ERROR ESTIMATES OF THE THIRD ORDER RUNGE-KUTTA ALTERNATING EVOLUTION DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS

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ABSTRACT. In this paper, we present the stability analysis and error estimates for the alternating evolution discontinuous Galerkin (AEDG) method with third order explicit Runge-Kutta temporal discretization for linear convection-diffusion equations. The scheme is shown stable under a CFL-like stability condition $c_0 \tau \leq \epsilon \leq c_1 h^2$. Here $\epsilon$ is the method parameter, and $h$ is the maximum spatial grid size. We further obtain the optimal $L^2$ error of order $O(\tau^3 + h^{k+1})$. Key tools include two approximation finite element spaces to distinguish overlapping polynomials, coupled global projections, and energy estimates of errors.

1. Introduction

In this paper, we present the stability analysis and a priori error estimates of Runge-Kutta alternating evolution discontinuous Galerkin (RKAEDG) method to smooth solutions of linear convection-diffusion equation

\begin{align}
\partial_t \phi + \alpha \partial_x \phi &= \beta \partial_x^2 \phi, \quad (x, t) \in [a, b] \times (0, T], \\
\phi(x, 0) &= \phi_0(x), \quad x \in [a, b],
\end{align}

(1.1a)

(1.1b)

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$ are given constants. We do not pay attention to boundary conditions in this paper, hence the solution is considered to be periodic; though other boundary conditions can also be studied along the same lines.

The AEDG method is a grid-based discontinuous Galerkin (DG) method, which was introduced by Liu and Pollack first in [5] for Hamilton-Jacobi equations, and further developed in [6] for nonlinear convection-diffusion equation

\begin{equation}
\partial_t \phi + \nabla x \cdot f(\phi) = \Delta_x a(\phi),
\end{equation}

(1.2)

in one and multi-dimensional setting, where $f(\phi)$ is a given flux function, and $a(\phi)$ a non-decreasing function. These and earlier works [11, 7] are all based on the alternating evolution (AE) framework introduced in [3]. The scheme construction is carried out by allowing the neighboring polynomials to overlap [4]. It is similar to the central DG methods [8, 9] in the sense that whenever a spatial derivative is evaluated, the neighboring polynomials (or the other representatives in the central DG schemes) are used. However, the AEDG method involves only one approximating polynomial near each grid point, independent of the spatial dimension, hence providing a unique high order approximation locally around each grid point.

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The one-dimensional semi-discrete AEDG scheme introduced in [6] has the following form:

\[
\int_{I_j} \left( \partial_t \Phi_j + \partial_x f(\Phi_j^{SN}) - \partial_x^2 a(\Phi_j^{SN}) \right) \eta dx = \left( - [f(\Phi_j^{SN})] \eta + [\partial_x a(\Phi_j^{SN})] \eta - [a(\Phi_j^{SN}) \partial_x \eta] \right) |_{x=x_j} \\
+ \frac{1}{\epsilon} \left( \int_{I_j} \Phi_j^{SN} \eta dx - \int_{I_j} \Phi_j \eta dx \right),
\]

where \(x_j\) is the grid point in cell \(I_j\), in which the numerical solution is denoted by \(\Phi_j\); \(\Phi_j^{SN}\) are sampled from neighboring polynomials \(\Phi_{j\pm 1}\), with \([g(\Phi_j^{SN})]|_{x_j}\) standing for the difference of two neighboring functions at \(x_j\) in the sense that \([g(\Phi_j^{SN})]|_{x_j} = g(\Phi_{j+1}(x_j^+) - g(\Phi_{j-1}(x_j^-))\).

In contrast to other DG methods, the stability analysis of the AEDG method is more subtle since stability property is less obvious from the scheme formulation. For linear convection-diffusion equation, the main results of both stability and optimal error estimates based on the stability result of the semi-discrete AEDG method has been proven if \(\epsilon \leq Qh^2\), for some \(Q\) and mesh size \(h\) in [6], in which the technical difficulty was resolved by a special regrouping of mixed terms combined with the use of some inverse inequalities.

Further in [10] the authors obtained the first optimal \(L^2\) error estimates based on the stability result established in [6] for the semi-discrete AEDG scheme (2.2). For the fully discrete scheme with forward Euler time discretization, the stability condition relating \(\epsilon\) to the time step \(\tau\) of the form \(c_0 \tau \leq \epsilon < Qh^2\) for some \(c_0 > 1\) is shown sufficient for obtaining the following optimal error estimate

\[
\sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_{j+1}} \left( \frac{|\Phi_{j+1}^n(x) - \phi(x,t^n)|^2 + |\Phi_j^n(x) - \phi(x,t^n)|^2}{2} \right) dx \leq C(\tau + h^{k+1})^2, \quad n\tau \leq T.
\]

Here of course \(\Phi_j^n\) is the numerical solution at time level \(n\) near grid \(x_j\), \(\tau\) is the time step, and the positive constant \(C\) is independent of \(\tau\), \(h\) and the numerical solution. This estimate differs from the usual \(L^2\) error since the AEDG method uses overlapping polynomials. These features require new techniques in the error estimates.

In this paper, we are interested in error estimates for the RKAEDG method approximating the smooth solutions of (1.1), with a third order explicit SSP Runge–Kutta time discretization [2]. A general discussion of the AEDG method and background references on the error estimates for the DG methods for convection-diffusion problems are given in the introduction to [10]. We have two objectives:

(i) to present the stability analysis of the RK3AEDG method;

(ii) to estimate the difference in \(L^2\) norm between the exact solution and the approximate ones.

The stability analysis for (i) is based on the AE formulation and carried out by identifying a sufficient condition on the time step restriction, relating to the method parameter \(\epsilon\).

The error estimates for (ii) are based on Taylor’s expansion and energy methods similarly to those for fully discrete DG schemes to hyperbolic conservation laws with Runge-Kutta time discretization in [14, 15, 16], but here the error analysis is carried out by solving a coupled system involving two bilinear operators, and we essentially use several tools developed in [10], including two approximation spaces \(V_h \times U_h\) associating with odd and even grids, respectively, with which the AE scheme can be reformulated using two bi-linear operators; the two global projections on \(V_h\) and \(U_h\), coupled through the \(\epsilon\)-dependent term dictated by the AEDG formulation, the projection errors, as well as the \(\epsilon\)-dependent energy norm in \(V_h \times U_h\), involving a special term of the form \(h^{-1}||u - v||\) for \((v, u) \in V_h \times U_h\). The error analysis for AEDG methods is more involved because the coupling between overlapping polynomials must be carefully handled. We also refer to [13, 12] for the error estimates of the fully discrete LDG algorithm with a third order Runge–Kutta time discretization to solve convection-diffusion equations.

The article is organized as follows: in section 2 we present both the semi-discrete and fully discrete AEDG schemes with third order Runge-Kutta time discretization for the one-dimensional linear convection-diffusion equation, and the main results of both stability and optimal \(L^2\) error estimates. In section 3 we reformulate the RK3AEDG scheme as a coupled system using two bi-linear operators, and then review several useful tools and known results from [10]. In section 4, we figure out a sufficient
condition on the time step restriction so that the RK3AEDG can be shown stable. Finally optimal $L^2$ error estimates are given in section 5.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset [a,b]$ equipped with the norm $\| \cdot \|_{m,p,D}$ and semi-norm $| \cdot |_{m,p,D}$. When $D = [a,b]$, we omit the index $D$; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\| \cdot \|_{m,p,D} = \| \cdot \|_{m,D}$, and $| \cdot |_{m,p,D} = | \cdot |_{m,D}$. We use either $\| \cdot \|_{0,D}$ or $\| \cdot \|$ when $D = [a,b]$ to denote the usual $L^2$ norm. We also use the notation $A \lesssim B$ to indicate that $A$ can be bounded by $B$ multiplied by a constant independent of the mesh size $\tau, h$. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. We will also use $C$ to denote a positive constant independent of $h$ and $\tau$, which may depend on solutions of (1.1).

### 2. Alternating evolution DG methods

The AEDG method consists of a semi-discrete formulation based on sampling of the AE system on alternating grids and a fully discrete version by using an appropriate Runge-Kutta solver.

#### 2.1. Setting of semi-discrete AEDG method.

Recall the AEDG method for the one-dimensional convection-diffusion equation

$$\partial_t \phi + \partial_x f(\phi) = \partial_x^2 a(\phi)$$

subject to initial data $\phi_0(x)$ and periodic boundary conditions.

Partition the spatial domain $[a,b]$ into a grid with grid points $\{x_j\}$ such that $x_1 = a, x_N = b$. Set $I_j = (x_{j-1}, x_{j+1})$ for $j = 1, 2, \ldots, N - 1$, while $I_1 = (x_0, x_2)$ in which $(x_0, x_1)$ is the periodic shift of $(x_{N-1}, x_N)$ and $h_j = \frac{2x_2 - x_0}{2}$, and we define the quantities

$$h = \max_{1 \leq j \leq N-1} h_j \quad \text{and} \quad \rho = \min_{1 \leq j \leq N-1} h_j.$$

For simplicity of presentation we would like to assume that the ratio of $h$ and $\rho$ is upper bounded by a fixed positive constant $\nu^{-1}$ when $h$ goes to zero so that $\nu h \leq \rho \leq h$. We shall analyze the uniform grid case $\nu = 1$, knowing that the techniques can be easily carried over to the case $\nu \neq 1$.

