ALTERNATING EVOLUTION SCHEMES FOR HAMILTON-JACOBI EQUATIONS

HAILIANG LIU, HASEENA SARAN AND MICHAEL POLLACK

Abstract. In this work, we propose a high resolution Alternating Evolution (AE) scheme to solve Hamilton-Jacobi equations. The construction of the AE scheme is based on an alternating evolution system of the Hamilton-Jacobi equation, following the idea previously developed for hyperbolic conservation laws. A semi-discrete scheme derives directly from a sampling of this system on alternating grids. Higher order accuracy is achieved by a combination of high-order non-oscillatory polynomial reconstruction from the obtained grid values and a time discretization with matching accuracy. Local AE schemes are made possible by choosing the scale parameter $\epsilon$ to reflect the local distribution of waves. The AE schemes have the advantage of easy formulation and implementation, and efficient computation of the solution. For the first and second order local AE schemes applied to Hamilton-Jacobi equations, we prove the numerical stability in the sense of satisfying the maximum principle. Numerical experiments for a set of Hamilton-Jacobi equations are presented to demonstrate both accuracy and capacity of these AE schemes.

1. Introduction

In this paper, we develop a new alternating evolution (AE) method to solve time-dependent Hamilton-Jacobi (HJ) equations. We describe the designing principle of our new AE scheme through the following form:

$$\phi_t + H(x, \nabla_x \phi) = 0, \quad \phi(x, 0) = \phi_0(x), \quad x \in \mathbb{R}^d, \quad t > 0.$$ 

Here, $d$ is the space dimension, $\phi \in \mathbb{R}^1$ is the unknown, and $H : \mathbb{R}^d \to \mathbb{R}^1$ is a nonlinear Hamiltonian. The Hamilton-Jacobi equation arises in many applications ranging from geometrical optics to differential games. These nonlinear equations typically develop discontinuous derivatives even with smooth initial conditions, the solutions of which are non-unique. In this paper, we are only interested in the viscosity solution [7, 6, 27], which is the unique physically relevant solution in some important applications.

The difficulty encountered for the satisfactory approximation of the exact solutions of these equations lies in the presence of discontinuities in the solution derivatives. An important class of finite difference
methods for computing the viscosity solution is the class of monotone schemes introduced by Crandall-Lions [8]. Unfortunately, monotone schemes are at most first-order accurate. The need for devising more accurate numerical methods for Hamilton-Jacobi equations has prompted the abundant research in this area in the last two decades, including essentially non-oscillatory (ENO) or weighted ENO (WENO) finite difference schemes, see e.g. [22, 23, 13, 17, 26, 29], and Central or Central-Upwind finite difference schemes, see e.g. [18, 16, 15, 4, 2, 3], as well as discontinuous Galerkin methods [12, 5, 28, 20].

The preliminary goal of this paper is to design symmetric high order finite difference schemes for Hamilton-Jacobi equations by following the alternating evolution framework introduced in [19] for hyperbolic conservation laws; see local AE schemes developed by Saran and Liu [24]. The global AE scheme [19] shares similar features to central schemes by Y. Liu [21] using overlapping cells. In such a framework, the general setting is to refine the original PDE by an alternating evolution (AE) system which involves two representatives: \{u, v\}. In the evolution of u, terms involving spatial derivatives are replaced by v’s derivatives and augmented by an additional relaxation term \((v-u)/\epsilon\), which serves to communicate the two representatives. The AE system for scalar hyperbolic conservation laws was shown in [19] to be capable of capturing the exact solution when initially both representatives are chosen as the given initial data. Such a feature allows for a sampling of two representatives over alternating grids. Using this alternating system as a ‘building base’, we apply standard approximation techniques to the AE system: high order accuracy is achieved by a combination of high-order non-oscillatory polynomial reconstruction in space and an ODE solver in time with matching accuracy, following those implemented for hyperbolic conservation laws in [24]. Preliminary results on AE schemes for Hamilton-Jacobi equations were reported in the second author’s thesis [1].

