We consider the question of the numerical stability of numerical methods in more detail in the next section.

2.4 Problems. 1) In calculating \( \sum_{\nu=1}^{n} a_{\nu} \) in fixed-point arithmetric, we can get an arbitrarily large relative error. If, however, all \( a_{\nu} \) are of the same sign, then it is bounded. Neglecting terms of higher order, derive an upper bound in this case.

2) Rearrange the following expressions so that their evaluation is stable:
   a) \( \frac{1}{1+x^2} - \frac{1-x^2}{1+x^2} \) for \( |x| \ll 1 \);
   b) \( \frac{\cos x}{x} \) for \( x \neq 0 \) and \( |x| \ll 1 \).

3) Suppose the sequence \((a_n)\) is defined by the following recurrence relation:
   \[
a_1 := 4, \quad a_{n+1} := \frac{\sqrt{1 + a_n^2/2^{2(n+1)}} - 1}{a_n} \cdot 2^{2(n+1)+1}.
   \]
   a) Rewrite the recurrence in an equivalent but stable form.
   b) Write a computer program to compute \( a_{30} \) using both formulae, and compare the results.

4) Prove that the sequence of numbers \( y_n = e^{-1} \int_{0}^{1} e^{x^n} dx \) can be computed using the recurrence
   \[
   (*) \quad y_{n+1} + (n+1)y_n = 1 \quad \text{for} \quad n = 0, 1, 2, \ldots \quad \text{and} \quad y_0 = \frac{1}{e} (e - 1).
   \]
   a) Using \((*)\), compute the numbers \( y_0, y_1, \ldots, y_{30} \) and interpret the results.
   b) Prove the sequence of numbers in \((*)\) converges to 0 as \( n \to \infty \). Thus \( y_0 \) can be computed by working backwards, starting with the approximation \( y_n = 0 \) for a given \( n \). Carry out this process for \( n = 5, 10, 15, 20, 30 \), and explain why it gives such a good result for \( y_0 \).

3. Error Analysis

As we saw in 2.3, in general, there may be several different ways of arranging the computation leading to a solution of a given problem. Competing