Centered at each grid $\{x_j\}$, the numerical approximation is a polynomial $\Phi|_{I_j} = \Phi_j(x) \in P_k$, where $P_k$ denotes a linear space of all polynomials of degree at most $k$:

$$P_k := \{ p | p(x)|_{I_j} = \sum_{0 \leq i \leq k} a_i(x-x_j)^i, \quad a_i \in \mathbb{R} \}.$$  

We denote $v(x^\pm) = \lim_{\epsilon \to 0^\pm} v(x + \epsilon)$, and $v^\pm = v(x^\pm)$. The jump at $x_j$ is $[v]|_{x_j} = v(x_j^+) - v(x_j^-)$. Note that the solution space here differs from the usual finite element space since it allows the overlapping of two neighboring polynomials of $\Phi_j$ and $\Phi_{j+1}$ over $I_j \cap I_{j+1} = [x_j, x_{j+1}] \neq \emptyset$.

The semi-discrete AEDG method introduced in [6] is to find $\Phi_j |_{I_j} \in P_k$ such that for all $\eta \in P_k(I_j)$,

$$\int_{I_j}(\partial_t \Phi_j + \partial_x f(\Phi_j^{SN}) - \partial_x^2 a(\Phi_j^{SN}))\eta dx = \left( -[f(\Phi_j^{SN})]\eta + [\partial_x a(\Phi_j^{SN})]\eta - [a(\Phi_j^{SN})]\eta_x \right)|_{x=x_j}$$

$$+ \frac{1}{\epsilon} \left( \int_{I_j} \Phi_j^{SN} \eta dx - \int_{I_j} \Phi_j \eta dx \right),$$

where $\Phi_j^{SN}$ is defined as

$$\Phi_j^{SN} = \begin{cases} 
\Phi_{j-1}(x), & x_{j-1} < x < x_j, \\
\Phi_{j+1}(x), & x_j < x < x_{j+1}
\end{cases}$$

with periodic boundary conditions. $\Phi_N(x)$ is regarded to be identical to $\Phi_1(x)$, which is computed over $I_1 = [x_0, x_2] = [a-h, a+h]$. Numerical solution on $[x_{N-1}, x_N]$ is simply taken from $\Phi_1$ over $[x_0, x_1]$. Note that $\Phi_1(x, 0) = \Phi_N(x, 0)$ for initial data.

The semi-discrete AEDG scheme is also shown to be conservative and stable for linear problems in [6].
Theorem 2.1. [6, Theorem 3.1, Theorem 3.2] Let $\Phi$ be computed from the AEDG scheme (2.2) for the linear convection-diffusion equation
\[
\partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi,
\]
with periodic boundary conditions. We have
(i) The scheme is conservative in the sense that
\[
\frac{d}{dt} \left( \sum_{j=1}^{N-1} \int_{x_{j+1}}^{x_j} \left( \frac{\Phi_{j+1}}{2} + \frac{\Phi_j}{2} \right) dx \right) = 0.
\]
(ii) The scheme using polynomials of degree $k \geq 1$ is $L^2$ stable if $\epsilon \leq Q h^2$. Moreover,
\[
\frac{d}{dt} \left( \sum_{j=1}^{N-1} \int_{x_{j+1}}^{x_j} \left( \frac{\Phi_{j+1}^2 + \Phi_j^2}{2} \right) dx \right) \leq -\beta \sum_{j=1}^{N-1} \int_{x_{j+1}}^{x_j} \left( \frac{(\partial_x \Phi_{j+1})^2 + (\partial_x \Phi_j)^2}{2} \right) dx
\]
\[
+ \left( \frac{1}{Q h^2} - \frac{1}{\epsilon} \right) \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (\Phi_{j+1} - \Phi_j)^2 dx
\]
with
\[
Q = \frac{1}{\beta(k+1)^2(17(k+1)^2 - 1)}.
\]

2.2. Fully discrete AEDG method with third order Runge-Kutta time discretization. We now turn to time discretization of (2.2). Let \( \{t^n\}, n = 0, 1, \ldots, K \) be a uniform partition of the time interval \([0, T]\) and denote the time step size as $\tau$. The initial data for $\Phi_j(x,0)$ is taken as the $L^2$ projection of $\phi_0$ on $I_j$ for $j = 1, \ldots, N - 1$:
\[
\int_{I_j} \Phi_j(x,0) dx = \int_{I_j} \phi_0(x) dx, \quad \forall \eta \in P_k(I_j), \quad j = 1, \ldots, N - 1.
\]
Denote $\Psi = [\psi_1, \psi_2, \ldots, \psi_J]^\top$ the unknown coefficients of the numerical solution against the basis in the DG space, the ODE system (2.2) can be written as
\[
\partial_t \Psi = L(\Psi),
\]
where $L(\cdot)$ is some spatial differential operator defined by (2.2).

We use the third order explicit SSP Runge-Kutta method [2] for time discretization. In details, let $\Psi^{n,0}$ be the solution at time level $n$, and $\Psi^{n,i}, i = 1, 2$ be the solution at intermediate step between $t^n$ and $t^{n+1}$, thus we can write
\[
\Psi^{n,1} = \Psi^{n,0} + \tau L(\Psi^{n,0}),
\]
\[
\Psi^{n,2} = \frac{3}{4} \Psi^{n,0} + \frac{1}{4} (\Psi^{n,1} + \tau L(\Psi^{n,1})),
\]
\[
\Psi^{n+1} = \frac{1}{3} \Psi^{n,0} + \frac{2}{3} (\Psi^{n,2} + \tau L(\Psi^{n,2})).
\]

Based on the above setting, we are able to show the scheme is stable under some restriction on the time step $\tau$ and $\epsilon$, and further obtain the optimal $L^2$ error estimates for (2.2) with third order time discretization (2.6). The main results are summarized in the following.

Theorem 2.2. Let $\Phi^n$ be the numerical solution computed from (2.6) with $\epsilon = c Q h^2$, $0 < c < 1$, then there exists $c_0 > 0$ such that for $h$ small,
\[
c_0 \tau \leq \epsilon = c Q h^2,
\]
we have
\[
\sum_{j=1}^{N-1} \int_{x_{j+1}}^{x_j} \frac{(\Phi_{j+1}^{n+1})^2 + (\Phi_j^{n+1})^2}{2} dx \leq \sum_{j=1}^{N-1} \int_{x_{j+1}}^{x_j} \frac{(\Phi_{j+1}^n)^2 + (\Phi_j^n)^2}{2} dx.
\]
Theorem 2.3. Let $\phi$ be the smooth solution to (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, and $\Phi^\eta_j \in P^k(I_j)$ ($k \geq 1$) be the numerical solution to (2.6) with $c_0 \tau \leq \epsilon = c Q h^2$ for some $c_0 > 0$ and $0 < c < 1$, then the following error estimate holds:

$$
(2.8) \quad N^{-1} \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\left| \Phi^\eta_{j+1}(x) - \phi(x, t^n) \right|^2 + \left| \Phi^\eta_j(x) - \phi(x, t^n) \right|^2}{2} dx \leq C(\tau^6 + h^{2k+2}), \quad n \tau \leq T,
$$

where $C$ is a constant uniformly in terms of $\tau, h$ and $n$.

We defer the proof of Theorem 2.2 to section 4 and Theorem 2.3 to section 5.

3. Scheme reformulation and useful tools

3.1. Scheme reformulation. Following [10], we introduce two solution spaces of piecewise polynomials as

$$
(3.1) \quad V_h = \{ \eta \in L^2, \eta \in P^k(I_j), \quad j = \text{odd} \}, \quad U_h = \{ \eta \in L^2, \eta \in P^k(I_j), \quad j = \text{even} \}.
$$

Note that for $N$ odd, the set $\{ j = \text{even} \} = \{2, 4, \ldots, N-1\}$, and $\{ j = \text{odd} \} = \{1, 3, \ldots, N-2\}$; For $N$ even, the set $\{ j = \text{even} \} = \{2, 4, \ldots, N-2\}$ and $\{ j = \text{odd} \} = \{1, 3, \ldots, N-1\}$. This way the periodic boundary condition is always satisfied through $\Phi_1 = \Phi_N$, with $\Phi_1 \in V_h$, no matter $N$ is odd or even.

Taking $f(w) = aw$ and $a(w) = \beta w$ in AEDG scheme (2.2), summing over $j = \text{odd}$ and $j = \text{even}$, respectively, we obtain a coupled system

$$
(3.2) \quad \langle \partial_t v; \xi \rangle + A_{21}(u, \xi) = \frac{1}{\epsilon} \langle u - v, \xi \rangle, \quad \xi \in V_h,
$$

$$
(3.3) \quad \langle \partial_t u; \eta \rangle + A_{12}(v, \eta) = \frac{1}{\epsilon} \langle v - u, \eta \rangle, \quad \eta \in U_h,
$$

where inner product is defined as $\langle u, \xi \rangle = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} w \xi dx$ and the two bilinear operators are defined by

$$
A_{21}(u, \xi) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \partial_x J(u) \xi dx + \sum_{j=\text{odd}} \left( [J(u)] \xi + \beta [u] \partial_x \xi \right)_{x_j}, \quad (u, \xi) \in U_h \times V_h,
$$

$$
A_{12}(v, \eta) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \partial_x J(v) \eta dx + \sum_{j=\text{even}} \left( [J(v)] \eta + \beta [v] \partial_x \eta \right)_{x_j}, \quad (v, \eta) \in V_h \times U_h,
$$

where $J(w) = \alpha w - \beta \partial_x w$. Note that for $N$ odd, $\int_{x_{N-1}}^{x_N} \partial_x J(v) \eta dx$ is defined by $\int_{x_{N-1}}^{x_N} \partial_x J(v) \eta dx$; and for $N$ even, $\int_{x_{N-1}}^{x_N} \partial_x J(v) \eta dx$ is defined by $\int_{x_{N-1}}^{x_N} \partial_x J(v) \xi dx$, using the periodicity of the numerical solution. We remark that the subscripts in the operator $A_{12}$ or $A_{21}$ indicate the odd and even (or even and odd) spaces to which the corresponding arguments belong.