For Hamilton-Jacobi equations, we consider the following alternating evolution system

\[
\begin{align*}
    u_t + H(x, \nabla_x u) &= \frac{1}{\epsilon}(v - u), \\
    v_t + H(x, \nabla_x v) &= \frac{1}{\epsilon}(u - v).
\end{align*}
\]

Here, \(\epsilon > 0\) is a scale parameter of user’s choice. At \(t = 0\), we take the initial data

\[
u_0(x) = v_0(x) = \phi_0(x).
\]

We sample the above system over alternating grids when performing the spatial discretization, leading to a class of high resolution semi-discrete AE schemes. One of the main differences in the formulation of AE schemes for Hamilton-Jacobi equations to that for conservation laws is that here we use grid point values instead of cell averages.

The semi-discrete schemes thus obtained has the form

\[
\frac{d}{dt} \Phi_\alpha + H(x_\alpha, \nabla x \Phi(x_\alpha)) + \frac{1}{\epsilon} \Phi_\alpha = -\frac{1}{\epsilon} p_\alpha(\Phi(x_\alpha)),
\]

where \(p_\alpha(\Phi(x))\) is the polynomial reconstruction based on grid values of \(\Phi\) in the domain centered at \(x_\alpha\). The scale parameter \(\epsilon\) and the time step size \(\Delta t\) are chosen to stabilize the time discretization. In the one-dimensional case, the fully discrete AE scheme of first order becomes

\[
\Phi_{k+1}^{n+1} = \left(1 - \frac{\Delta t}{\epsilon}\right) \Phi_k^n + \frac{\Delta t}{\epsilon} \left[\frac{1}{2} (\Phi_{k+1}^n + \Phi_{k-1}^n) - \epsilon H \left(x_k, \frac{\Phi_{k+1}^n - \Phi_{k-1}^n}{2\Delta x}\right)\right], \quad \epsilon \geq \Delta t,
\]

which when \(\epsilon = \Delta t\) is taken, reduces to the celebrated Lax-Friedrichs scheme

\[
\Phi_{k+1}^n = \frac{1}{2} (\Phi_{k+1}^n + \Phi_{k-1}^n) - \epsilon H \left(x_k, \frac{\Phi_{k+1}^n - \Phi_{k-1}^n}{2\Delta x}\right).
\]

Thus the class of AE schemes can be viewed as a natural high order extension of the Lax-Friedrichs scheme. The main feature of our scheme is its simplicity and easy formulation of semi-discrete schemes of any desired order.

It should be noted that even though our AE schemes are derived based on sampling the alternating evolution system, we do not solve the system directly. The AE system simply provides a systematic
way for developing computation schemes of both semi-discrete and fully discrete form for the original problem, instead of as an approximation system at the continuous level.

The article is organized as follows: in section 2, we formulate the AE method for one dimensional Hamilton-Jacobi equations. We then give a rigorous proof of $L^\infty$ stability by the AE method of both first and second order in section 3. Extensions to the multi-D case is given in section 4, including stability for the AE method of first and second order. In section 5, we show numerical results, which includes both one and two dimensional problems. The results illustrate accuracy, efficiency, and high resolution near kinks. The last section 6 ends this paper with our concluding remarks.

2. Alternating evolution methods

Our new numerical schemes for HJ equations consist of a semi-discrete formulation based on sampling of the AE system on alternating grids and a fully discrete version by further using an appropriate Runge-Kutta solver.

To illustrate, we start with the one-dimensional Hamilton-Jacobi equation of the form

$$\phi_t + H(x, \phi_x) = 0.$$  \hfill (2.1)

The ‘building base’ is the following AE system

$$u_t + H(x, u_x) = \frac{1}{\epsilon} (v - u), \quad \hfill (2.2)$$

2.1. Semi-discrete formulation. Consider a uniform discretization \{x_k, k \in \mathbb{Z}\} with grid size $\Delta x$. We sample the AE approximate system (2.1)-(2.2) at the even and odd grid points respectively to obtain

$$\frac{d}{dt} u_{2i}(t) = -\frac{1}{\epsilon} u_{2i} + \frac{1}{\epsilon} L[v](x_{2i}, t),$$

$$\frac{d}{dt} v_{2i+1}(t) = -\frac{1}{\epsilon} v_{2i+1} + \frac{1}{\epsilon} L[u](x_{2i+1}, t),$$

where

$$L[\Phi](x, t) = \Phi - \epsilon H(x, \Phi_x).$$

We assume that we have computed the solution at $t$, denoted by

$$\Phi_k = \begin{cases} u_k, & k = 2i, \\ v_k, & k = 2i + 1. \end{cases}$$

To increase the order of accuracy, we reconstruct a continuous, piecewise polynomial $p_k[\Phi](x)$ over each interval $I_k := [x_{k-1}, x_{k+1}]$ such that

$$p_k[\Phi](x_{k\pm1}) = \Phi_{k\pm1}.$$  \hfill (2.3)

