_{\mu\nu} := 1/(\mu + \nu - 1), \quad 1 \leq \mu, \nu \leq n, \quad b_\mu := 1/(n + \mu - 1), \quad 1 \leq \mu \leq n.$$

2. The Cholesky Decomposition

For general nonsingular $n \times n$ matrices, pivot the corresponding LR decomposition, certain matrices, pivoting is not needed, but it is difficult to use in practice because it requires too much computation. In this section, we show that LR decompositions of positive definite matrices can be computed without pivoting, and discuss a sparsity pattern for them.

2.1 Review of Positive Definite Matrices

Some useful properties of positive definite matrices follow from linear algebra, e.g. G. Strang (1976).

Definition. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if $x^T A x > 0$ for all vectors $x \in \mathbb{R}^n$ with $x \neq 0$.

The positive definiteness of a matrix can be characterized by the following two equivalent conditions:

(i) there exists a nonsingular matrix $W$ with $A = W^T W$ and
(ii) all principal minors $\text{det} A_{\mu\mu}$, $1 \leq \mu \leq n$, of $A$ are necessary and sufficient for the symmetric matrix $A$ to be positive definite.

Moreover, positive definite matrices have the following properties.

Let $A \in \mathbb{R}^{n \times n}$ be positive definite. Then $A$ exists, is symmetric, and is positive definite. In particular, a submatrix $A_{\mu\mu}$ of $A$ with $1 \leq \mu \leq n$ is symmetric positive definite.

2.2 The Cholesky Decomposition

In view of the matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if there exists a triangular decomposition of the form $A = L^T L$, where $L$ is upper triangular.

Theorem. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and let $A = \ell_{\mu\mu} > 0$, $1 \leq \mu \leq n$.

Proof. We proceed by induction on $n$. For $n = 1$, and so $L = L^T = (\sqrt{a_{11}})$.

Now let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Suppose the assertion holds for $n - 1$. We partition the matrix $A$ as

$$A = \begin{pmatrix} A_{nn-1} & b \\ b^T & a_{nn} \end{pmatrix}.$$