For notational convenience, in the following we use $\| \partial_x v \|^2 := \int_u^v (\partial_x v)^2 dx$ and $\| \partial_x u \|^2 := \int_u^v (\partial_x u)^2 dx$ to denote

$$
(3.4) \quad \sum_{j=\text{odd}} \int_{x_j}^{x_{j+1}} |\partial_x v|^2 dx, \quad \sum_{j=\text{even}} \int_{x_j}^{x_{j+1}} |\partial_x u|^2 dx
$$

respectively if $(v, u) \in V_h \times U_h$, unless otherwise stated. Also, we define the $L^2$ norm and energy norm of $(v, u) \in V_h \times U_h$ as

$$
(3.5) \quad \| (v, u) \|^2 := \| v \|^2 + \| u \|^2
$$

and

$$
(3.6) \quad \| (v, u) \|^2 := \| (v, u) \|^2 + \| (\partial_x v, \partial_x u) \|^2 + \frac{1}{h^2} \| v - u \|^2,
$$

respectively, where $\| \cdot \|$ is the $L^2$ norm for functions in $V_h$ and $U_h$ shown in (3.4).

For semi-discrete AEDG scheme (2.2), the stability result obtained in [6] is as follows.
Lemma 3.1. For any \((v, u) \in V_h \times U_h\), we have
\[
A_{21}(u, v) + A_{12}(v, u) \geq \frac{\beta}{2} \| (\partial_x v, \partial_x u) \|^2 - \frac{1}{Q h^2} \| v - u \|^2,
\]
where \(Q\) is defined in (2.4).

The AEDG method with the third order SSP Runge-Kutta method for time discretization gives the RK3AEDG scheme (2.6), which can be rewritten as
\begin{align}
(3.10) & \quad \langle v^{n+1}, \xi \rangle = \langle v^n, \xi \rangle - \tau A_{21}(u^n, \xi) + \frac{\tau}{\epsilon} (u^n - v^n, \xi), \quad (\xi, \eta) \in V_h \times U_h, \\
(3.11) & \quad \langle v^n, \eta \rangle = \langle v^n, \eta \rangle - \tau A_{12}(v^n, \eta) + \frac{\tau}{\epsilon} (u^n - v^n, \eta), \quad (v^n, u^n) \in V_h \times U_h;
\end{align}

\begin{align}
(3.8a) & \quad \langle v^{n+2}, \xi \rangle = \frac{1}{4} \langle 3v^n + v^{n+1}, \xi \rangle - \frac{\tau}{4} A_{21}(u^{n+1}, \xi) + \frac{\tau}{4} \epsilon (u^{n+1} - v^{n+1}, \xi), \quad (\xi, \eta) \in V_h \times U_h, \\
(3.8b) & \quad \langle v^{n+1}, \eta \rangle = \frac{1}{4} \langle 3v^n + u^{n+1}, \eta \rangle - \frac{\tau}{4} A_{12}(v^{n+1}, \eta) + \frac{\tau}{4} \epsilon (v^{n+1} - u^{n+1}, \eta), \quad (v^{n+1}, u^{n+1}) \in V_h \times U_h;
\end{align}

\begin{align}
(3.9a) & \quad \langle v^{n+1}, \xi \rangle = \frac{1}{3} \langle v^n + 2v^{n+2}, \xi \rangle - \frac{2\tau}{3} A_{21}(u^{n+2}, \xi) + \frac{2\tau}{3} \epsilon (u^{n+2} - v^{n+2}, \xi), \quad (\xi, \eta) \in V_h \times U_h, \\
(3.9b) & \quad \langle v^{n+2}, \eta \rangle = \frac{1}{3} \langle v^n + 2u^{n+2}, \eta \rangle - \frac{2\tau}{3} A_{12}(v^{n+2}, \eta) + \frac{2\tau}{3} \epsilon (v^{n+2} - u^{n+2}, \eta), \quad (v^{n+2}, u^{n+2}) \in V_h \times U_h.
\end{align}

3.2. Projection and projection errors. In this subsection, we review the global projections introduced in [10] and the associated properties proved therein.

Suppose \(w\) is a smooth periodic function, the two coupled projections \((\Pi_v w, \Pi_u w) \in V_h \times U_h\) introduced in [10] is as follows
\begin{align}
(3.10) & \quad \langle \Pi_v w - w, \xi \rangle + A_{21}(\Pi_u w - w, \xi) = \frac{1}{\epsilon} \langle \Pi_v w - \Pi_v w, \xi \rangle, \quad \xi \in V_h, \\
(3.11) & \quad \langle \Pi_u w - w, \eta \rangle + A_{12}(\Pi_v w - w, \eta) = \frac{1}{\epsilon} \langle \Pi_u w - \Pi_u w, \eta \rangle, \quad \eta \in U_h.
\end{align}

Here, we again construct \(\Pi_v w, \Pi_u w\) over the extended cell \(I_1 = [x_0, x_2]\), and set
\[
\Pi_v w|_{[x_{N-1}, x_N]} = 0, \quad N = \text{odd},
\]
\[
\Pi_v w|_{[x_{N-1}, x_N]} = \Pi_v w|_{[x_0, x_1]}, \quad N = \text{even}.
\]
Such extension for both \(\Pi_v w\) and \(\Pi_u w\) is made so that they become periodic.

Theorem 3.2. [10, Lemma 3.2, Theorem 3.3, Theorem 4.4] Let \(w\) be a smooth periodic function that belongs to \(H^m\), if \(\epsilon = c h^2\) for \(0 < c < 1\), then the two projection operators \(\Pi_v, \Pi_u\) defined in (3.10)-(3.11) have the following projection error
\begin{align}
(3.12) & \quad \| (\Pi_v w - w, \Pi_u w - w) \|_E \leq C h^{\min(k, m)} |w|_{m}, \\
(3.13) & \quad \| \Pi_v w - w \| + \| \Pi_u w - w \| \leq C h^{\min(k+1, m)} |w|_{m},
\end{align}
where \(C\) is a constant independent of \(h\). Here \(\| \cdot \|\) denotes the \(L^2\) norm in \([a, b]\).

For the local approximation results, see, e.g.,[1, Lemma 4.3.8], which will be used for the energy estimates.

Lemma 3.3. [10, Lemma 4.1] If \(w \in H^m(\Omega)\) is a periodic function, then there exists polynomials \((v_1, u_1) \in V_h \times U_h\) that satisfy optimal approximation properties, i.e., \(v_1 \in V_h, u_1 \in U_h\) are polynomials in \(I_j\), for \(j = \text{odd}\) and \(j = \text{even}\), respectively,
\[
|w - v_1|_{s, I_j} \leq C h^{\min(m+k+1) - s} |w|_{m, I_j}, \quad j = \text{odd},
\]
\[
|w - u_1|_{s, I_j} \leq C' h^{\min(m+k+1) - s} |w|_{m, I_j}, \quad j = \text{even},
\]
for $0 \leq s \leq \min\{m, k + 1\}$, where $C, C'$ are two constants independent of mesh size $h$.

**Lemma 3.4.** [10, Lemma 4.2] Let $I = [c, d] \subset [a, b]$ be an interval of length $|I|$, and $v \in P^m(I)$, then

\[
\max\{|v(c)|, |v(d)|\} \leq (m + 1)|I|^{-1/2}\|v\|_{0, I},
\]

\[
||\partial_x v||_{0, I} \leq (m + 1)\sqrt{m(m + 2)}|I|^{-1/2}\|v||_{0, I},
\]

\[
\|v(\cdot)|^2_{0, I} \leq \frac{\sqrt{5}}{2} + \frac{1}{2}(|I|^{-1}\|v\|^2_{0, I} + |I|\|\partial_x v\|^2_{0, I}),
\]

if $v \in H^1(I)$.

We shall also use the following bound of two bilinear operators $A_{21}$ and $A_{12}$.

**Lemma 3.5.** [10, Lemma 5.1] For any $(\xi, \eta), (v, u) \in V_h \times U_h$, it holds

\[
A_{21}(u, \xi) \leq h^{-1}\Gamma(\|u\| + h^{-1}\|v - u\|)\|\xi\|,
\]

\[
A_{12}(v, \eta) \leq h^{-1}\Gamma(\|v\| + h^{-1}\|v - u\|)\|\eta\|,
\]

where

\[
\Gamma := \max\{|\alpha|d + 2(2 + 1)|1 + (2 + 1)/2(\alpha h + \beta)\},
\]

with $\gamma_k = (k + 1)\sqrt{k(k + 2)}$.