The numerical approximation for $L$ is then realized by $L[p_k[\Phi]](x_k, t)$, which leads to the following semi-discrete scheme

The fully discrete scheme follows from applying an appropriate Runge-Kutta solver to (2.3). For a computational domain $[a, b]$ with $x_0 = a, x_N = b$ and $\Delta x = (b - a)/N$, we summarize the algorithm as follows:

Algorithm 2.1.

1. Initialization: at any node, compute the initial data as $\Phi_k^0 = \phi_0(x_k)$, $k = 0, 1, \ldots, N.$
2. Reconstruction: from \( \{ \Phi_k \} \) construct \((r + 1)\)th order non-oscillatory polynomial \( p_k[\Phi](x) \) and \( r \)th order polynomial \( \partial_x p_k[\Phi](x) \) on \( I_k \), \( k = 1, 2, \ldots, N \), then let

\[
L_k[\Phi] = p_k[\Phi](x_k) - \epsilon H(x_k, \partial_x p_k[\Phi](x_k)), \quad k = 1, 2, \ldots, N.
\]

3. Evolution: obtain \( \Phi^{n+1} \) from \( \Phi^n \) by the following TVD Runge-Kutta type procedure [23]:

\[
\Phi_k^{(l)} = \sum_{i=0}^{l-1} (\alpha_{li} \Phi_k^{(i)}) - \frac{\beta_{li}}{\epsilon} \left[ \Phi_k^{(i)} + L_k[\Phi^{(i)}] \right], \quad l = 1, \ldots, r,
\]

\[
\Phi_k^{(0)} = \Phi_k^{(n)}, \quad \Phi_k^{n+1} = \Phi_k^{(r)}.
\]

In the AE schemes up to the third order, \( \epsilon \) is chosen such that the stability condition,

\[
\Delta t \leq \epsilon \leq Q \frac{\Delta x}{\max |H_p(x,p)|},
\]

is satisfied. The choice of \( Q \) depends on the order of the scheme, see (3.1) and (3.4) in Section 3.

2.2. Fully discrete AE schemes. We now present AE schemes of the first, second and third order for HJ equations, and then discuss advantages of using a local parameter \( \epsilon = \epsilon_j \).

2.2.1. First order scheme. The first order scheme requires a continuous linear polynomial reconstruction in \( I_k \) using \( \Phi_{k\pm 1} \), which is,

\[
p_k^1[\Phi](x) = \Phi_{k-1} + s_k(x - x_{k-1}), \quad s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x},
\]

and hence,

\[
p_k^1[\Phi](x_k) = \frac{\Phi_{k+1} + \Phi_{k-1}}{2} \quad \text{and} \quad \partial_x p_k^1[\Phi](x_k) = s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x}.
\]

This, when combined with a forward Euler in time discretization gives the first order numerical scheme

\[
\Phi_k^{n+1} = (1 - \kappa)\Phi_k^n + \kappa \left[ \frac{1}{2}(\Phi_{k+1}^n + \Phi_{k-1}^n) - \epsilon H(x_k, \frac{\Phi_{k+1}^n - \Phi_{k-1}^n}{2\Delta x}) \right],
\]

where \( \kappa := \frac{\Delta t}{\epsilon} < 1 \). When \( \epsilon \leq \frac{\Delta x}{\max |H_p(x)|} \), this forms a class of monotone schemes, with the celebrated Lax-Friedrichs scheme being a special case of \( \kappa = 1 \).

