Subtraction: \( X = A - B, X := \{ x \in \mathbb{R} \mid x = a - b \text{ with } a \in A \text{ and } b \in B \} \), i.e. \( X = [A - B], A - [B] \).
Multiplication: \( X = A \cdot B, X := \{ x \in \mathbb{R} \mid x = a \cdot b \text{ with } a \in A \text{ and } b \in B \} \).
Division: \( X = A/B, \text{ if } 0 \notin B, X := \{ x \in \mathbb{R} \mid x = a/b \text{ with } a \in A \text{ and } b \in B \} \).

If we replace \( \mathbb{R} \) by the set of machine numbers, we get a corresponding set of machine intervals. In this case we can define arithmetic operations as above, except that we have to take account of roundoff, which means that the result intervals must be enlarged.

Interval arithmetic has been heavily studied, and many of the methods in numerical analysis have been built into the theory. The main problem is to develop methods such that the intervals in the calculation do not grow too much. This requires a clever combination of conventional techniques with those from interval analysis. For details, see the extensive literature and the book of R. E. Moore [1966].

In present-day computers, which are capable of carrying out millions of arithmetic operations per second, it is essential to have an effective control over roundoff errors. Interval arithmetic provides the possibility of doing this, especially since there now exist computers with special hardware for carrying out the arithmetic. Moreover, there are compilers which lead to programs which can be executed in sufficiently high precision arithmetic to provide rigorous error bounds along with the result.

3.5 Problems. 1) Determine the maximal (absolute and relative) error in \( y = \frac{x_1 x_2}{\sqrt{x_3}} \) for \( x_1 = 2.0 \pm 0.1, x_2 = 3.0 \pm 0.2, x_3 = 1.0 \pm 0.1 \) using an error analysis with differentials (cf. 3.1). Compute the condition numbers. Which variable contributes the most to the error?
2) Consider the linear system of equations
\[
\begin{align*}
    a_{11} x_1 + a_{12} x_2 &= b_1, \\
    a_{21} x_1 + a_{22} x_2 &= b_2,
\end{align*}
\]
with \( a_{\mu\nu}, b_\nu \in \mathbb{R} \).

a) Suppose the coefficients \( a_{11} = a_{22} = 1.9, a_{12} = a_{21} = -1.7 \) and the right-hand sides \( b_1 = 1.2, b_2 = 1.5 \) are subject to errors whose size is not larger than \( 5 \cdot 10^{-2} \). Find the sharpest possible bounds for the solution.

b) Consider the solution \( x = (x_1, x_2) \) as a function of the coefficients and the right-hand side:
\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \varphi(a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2).
\]
Compute the condition numbers of this problem, and give sufficient conditions for it to be well-conditioned and poorly-conditioned, respectively.
c) What are the condition numbers corresponding to the values given in part a) of this problem?

3) The condition number of a problem of the form \( y = \varphi(x) \) defined by \( \varphi : D \subset \mathbb{R}^n \to \mathbb{R}^m \) can also be determined experimentally (ignoring roundoff errors) by approximating the differential quotient

\[
\Delta_{\mu
u}(\epsilon) := \frac{\varphi(x_1, \ldots, x_{\nu-1}, x_{\nu} + \epsilon, x_{\nu+1}, \ldots, x_n) - \varphi(x)}{\epsilon}
\]

For example, to do this, we should choose \( \epsilon \) so that \(|\Delta_{\mu
u}(\epsilon) - \Delta_{\mu
u}(-\epsilon)| \ll 1\). Apply this method to the linear system of equations in 2a), and compare with the results from 2c).

4) Suppose we want to compute the product \( P_n := \prod_{\mu=1}^{n} a_{\mu} \) of real numbers \( a_{\mu} \) using the following recurrence:

\[
\begin{align*}
P_1 & := a_1, \\
P_\nu & := P_{\nu-1} \cdot a_\nu, \quad 2 \leq \nu \leq n.
\end{align*}
\]

Carry out an exact forward analysis, assuming the calculation is done with floating-point arithmetic with base \( B \) and mantissa length \( t \). Is there possibly a better way to compute the product?

5) Let \( x \) and \( y \) be vectors in \( \mathbb{R}^n \). Carry out a forward error analysis for the problem of computing the scalar product

\[
\langle x, y \rangle := \sum_{\nu=1}^{n} x_\nu \cdot y_\nu.
\]

What does the result say for \( n = 3 \)?

6) Carry out a backward analysis for the computation of the product \( P_n := \prod_{\mu=1}^{n} a_{\mu} \) using the method in Problem 4).

7) Let \( A, B, C \in \mathbb{R} \) be closed intervals in \( \mathbb{R} \). Show:
   a) The subdistributivity law

\[
A \cdot (B + C) \subset A \cdot B + A \cdot C
\]

holds.

b) If \( B \cdot C > 0 \) (i.e., all elements of \( B \cdot C \) are positive), then the distributivity law

\[
A \cdot (B + C) = A \cdot B + A \cdot C
\]

holds.

8) Use interval arithmetic to find bounding intervals for the values of the following functions:
   a) \( f(x) = x(1 - x), \quad 0 \leq x \leq 1 \);   b) \( f(x) = x/(1 - x), \quad 0 \leq x \leq 1 \);
   c) \( f(x) = x^2 + x^3 - 6x^2 + 0.11x - 0.006, \quad 0 \leq x \leq 0.2 \).