### 4. Stability Analysis

In this section, we present the $L^2$ stability analysis in relating time step $\tau$ to $\epsilon$ for the RK3AEDG method. We start by introducing some notations:

\[
\mathcal{L}^{n, 1}_v = v^{n, 1} - v^n, \mathcal{L}^{n, 2}_v = 2v^{n, 2} - v^{n, 1} - v^n, \mathcal{L}^{n, 3}_v = v^{n, 1} - 2v^{n, 2} + v^n.
\]

\[
\mathcal{L}^{n, 1}_u = u^{n, 1} - u^n, \mathcal{L}^{n, 2}_u = 2u^{n, 2} - u^{n, 1} - u^n, \mathcal{L}^{n, 3}_u = u^{n, 1} - 2u^{n, 2} + u^n.
\]

**Lemma 4.1.** For the fully discrete RK3AEDG method (3.7)-(3.9), we can write

\[
\langle \mathcal{L}^{n, 1}_v, \xi \rangle = -\tau A_{21}(u^n, \xi) + \frac{\tau}{\epsilon}\langle u^n - v^n, \xi \rangle, \quad \xi \in V_h,
\]

\[
\langle \mathcal{L}^{n, 2}_v, \xi \rangle = -\frac{\tau}{2}A_{21}(\mathcal{L}^{n, 1}_u, \xi) + \frac{\tau}{2\epsilon}\langle \mathcal{L}^{n, 1}_u - \mathcal{L}^{n, 1}_v, \xi \rangle, \quad \xi \in V_h,
\]

\[
\langle \mathcal{L}^{n, 3}_v, \xi \rangle = -\frac{\tau}{3}A_{21}(\mathcal{L}^{n, 2}_u, \xi) + \frac{\tau}{3\epsilon}\langle \mathcal{L}^{n, 2}_u - \mathcal{L}^{n, 2}_v, \xi \rangle, \quad \xi \in V_h,
\]

\[
\langle \mathcal{L}^{n, 1}_u, \eta \rangle = -\tau A_{12}(v^n, \eta) + \frac{\tau}{\epsilon}\langle v^n - u^n, \eta \rangle, \quad \eta \in U_h,
\]

\[
\langle \mathcal{L}^{n, 2}_u, \eta \rangle = -\frac{\tau}{2}A_{12}(\mathcal{L}^{n, 1}_v, \eta) + \frac{\tau}{2\epsilon}\langle \mathcal{L}^{n, 1}_v - \mathcal{L}^{n, 2}_v, \eta \rangle, \quad \eta \in U_h,
\]

\[
\langle \mathcal{L}^{n, 3}_u, \eta \rangle = -\frac{\tau}{3}A_{12}(\mathcal{L}^{n, 2}_v, \eta) + \frac{\tau}{3\epsilon}\langle \mathcal{L}^{n, 2}_v - \mathcal{L}^{n, 3}_v, \eta \rangle, \quad \eta \in U_h.
\]

**Proof.** Using notations in (4.1a), (4.2a) is straightforward from (3.7a). We can obtain (4.2b) by calculating $2 \times (3.8a) - \frac{1}{2} \times (3.7a)$. To prove (4.2c), substituting the left hand side of (3.7a) into (3.8a), we have

\[
\langle v^n - v^n, \xi \rangle = -\frac{\tau}{4}A_{21}(u^{n, 1} + u^n, \xi) + \frac{\tau}{4\epsilon}(u^{n, 1} + u^n - (u^{n, 1} + v^n), \xi),
\]

then by applying (3.9a) - $\frac{1}{4} \times (4.3)$, we get (4.2c). In entirely same manner, we can obtain the claimed relations (4.2d-f) for $\mathcal{L}^{n, 1}_u, \mathcal{L}^{n, 2}_u, \mathcal{L}^{n, 3}_u$ too.

Stability result in Theorem 2.2 can be reformulated in terms of $(v^n, u^n)$ as follows.

**Theorem 4.2.** Let $v^n$ and $u^n$ be the numerical solutions computed from (3.7)-(3.9) with $\epsilon = cQh^2$, $0 < c < 1$, there exists $c_0 > 0$ such that for $h$ small,

\[
c_0\tau \leq \epsilon = cQh^2,
\]

then

\[
\|(v^{n+1}, u^{n+1})\| \leq \|(v^n, u^n)\|.
\]
Proof. Taking \((\xi, \eta) = (v^n, u^n)\) in (3.7), \((\xi, \eta) = (4v^{n,1}, 4u^{n,1})\) in (3.8), \((\xi, \eta) = (6v^{n,2}, 6u^{n,2})\) in (3.9) and summing up the resulting relations, we obtain

\begin{equation}
-\tau (A_{21}(u^n, v^n) + A_{21}(u^{n,1}, v^{n,1}) + 4A_{21}(v^{n,2}, v^{n,2}) + A_{12}(v^n, u^n) + A_{12}(v^{n,1}, u^{n,1})) - 4\tau A_{12}(v^{n,2}, u^{n,2}) - \frac{T}{\epsilon} ||v^n - u^n||^2 - \frac{T}{\epsilon} ||v^{n,1} - u^{n,1}||^2 - \frac{4\tau}{\epsilon} ||v^{n,2} - u^{n,2}||^2 = \int_a^b Vdx + \int_a^b Ud\tau,
\end{equation}

in which \(V\) and \(U\) can be written as

\begin{equation}
V = -2\epsilon v^n - (v^n)^2 + 4v^{n,1} v^{n,2} - (v^{n,1})^2 + 6v^{n,2} v^{n,1} + 2\epsilon v^n v^{n,2} - 4(v^{n,2})^2
\end{equation}

\begin{equation}
= 3((v^{n+1})^2 - (v^n)^2) - (2v^{n,2} - v^{n,1} - v^n)^2 - 3(v^{n+1} - v^n)(v^{n+1} - v^n) - 2\epsilon v^n v^{n,2} + v^n),
\end{equation}

\begin{equation}
U = -2\epsilon u^n - (u^n)^2 + 4u^{n,1} u^{n,2} - (u^{n,1})^2 + 6u^{n,2} u^{n,1} + 2\epsilon u^n u^{n,2} - 4(u^{n,2})^2
\end{equation}

\begin{equation}
= 3((u^{n+1})^2 - (u^n)^2) - (2u^{n,2} - u^{n,1} - u^n)^2 - 3(u^{n+1} - u^n)(u^{n+1} - u^n) - 2\epsilon u^n u^{n,2} + u^n).
\end{equation}

Substituting (4.6) and (4.7) into (4.5), we have

\begin{equation}
3||(v^{n+1}, u^{n+1})||^2 - 3||(v^n, u^n)||^2 = \Pi_1 + \Pi_2,
\end{equation}

where \(\Pi_i (i = 1, 2)\) are defined by

\begin{equation}
\Pi_1 = -\tau (A_{21}(u^n, v^n) + A_{21}(u^{n,1}, v^{n,1}) + 4A_{21}(v^{n,2}, v^{n,2}) + A_{12}(v^n, u^n) + A_{12}(v^{n,1}, u^{n,1})) - 4\tau A_{12}(v^{n,2}, u^{n,2}) - \frac{T}{\epsilon} ||v^n - u^n||^2 - \frac{T}{\epsilon} ||v^{n,1} - u^{n,1}||^2 - \frac{4\tau}{\epsilon} ||v^{n,2} - u^{n,2}||^2,
\end{equation}

and

\begin{equation}
\Pi_2 = \int_a^b [(2v^{n,2} - v^{n,1} - v^n)^2 + 3v^{n+1} - v^n)(v^{n+1} - v^n)] dx + \int_a^b [(2u^{n,2} - u^{n,1} - u^n)^2 + 3u^{n+1} - u^n)(u^{n+1} - u^n)] dx
\end{equation}

\begin{equation}
= (L_v^{n,2}, L_v^{n,2}) + 3(L_v^{n,1}, L_v^{n,1}) + (L_v^{n,2}, L_v^{n,2}) + 3(L_u^{n,1}, L_u^{n,2} + L_u^{n,2}) + L_u^{n,2} + L_u^{n,3}) + (L_u^{n,3}, L_u^{n,3}).
\end{equation}

Here we have used notations in (4.1) and the fact that \(v^{n+1} - v^n = L_v^{n,1} + L_v^{n,2} + L_v^{n,3}\) (similarly for \(u^{n+1} - u^n\) in the last equality of (4.10).

From Lemma 3.1 and the fact \(\frac{1}{\sqrt{2\tau}} = \xi\), it follows that

\begin{equation}
\Pi_1 \leq -\frac{\beta\tau}{\epsilon} ||(\partial_x v^n, \partial_x u^n)||^2 - \frac{T}{\epsilon} ||v^n - u^n||^2 - \frac{\beta\tau}{\epsilon} ||(\partial_x v^{n,1}, \partial_x u^{n,1})||^2 - \frac{T}{\epsilon} ||v^{n,1} - u^{n,1}||^2 - 2\beta\tau ||(\partial_x v^{n,2}, \partial_x u^{n,2})||^2 - \frac{4\tau}{\epsilon} ||(\partial_x v^{n,2}, \partial_x u^{n,2})||^2 \leq 0.
\end{equation}

Next we show that under some restriction on \(\tau\) and \(\epsilon\), \(\Pi_2\) can be bounded by \(|\Pi_1|\) for \(0 < \epsilon < 1\). To this aim, we denote

\begin{equation}
\Pi_2 = \Pi_2^1 + \Pi_2^2 + \Pi_2^3,
\end{equation}

where

\begin{equation}
\Pi_2^1 = (L_v^{n,2}, L_v^{n,2}) + 3(L_v^{n,1}, L_v^{n,1}) + (L_v^{n,2}, L_v^{n,2}) + 3(L_u^{n,1}, L_u^{n,1}),
\end{equation}

\begin{equation}
\Pi_2^2 = 3(L_v^{n,3}, L_v^{n,3}) + 3(L_u^{n,3}, L_u^{n,3}),
\end{equation}

\begin{equation}
\Pi_2^3 = 3(L_v^{n,3}, L_v^{n,3}) + 3(L_u^{n,3}, L_u^{n,3}).
\end{equation}
Regrouping terms in $\Pi_2$ we have

\[
\Pi_2 = - (L^{n,2}_v, L^{n,2}_v) + 2(L^{n,2}_v, L^{n,2}_u) + 3(L^{n,3}_v, L^{n,1}_v)
- \frac{1}{\epsilon} \Pi_2 (L^{n,2}_u - L^{n,1}_v, L^{n,2}_v) + \frac{\tau}{\epsilon} (L^{n,1}_u - L^{n,1}_v, L^{n,2}_v)
- \frac{1}{\epsilon} \Pi_2 (L^{n,2}_u - L^{n,2}_v, L^{n,1}_v) + \frac{\tau}{\epsilon} (L^{n,1}_u - L^{n,1}_v, L^{n,2}_v)
- \frac{1}{\epsilon} \Pi_2 (L^{n,2}_u - L^{n,2}_v, L^{n,1}_v) + \frac{\tau}{\epsilon} (L^{n,1}_u - L^{n,1}_v, L^{n,2}_v),
\]

(4.13)

where we have used (4.2b, 4.2c) and (4.2e, 4.2f) in the last equality.

For those bilinear forms in (4.13), from Lemma 3.1, we can derive

\[
-A_{21}(L^{n,2}_u - L^{n,2}_v, L^{n,1}_u) \leq \frac{\beta}{2} \frac{1}{h^2} \left( \| \partial_x L^{n,2}_v, \partial_x L^{n,1}_u \|^2 + \| \partial_{x} L^{n,1}_u, \partial_y L^{n,1}_u \|^2 \right)
\]

(4.14)

and

\[
-A_{21}(L^{n,2}_u - L^{n,2}_v, L^{n,1}_v) \leq \frac{\beta}{2} \frac{1}{h^2} \left( \| \partial_x L^{n,1}_u, \partial_x L^{n,2}_u \|^2 + \| \partial_{y} L^{n,2}_v, \partial_y L^{n,1}_u \|^2 \right)
\]

(4.15)

Choosing $\xi = L^{n,1}_v$ in (4.2a), by Lemma 3.5 and the Cauchy-Schwartz inequality, we have

\[
\| L^{n,1}_v \|_2^2 = - \frac{\tau}{\epsilon} \frac{1}{h} (\| \partial_x L^{n,1}_u \| + h^{-1} \| v^n - u^n \|) \Pi L^{n,1}_v + \frac{\tau}{\epsilon} \| v^n - u^n \| \| L^{n,1}_v \|
\]

then, canceling the common term $\| L^{n,1}_v \|$ and noting $\frac{1}{\Pi} = \frac{\Omega}{\epsilon}$, we write

\[
\| L^{n,1}_v \| \leq \frac{\tau}{h} \| \partial_{x} u^n \| + \frac{\tau}{\epsilon} (1 + cQ \Gamma) \| v^n - u^n \|
\]

(4.17)

Since we can obtain a similar estimate as (4.17) for $\| L^{n,1}_u \|$, we have

\[
\| (L^{n,1}_u, L^{n,1}_u) \|_2^2 \leq \frac{2 \tau^2 T^2}{h^2} \| (\partial_x v^n, \partial_x u^n) \|^2 + \frac{4 \tau^2}{c^2} (1 + cQ \Gamma) \| v^n - u^n \|^2.
\]

(4.18)

Thus substituting (4.14) and (4.15) into (4.13), noticing $\epsilon = cQh^2$, we estimate $\Pi_2$ as

\[
\Pi_2 \leq - \frac{\beta}{2} \frac{1}{h^2} \left( \| \partial_x L^{n,2}_v, \partial_x L^{n,1}_u \|^2 + \| \partial_{x} L^{n,1}_u, \partial_x L^{n,2}_u \|^2 \right)
+ \frac{\tau}{\epsilon} \frac{1}{h} (\| \partial_x L^{n,1}_u \| + h^{-1} \| v^n - u^n \|) \Pi L^{n,1}_v + \frac{\tau}{\epsilon} \| v^n - u^n \| \| L^{n,1}_v \|
\]

(4.19)

where $\delta \in (0, 1)$ in the second inequality is a constant to be specified, and we have used (4.18) in the last inequality (4.19).
For $\Pi^2_2$, choosing $\xi = \mathcal{L}^n_v$ in (4.2c) and $\eta = \mathcal{L}^n_u$ in (4.2f), summing the results up, we have

\[
\Pi^2_2 = 3\langle \mathcal{L}^{n,3}_v, \mathcal{L}^n_u \rangle + 3\langle \mathcal{L}^{n,3}_u, \mathcal{L}^n_v \rangle \\
= -\tau A_{11}(\mathcal{L}^n_u, \mathcal{L}^n_v) + \frac{T}{c} (\mathcal{L}^n_v - \mathcal{L}^n_u, \mathcal{L}^n_v) \\
- \tau A_{12}(\mathcal{L}^n_v, \mathcal{L}^n_u) + \frac{T}{c} (\mathcal{L}^n_v - \mathcal{L}^n_u, \mathcal{L}^n_u) \\
\leq -\frac{\beta \tau^2}{2} \|(\partial_x \mathcal{L}^n_u, \partial_x \mathcal{L}^n_u)\|^2 - \frac{T}{c} (1 - \epsilon) \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2,
\]

(4.20)

where we have used Lemma 3.1 and $\epsilon = cQh^2$ in the above inequality (4.20).

For $\Pi^2_2$, choosing $\xi = \mathcal{L}^{n,3}_v$ in (4.2c), in a similar way to get (4.17) and (4.18), we obtain

\[
\|\mathcal{L}^{n,3}_v\| \leq \frac{\tau \Gamma}{3h} \|\partial_z \mathcal{L}^n_u\| + \frac{T}{3c} (1 + cQ) \|\mathcal{L}^n_v - \mathcal{L}^n_u\|
\]

(4.21)

and

\[
\|(\mathcal{L}^{n,3}_v, \mathcal{L}^{n,3}_u)\|^2 \leq \frac{2\tau^2 \Gamma^2}{9h^2} \|\mathcal{L}^n_v, \mathcal{L}^n_u\|^2 + \frac{4\tau^2}{3c^2} (1 + cQ)^2 \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2.
\]

(4.22)

Thus, $\Pi^2_2$ can be bounded by

\[
\Pi^2_2 = 3\|(\mathcal{L}^{n,3}_v, \mathcal{L}^{n,3}_u)\|^2 \\
\leq \frac{2\tau^2 \Gamma^2}{3h^2} \|\mathcal{L}^n_v, \mathcal{L}^n_u\|^2 + \frac{4\tau^2}{3c^2} (1 + cQ)^2 \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2.
\]

(4.23)

Thus substituting (4.19), (4.20) and (4.23) into (4.12), we have

\[
\Pi_2 \leq -\left(1 - \frac{2\epsilon T}{c}\right) \|(\mathcal{L}^{n,2}_v, \mathcal{L}^{n,2}_u)\|^2 - \frac{\beta \tau}{2} \|\partial_x \mathcal{L}^{n,1}_v, \partial_x \mathcal{L}^{n,1}_u\|^2 - \left(\beta T - \frac{2\tau^2 \Gamma^2}{3h^2}\right) \|\partial_x \mathcal{L}^{n,2}_v, \partial_x \mathcal{L}^{n,2}_u\|^2 \\
+ \frac{4\tau}{c} (c + \frac{1}{\delta}) \left(\frac{\tau \Gamma^2}{h^2} \|\partial_z v^n, \partial_z u^n\|^2 + \frac{2\epsilon T}{c^2} (1 + cQ)^2 \|v^n - u^n\|^2\right) \\
- \frac{T}{c} (1 - \delta - \frac{4\tau}{3c} (1 + cQ)^2) \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2.
\]

(4.24)

Combining those estimates in (4.11) and (4.24) and collecting the common terms, we obtain

\[
\Pi_1 + \Pi_2 \leq -\tau \left(\frac{\beta}{2} - (c + \frac{1}{\delta}) \frac{4\tau^2 \Gamma^2}{c h^2}\right) \|(\partial_x v^n, \partial_x u^n)\|^2 - \frac{T}{c} \left(1 - c - (\frac{1}{\delta} + 1) cQ\right) \left(\frac{8\tau^2}{c^2}\right) \|v^n - u^n\|^2 \\
- \frac{\beta \tau}{2} \|\partial_x v^n, \partial_x u^n\|^2 + 4\|\partial_x v^n, \partial_x u^n\|^2 + \|\partial_x \mathcal{L}^{n,1}_v, \partial_x \mathcal{L}^{n,1}_u\|^2 \|\partial_x \mathcal{L}^{n,2}_v, \partial_x \mathcal{L}^{n,2}_u\|^2 \\
- \frac{T}{c} (1 - \frac{4\tau}{3c} (1 + cQ)^2) \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2 \\
- \left(\beta T - \frac{2\tau^2 \Gamma^2}{3h^2}\right) \|(\partial_x \mathcal{L}^{n,2}_v, \partial_x \mathcal{L}^{n,2}_u)\|^2 - \frac{T}{c} \left(1 - c - \frac{4 \tau}{3c} (1 + cQ)^2\right) \|\mathcal{L}^n_v - \mathcal{L}^n_u\|^2.
\]

(4.25)

Recall (4.8) we see that the desired stability will follow if each term on the the right side of the above inequality is nonpositive, this is indeed so if

\[
\begin{cases}
\beta - (c + \frac{1}{\delta}) \frac{4\tau^2 \Gamma^2}{c h^2} \geq 0, \\
1 - c - (\frac{1}{\delta} + 1) cQ \left(\frac{8\tau^2}{c^2}\right) \geq 0, \\
1 - \frac{2\epsilon T}{c} \geq 0, \\
\beta T - \frac{2\tau^2 \Gamma^2}{3h^2} \geq 0, \\
1 - c - \frac{4\tau}{3c} (1 + cQ)^2 \geq 0
\end{cases}
\]

(4.26)
for any $\delta \in (0, 1 - c)$. These are implied by (4.4) if we choose

$$c_0 = \max \left\{ \left( \frac{8cQ\Gamma^2(c + \frac{1}{2})}{\beta} \right)^{1/2}, 2(1 + cQ\Gamma) \left( \frac{2(c + \frac{1}{2})}{1 - c} \right)^{1/2}, 2c, \frac{2cQ\Gamma^2}{3\beta}, \frac{4(1 + cQ\Gamma)^2}{3(1 - c - \delta)} \right\}$$

with $\delta = \frac{1-c}{2}$.

\[\square\]

5. Error estimates

In this section, based on the stability analysis presented in section 4, we obtain the optimal $L^2$ error estimates of the fully discrete AEDG method (3.7)-(3.9). We first prepare the error function and two lemmas for later use.

5.1. Reference solution and error representation. Set $\phi^{n,0} = \phi(x,t^n)$, according to the 3-stage RK3AEDG method (3.7)-(3.9), we define the reference solutions of (1.1) as

$$\phi^{n,1} = \phi^{n,0} - \tau(\alpha \partial_x - \beta \partial^2_x)\phi^{n,0},$$

$$\phi^{n,2} = \frac{3}{4}\phi^{n,0} + \frac{1}{4}\phi^{n,1} - \frac{\tau}{4}(\alpha \partial_x - \beta \partial^2_x)\phi^{n,1}.$$  

Lemma 5.1. If $||\partial^4_t \phi||$ is bounded uniformly for any $t \in [0, T]$, we have

$$\phi(x,t^n + \tau) = \frac{1}{3}\phi^{n,0} + \frac{2}{3}\phi^{n,2} - \frac{2\tau}{3}(\alpha \partial_x - \beta \partial^2_x)\phi^{n,2} + F(n;x),$$

where $F(n;x)$ is the local truncation error in time and $||F(n;x)|| = O(\tau^4)$ uniformly for any time $t \in [0, T]$.

Proof. By Taylor’s expansion in variable $t$,

$$\phi(x,t^n + \tau) = \phi(x,t^n) + \tau\partial_x \phi(x,t^n) + \frac{\tau^2}{2}\partial^2_x \phi(x,t^n) + \frac{\tau^3}{6}\partial^3_x \phi(x,t^n) + \frac{\tau^4}{24}\partial^4_x \phi(x,t'),$$

where $t' \in (t^n, t^{n+1})$. The right hand side of (5.2) (RHS) with notations in (5.1) reduces to

$$RHS = \phi^{n,0} - \tau(\alpha \partial_x - \beta \partial^2_x)\phi^{n,0} + \frac{\tau^2}{2}(\alpha \partial_x - \beta \partial^2_x)^2\phi^{n,0} - \frac{\tau^3}{6}(\alpha \partial_x - \beta \partial^2_x)^3\phi^{n,0} + F(n;x).$$

Using the fact that $\phi(x,t^n) = \phi^{n,0}$ and $\phi^{n,0}$ is a solution of (1.1a), we have

$$F(n;x) = \frac{\tau^4}{24}\partial^4_x \phi(x,t').$$

This completes the proof. \[\square\]

Since all reference solutions $\phi^{n,i}, i = 1, 2$ are smooth in $[a, b]$, the consistency of the AEDG scheme (see [6]) yields

$$\begin{cases}
\langle \phi^{n,1}, \xi \rangle = \langle \phi^{n,0}, \xi \rangle - \tau A_{21}(\phi^{n,0}, \xi) + \frac{\tau}{\epsilon}\langle \phi^{n,0} - \phi^{n,0}, \xi \rangle, \\
\langle \phi^{n,1}, \eta \rangle = \langle \phi^{n,0}, \eta \rangle - \tau A_{12}(\phi^{n,0}, \eta) + \frac{\tau}{\epsilon}\langle \phi^{n,0} - \phi^{n,0}, \eta \rangle,
\end{cases}$$

(5.5)

$$\begin{cases}
\langle \phi^{n,2}, \xi \rangle = \frac{1}{4}\langle 3\phi^{n,0} + \phi^{n,1}, \xi \rangle - \frac{\tau}{4}A_{21}(\phi^{n,1}, \xi) + \frac{\tau}{4\epsilon}\langle \phi^{n,1} - \phi^{n,1}, \xi \rangle, \\
\langle \phi^{n,2}, \eta \rangle = \frac{1}{4}\langle 3\phi^{n,0} + \phi^{n,1}, \eta \rangle - \frac{\tau}{4}A_{12}(\phi^{n,1}, \eta) + \frac{\tau}{4\epsilon}\langle \phi^{n,1} - \phi^{n,1}, \eta \rangle,
\end{cases}$$

(5.6)
If the time step satisfies

\[
\frac{\epsilon}{\delta t} \leq C \epsilon h^2
\]

where \( (\xi, \eta) \in V_h \times U_h \) at each stage.

Noting that \( \nu^{n,0} = u^n, u^{n,0} = u^n \), we split the solution errors as follows.

\[
\phi^{n,i} - \nu^{n,i} = e_1^{n,i} - \epsilon_1^{n,i}, \quad \phi^{n,i} - u^{n,i} = e_2^{n,i} - \epsilon_2^{n,i}
\]

for \( i = 0, 1, 2 \), where

\[
e_1^{n,i} = \Pi_4 \phi^{n,i} - \nu^{n,i}, \quad \epsilon_1^{n,i} = \Pi_4 \phi^{n,i} - \phi^{n,i},
\]

\[
e_2^{n,i} = \Pi_4 \phi^{n,i} - u^{n,i}, \quad \epsilon_2^{n,i} = \Pi_4 \phi^{n,i} - \phi^{n,i}.
\]

Each equation in scheme (3.7)-(3.9) when subtracted from the corresponding relation in (5.5)-(5.7), leads to

\[
\begin{align}
\langle e_1^{n,1} - e_1^{n,0}, \xi \rangle + \tau A_21(e_2^{n,1}, \xi) &= \langle e_1^{n,1} - e_1^{n,0}, \xi \rangle + \tau A_21(e_2^{n,0}, \xi) - \frac{\tau}{\epsilon} (u^{n,0} - \nu^{n,0}, \xi), \\
\langle e_2^{n,1} - e_2^{n,0}, \eta \rangle + \tau A_12(e_1^{n,0}, \eta) &= \langle e_2^{n,1} - e_2^{n,0}, \eta \rangle + \tau A_12(e_1^{n,1}, \eta) - \frac{\tau}{\epsilon} (u^{n,0} - \nu^{n,0}, \eta), \\
\langle 4e_1^{n,2} - 3e_1^{n,0} - e_1^{n,1}, \xi \rangle + \tau A_21(e_2^{n,1}, \xi) &= \langle 4e_1^{n,1} - 3e_1^{n,0} - e_1^{n,1}, \xi \rangle \\
&\quad + \tau A_21(e_2^{n,1}, \xi) - \frac{\tau}{\epsilon} (u^{n,1} - \nu^{n,1}, \xi), \\
\langle 4e_2^{n,2} - 3e_2^{n,0} - e_2^{n,1}, \eta \rangle + \tau A_12(e_1^{n,1}, \eta) &= \langle 4e_2^{n,1} - 3e_2^{n,0} - e_2^{n,1}, \eta \rangle \\
&\quad + \tau A_12(e_1^{n,1}, \eta) - \frac{\tau}{\epsilon} (u^{n,1} - \nu^{n,1}, \eta), \\
\langle 6e_1^{n+1} - 2e_1^{n,0} - 4e_1^{n,2}, \xi \rangle + 4\tau A_21(e_2^{n,2}, \xi) &= \langle 6e_1^{n+1} - 2e_1^{n,0} - 4e_1^{n,2}, \xi \rangle + 4\tau A_21(e_2^{n,1}, \xi) \\
&\quad - \frac{4\tau}{\epsilon} (u^{n,2} - \nu^{n,2}, \xi) + \langle 6F(n; x), \xi \rangle, \\
\langle 6e_2^{n+1} - 2e_2^{n,0} - 4e_2^{n,2}, \eta \rangle + 4\tau A_12(e_1^{n,2}, \eta) &= \langle 6e_2^{n+1} - 2e_2^{n,0} - 4e_2^{n,2}, \eta \rangle + 4\tau A_12(e_1^{n,1}, \eta) \\
&\quad - \frac{4\tau}{\epsilon} (u^{n,2} - \nu^{n,2}, \eta) + \langle 6F(n; x), \eta \rangle
\end{align}
\]

for \( (\xi, \eta) \in V_h \times U_h \) at each stage, respectively.

**Lemma 5.2.** If the time step satisfies \( \epsilon \delta t \leq \epsilon = C\epsilon h^2 \), then the following inequalities hold true

\[
\begin{align}
\|e_1^{n,1}\|_2^2 - \langle e_1^{n,1}, e_1^{n,1} \rangle + \tau A_21(e_2^{n,1}, e_1^{n,1}) &= \|e_1^{n,1}\|_2^2 - \langle e_1^{n,1}, e_1^{n,1} \rangle \\
&\quad + \tau A_21(e_2^{n,1}, e_1^{n,1}) - \frac{\tau}{\epsilon} (u^{n,0} - \nu^{n,0}, e_1^{n,1}).
\end{align}
\]

Using the Young’s inequality, \( ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \) with \( \delta = 3/2 \), we have

\[
\langle e_1^{n,0}, e_1^{n,1} \rangle = \frac{3}{2} \|e_1^{n,0}\|_2^2 + \frac{1}{6} \|e_1^{n,1}\|_2^2.
\]
From Lemma 3.5 it follows

\[
\tau A_2(\epsilon^{n,0}_1, \epsilon^{n,1}_1) \leq \tau \left( \frac{\Gamma}{h} \| \partial_x \epsilon^{n,0}_2 \| + \frac{\Gamma}{h^2} (\| \epsilon^{n,0}_1 \| + \| \epsilon^{n,0}_2 \|) \right) \| \epsilon^{n,1}_1 \|
\]

\[
\leq \frac{C \tau}{h^2} (\| \epsilon^{n,0}_1 \| + \| \epsilon^{n,0}_2 \|) \| \epsilon^{n,1}_1 \|
\]

\[
\leq \frac{1}{6} \| \epsilon^{n,1}_1 \|^2 + C \| (\epsilon^{n,0}_1, \epsilon^{n,0}_2) \|^2,
\]

where we have used the inverse inequality (3.14b) in the second inequality and \( \tau \leq cQh^2/c_0 \) in the last inequality.

Again from Lemma 3.5 and the projection error in Theorem 3.2, we obtain

\[
\tau A_2(\epsilon^{n,0}_2, \epsilon^{n,1}_1) \leq \tau \left( \frac{\Gamma}{h} \| \partial_x \epsilon^{n,0}_2 \| + \frac{\Gamma}{h^2} (\| \epsilon^{n,0}_1 \| + \| \epsilon^{n,0}_2 \|) \right) \| \epsilon^{n,1}_1 \|
\]

\[
\leq \frac{C \tau}{h^2} (\| \epsilon^{n,0}_1 \| + \| \epsilon^{n,0}_2 \| + h \| \partial_x \epsilon^{n,0}_2 \|) \| \epsilon^{n,1}_1 \|
\]

\[
\leq \frac{1}{6} \| \epsilon^{n,1}_1 \|^2 + Ch^{2k+2}.
\]

Using the \( L^2 \) projection error in (3.13), we obtain

\[
\langle \epsilon^{n,1}_1 - \epsilon^{n,0}_1, \epsilon^{n,1}_1 \rangle \leq (\| \epsilon^{n,1}_1 \| + \| \epsilon^{n,0}_1 \|) \| \epsilon^{n,1}_1 \| \leq \frac{1}{6} \| \epsilon^{n,1}_1 \|^2 + Ch^{2k+2},
\]

similarly,

\[
\langle u^{n,0} - \epsilon^{n,0}_1, \epsilon^{n,1}_1 \rangle = \langle \epsilon^{n,0}_1 - \epsilon^{n,0}_1 - (\epsilon^{n,0}_2 - \epsilon^{n,0}_2), \epsilon^{n,1}_1 \rangle
\]

\[
\leq (Ch^{k+1} + \| \epsilon^{n,0}_1 \| + \| \epsilon^{n,0}_1 \|) \| \epsilon^{n,1}_1 \|
\]

\[
\leq \frac{1}{6} \| \epsilon^{n,1}_1 \|^2 + Ch^{2k+2}.
\]

Plugging (5.14)-(5.18) into (5.13) we arrive at

\[
\| \epsilon^{n,1}_1 \|^2 \leq C \| (\epsilon^{n,0}_1, \epsilon^{n,0}_2) \|^2 + Ch^{2k+2}.
\]

Likewise, we have

\[
\| \epsilon^{n,1}_1 \|^2 \leq C \| (\epsilon^{n,0}_1, \epsilon^{n,0}_2) \|^2 + Ch^{2k+2}.
\]

Taking summation of (5.19) and (5.20) leads to (5.13a). In a similar manner, we can also prove (5.12b).

5.2. Error estimates of RK3AEDG in \( L^2 \) norm. Based on the stability analysis and the error representation, we set out to derive the error estimates of scheme (3.7)-(3.9). The result stated in Theorem 2.3 can be reformulated in terms of \( (v^n, u^n) \) as follows.

**Theorem 5.3.** Let \( \phi \) be the smooth solution of (2.1) subject to initial data \( \phi_0(x) \) and periodic boundary conditions, \( (v^n, u^n) \in V_h \times U_h \) be the numerical solution computed through the fully discrete scheme (3.7)-(3.9), then we have the following error estimate:

\[
\| \phi(t^n, \cdot) - \n^{n} \| + \| \phi(\cdot, t^n) - u^n(\cdot) \| \leq C(\tau^3 + h^{k+1}), \quad n\tau \leq T,
\]

where \( C \) is a constant uniformly in terms of \( \tau, h \), and \( n \).

**Proof.** Taking \( w = \phi^{n,i} \) with \( \xi = e^{n,i}_1 \) in (3.10), and \( \eta = e^{n,i}_2 \) in (3.11) for \( i = 0, 1, 2 \), respectively, we have

\[
\langle e^{n,i}_1, e^{n,i}_1 \rangle + A_{21}(e^{n,i}_2, e^{n,i}_1) = \frac{1}{\epsilon} (e^{n,i}_2 - e^{n,i}_1, e^{n,i}_1),
\]

\[
\langle e^{n,i}_2, e^{n,i}_2 \rangle + A_{12}(e^{n,i}_1, e^{n,i}_2) = \frac{1}{\epsilon} (e^{n,i}_1 - e^{n,i}_2, e^{n,i}_2).
\]
This together with $v^{n,i} - u^{n,i} = \epsilon v^{n,i} - \epsilon u^{n,i} = (\epsilon v^{n,i} - \epsilon u^{n,i})$ gives

$$A_{21}(e_{1}^{n,i}, e_{1}^{n,i}) + A_{12}(e_{1}^{n,i}, e_{2}^{n,i}) = \frac{1}{\epsilon} (\epsilon v^{n,i} - \epsilon u^{n,i}, e_{1}^{n,i} - e_{2}^{n,i}) - \langle e_{1}^{n,i}, e_{1}^{n,i} \rangle - (\epsilon v^{n,i} - \epsilon u^{n,i}) = -\frac{1}{\epsilon} \|e_{1}^{n,i} - e_{2}^{n,i}\|^2 - \frac{1}{\epsilon} \langle v^{n,i} - u^{n,i}, e_{1}^{n,i} - e_{2}^{n,i} \rangle - \langle e_{1}^{n,i}, e_{1}^{n,i} \rangle - (\epsilon v^{n,i} - \epsilon u^{n,i}),$$

which is equivalent to

(5.22)

$$A_{21}(e_{1}^{n,i}, e_{1}^{n,i}) + A_{12}(e_{1}^{n,i}, e_{2}^{n,i}) + \frac{1}{\epsilon} (v^{n,i} - u^{n,i}, e_{1}^{n,i} - e_{2}^{n,i}) = -\frac{1}{\epsilon} \|e_{1}^{n,i} - e_{2}^{n,i}\|^2 - \langle e_{1}^{n,i}, e_{1}^{n,i} \rangle - (\epsilon v^{n,i} - \epsilon u^{n,i}).$$

Taking $(\xi, \eta) = (e_{1}^{n,0}, e_{2}^{n,0})$, $(\xi, \eta) = (e_{1}^{n,1}, e_{2}^{n,1})$ and $(\xi, \eta) = (e_{1}^{n,2}, e_{2}^{n,2})$ in each stage of (5.9)-(5.11), respectively, summing the result up, noticing (5.22), according to the stability analysis (4.8), we obtain

(5.23)

$$3\|e_{1}^{n+1}, e_{2}^{n+1}\|^2 = 3\|e_{1}^{n}, e_{2}^{n}\|^2 + \Pi_{1}' + \Pi_{2}' + G_{1} + G_{2},$$

where

(5.24)

$$\Pi_{1}' = -\tau \left( A_{21}(e_{1}^{n,0}, e_{1}^{n,0}) + A_{21}(e_{1}^{n,1}, e_{1}^{n,1}) + 4A_{21}(e_{1}^{n,2}, e_{1}^{n,2}) \right) - \tau \left( A_{12}(e_{1}^{n,0}, e_{2}^{n,0}) + A_{12}(e_{1}^{n,1}, e_{2}^{n,1}) + 4A_{12}(e_{1}^{n,2}, e_{2}^{n,2}) \right) - \frac{\tau}{\epsilon} \|e_{1}^{n,0} - e_{2}^{n,0}\|^2 - \frac{\tau}{\epsilon} \|e_{1}^{n,1} - e_{2}^{n,1}\|^2 - \frac{4\tau}{\epsilon} \|e_{1}^{n,2} - e_{2}^{n,2}\|^2,$$

(5.25)

$$\Pi_{2}' = \int_{a}^{b} \left[ (2\epsilon_{1}^{n,2} - \epsilon_{1}^{n,1} - \epsilon_{1}^{n,0})^2 + 3(\epsilon_{1}^{n+1} - \epsilon_{1}^{n}) \right] dx + \int_{a}^{b} \left[ (2\epsilon_{2}^{n,2} - \epsilon_{2}^{n,1} - \epsilon_{2}^{n,0})^2 + 3(\epsilon_{2}^{n+1} - \epsilon_{2}^{n}) \right] dx,$$

(5.26)

$$G_{1} = \langle e_{1}^{n,1} - e_{1}^{n,0}, e_{1}^{n,0} \rangle + \langle e_{2}^{n,1} - e_{2}^{n,0}, e_{2}^{n,0} \rangle + \langle 4e_{1}^{n,2} - 3e_{1}^{n,0} - \epsilon_{1}^{n,1}, e_{1}^{n,1} \rangle + \langle 4e_{2}^{n,2} - 3e_{2}^{n,0} - \epsilon_{2}^{n,1}, e_{2}^{n,1} \rangle + \langle 6e_{1}^{n,1} - 2\epsilon_{1}^{n,0} - 4\epsilon_{1}^{n,2}, e_{1}^{n,1} \rangle + \langle 6e_{2}^{n,1} - 2\epsilon_{2}^{n,0} - 4\epsilon_{2}^{n,2}, e_{2}^{n,1} \rangle,$$

and

(5.27)

$$G_{2} = -\tau \sum_{i=0}^{1} \langle e_{1}^{n,i}, e_{1}^{n,i} \rangle + \langle e_{2}^{n,i}, e_{2}^{n,i} \rangle - 4\tau \langle e_{1}^{n,2}, e_{1}^{n,2} \rangle - 4\tau \langle e_{2}^{n,2}, e_{2}^{n,2} \rangle + (6F(n,x), e_{1}^{n,1} + e_{2}^{n,1}).$$

From the stability analysis in leading to (4.8) and (4.25), under the stability condition (4.4), we have

(5.28)

$$\Pi_{1}' + \Pi_{2}' \leq 0.$$

Noticing the fact that $\phi^{n,0}$ is a solution of (1.1), using (5.1) and the Taylor expansion, we obtain

(5.29a)

$$\phi^{n,1} - \phi^{n,0} = -\tau (\alpha \partial_{x} - \beta \partial_{x}^{2}) \phi^{n,0} = \tau \partial_{t} \phi^{n,0},$$

(5.29b)

$$4\phi^{n,2} - 3\phi^{n,0} - \phi^{n,1} = -\tau (\alpha \partial_{x} - \beta \partial_{x}^{2}) \phi^{n,1}$$

$$= -\tau (\alpha \partial_{x} - \beta \partial_{x}^{2}) \phi^{n,0} + \tau^{2} (\alpha \partial_{x} - \beta \partial_{x}^{2})^{2} \phi^{n,0},$$

(5.29c)

$$3\phi^{n+1} - \phi^{n,0} - 2\phi^{n,2} = 3\phi^{n+1} - 3\phi^{n,0} + \tau (\alpha \partial_{x} - \beta \partial_{x}^{2}) \phi^{n,0} - \frac{\tau^{2}}{2} (\alpha \partial_{x} - \beta \partial_{x}^{2})^{2} \phi^{n,0}$$

$$= 2\tau (\alpha \partial_{x} - \beta \partial_{x}^{2}) \phi^{n,0} - \tau^{2} (\alpha \partial_{x} - \beta \partial_{x}^{2})^{2} \phi^{n,0} - \frac{\tau^{3}}{2} (\alpha \partial_{x} - \beta \partial_{x}^{2})^{3} \phi^{n,0}.$$
Applying the projection error (3.13) and (5.29), we have

\begin{align}
(5.30a) \quad \|e_1^{n,i} - e_1^{n,0}\| &= \| \Pi_v (\phi^{n,1} - \phi^{n,0}) - (\phi^{n,1} - \phi^{n,0}) \| \\
&\leq C \tau h^{\min\{k+1,m\}} \| \partial_t \phi^{n,0} \|_{m} \leq C \tau h^{\min\{k+1,m\}} \| \phi^{n,0} \|_{m+2},
\end{align}

\begin{align}
(5.30b) \quad \|4e_1^{n,2} - 3e_1^{n,0} - e_1^{n,1}\| &= \| \Pi_v (4\phi^{n,0} - 3\phi^{n,0} - \phi^{n,1}) - (4\phi^{n,0} - 3\phi^{n,0} - \phi^{n,1}) \| \\
&\leq C \tau h^{\min\{k+1,m\}} \| \phi^{n,0} \|_{m+4},
\end{align}

\begin{align}
(5.30c) \quad \|6e_1^{n+1} - 2e_1^{n,0} - 4e_1^{n,2}\| &= \| \Pi_v (6\phi^{n+1} - 2\phi^{n,0} - 4\phi^{n,2}) - (6\phi^{n+1} - 2\phi^{n,0} - 4\phi^{n,2}) \| \\
&\leq C \tau h^{\min\{k+1,m\}} \| \phi^{n,0} \|_{m+6}.
\end{align}

Same estimates as (5.30a)-(5.30c) hold true for \(e_2^{n,i}\).

From the Young’s inequality, for \(w \in L^2\), we have

\begin{equation}
(5.31) \quad \langle w, e_j^{n,i} \rangle \leq \| e_j^{n,i} \|^2 + \frac{1}{4} \| w \|^2,
\end{equation}

where \(i = 0, 1, 2\) and \(j = 1, 2\).

This when applied to each term in \(G_1\) and taking \(m = k + 1\), together with (5.30a-c) gives

\begin{equation}
(5.32) \quad |G_1| \leq \sum_{i=0}^{2} \tau \| (e_1^{n,i}, e_2^{n,i}) \|^2 + C \tau h^{2k+2}.
\end{equation}

Now we turn to the estimate of \(G_2\), again from (5.31),

\begin{equation}
(5.33) \quad |G_2| \leq \sum_{i=0}^{2} \| (e_1^{n,i}, e_2^{n,i}) \|^2 + \frac{\tau}{4} \left( \sum_{i=0}^{2} \| (e_1^{n,i}, e_2^{n,i}) \|^2 + 16 \| (e_1^{n,i}, e_2^{n,i}) \|^2 + \frac{1}{\tau^2} \| F(n; \cdot) \|^2 \right)
\end{equation}

\begin{equation}
\leq \sum_{i=0}^{2} \| (e_1^{n,i}, e_2^{n,i}) \|^2 + C \tau (h^{2k+2} + \tau^6)
\end{equation}

thus, collecting the estimate in (5.32) and (5.33), it is easy to see that

\begin{equation}
(5.34) \quad |G_1 + G_2| \leq 2\tau \sum_{i=0}^{2} \| (e_1^{n,i}, e_2^{n,i}) \|^2 + C \tau (h^{2k+2} + \tau^6)
\end{equation}

\begin{equation}
\leq C \tau \| (e_1^{n,0}, e_2^{n,0}) \|^2 + C \tau (h^{2k+2} + \tau^6),
\end{equation}

where we have used (5.12) in the last inequality.

Plugging (5.28) and (5.34) into (5.23) leads to

\begin{equation}
\| (e_1^{n+1}, e_2^{n+1}) \|^2 \leq (1 + C \tau) \| (e_1^{n}, e_2^{n}) \|^2 + C \tau (h^{2k+2} + \tau^6),
\end{equation}

this gives

\begin{equation}
\| (e_1^{n}, e_2^{n}) \| \leq (1 + C \tau)^{n/2} \| (e_1^{0}, e_2^{0}) \| + (1 + C \tau)^{n/2} (h^{k+1} + \tau^3).
\end{equation}

From the choice of the initial data in (2.5), projection error in Theorem 3.2 and local approximation property in Lemma 3.3, we have

\begin{equation}
(5.35) \quad \| e_1^{0} \| = \| e_1(\cdot, 0) \| = \| \Pi_v \phi_0 - v(\cdot, 0) \| \leq \| \Pi_v \phi_0 - \phi_0 \| + \| \phi_0 - v(\cdot, 0) \| \leq C h^{k+1},
\end{equation}

similarly \( \| e_2^{0} \| \leq C h^{k+1} \).

Further using the initial error as given in (5.35), so that

\begin{equation}
\| (e_1^{n}, e_2^{n}) \| \leq C (h^{k+1} + \tau^3),
\end{equation}

This together with the projection error for \( e_1^{n} \) when inserted into (5.8) yields the desired estimate (5.21). \( \Box \)
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