2.2.2. Second order scheme. A second order continuous, piecewise quadratic reconstruction gives

\[
p_k^2[\Phi](x) = \Phi_{k-1} + s_k(x - x_{k-1}) + \frac{s_k'}{2}(x - x_{k-1})(x - x_{k+1}),
\]

where \( s_k \) is the first numerical derivative defined as before and \( s_k' \) is an approximation to the exact second derivative \( \partial_x s_k \). We then obtain,

\[
p_k^2[\Phi](x_k) = \frac{\Phi_{k+1} + \Phi_{k-1}}{2} - \frac{s_k'}{2}(\Delta x)^2 \quad \text{and} \quad \partial_x p_k^2[\Phi](x_k) = s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x},
\]

which when combined with the second order Runge-Kutta method (Heun’s method) gives,

\[
\Phi_k^{*} = (1 - \kappa)\Phi_k^n + \kappa L_k[\Phi^n],
\]

(2.5)

\[
\Phi_k^{n+1} = \frac{1}{2}\Phi_k^n + \left( \frac{1 - \kappa}{2} \right) \Phi_k^{*} + \frac{\kappa}{2} L_k[\Phi^{*}],
\]

(2.6)

where

\[
L_k[\Phi] = \left[ \frac{1}{2}(\Phi_{k-1}^n + \Phi_{k+1}^n) - \frac{(\Delta x)^2}{2} s_k \right] - \epsilon H(x_k, \frac{\Phi_{k+1}^n - \Phi_{k-1}^n}{2\Delta x}),
\]

(2.7)
Here, the standard minmod function is defined as
\[ s_k' = \minmod \left\{ \frac{s_{k+2} - s_k}{2\Delta x}, \frac{s_k - s_{k-2}}{2\Delta x} \right\}. \]

Here, the standard minmod function is defined as
\[ \minmod(a, b) = \frac{1}{2} (\text{sign}(a) + \text{sign}(b)) \min(|a|, |b|). \]

2.2.3. Third order scheme. We formulate the third order scheme by using a cubic polynomial reconstruction. In order that the solution is non-oscillatory, we obtain the reconstructed polynomials by using the ENO interpolation technique [11, 23]. A polynomial reconstruction on the cell \( I_k \) is given as
\[ p_k^3[\Phi](x) = \Phi_{k-1} + s_k(x-x_{k-1}) + \frac{s_k}{2}(x-x_{k-1})(x-x_{k+1}) + \frac{s_k''}{6}(x-x_{k-1})(x-x_{k+1})(x-x^*), \]
with \( x^* = x_{k-3} \) or \( x^* = x_{k+3} \) depending on which grid point value is used in obtaining the second numerical derivative in the ENO procedure. This gives,
\[ \partial_x p_k^3[\Phi](x_k) = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x} - \frac{s_k''}{6}(\Delta x)^2, \]
which when combined with the third order Runge-Kutta method gives,
\[ \Phi_k^{(1)} = (1 - \kappa) \Phi_k^n + \kappa L_k[\Phi^n], \]
\[ \Phi_k^{(2)} = \frac{3}{4} \Phi_k^n + \frac{1}{4} (1 - \kappa) \Phi_k^{(1)} + \frac{1}{4} \kappa L_k[\Phi^{(1)}], \]
\[ \Phi_k^{n+1} = \frac{1}{3} \Phi_k^n + \frac{2}{3} (1 - \kappa) \Phi_k^{(2)} + \frac{2}{3} \kappa L_k[\Phi^{(2)}], \]
with
\[ L_k[\Phi] = \left[ \frac{1}{2} (\Phi_{k-1} + \Phi_{k+1}) - \frac{s_k'}{2}(\Delta x)^2 - \frac{s_k''}{6}(\Delta x)^2(x_k - x^*) \right] - \epsilon H(\Phi, s_k - \frac{s_k''}{6}(\Delta x)^2). \]

2.3. Local AE schemes. In practice, it is preferred to use local AE schemes, in which local speeds instead of global speeds are explored in the scheme formulation for \( \epsilon \). The notation \( M_k \) is used to denote the range
\[ M_k = \left[ \min_{I_k} \partial_x P[\Phi](x), \max_{I_k} \partial_x P[\Phi](x) \right], \]
where \( I_k = [x_{k-1}, x_{k+1}] \). Depending on the way \( \epsilon_k \) is defined, we can formulate two local AE schemes.

1. Local AE1 scheme: We choose \( \epsilon_k \) such that
\[ \epsilon_k \leq Q \max_{p \in M_k, x \in I_k} \frac{\Delta x}{|H_p(x, p)|}. \]

2. Local AE2 scheme: We choose \( \epsilon_k \) such that
\[ \epsilon_k \leq Q \max_{x \in I_k} \frac{\Delta x}{\max_{p \in M_k} \int_{M_k} |H_p(x, p)| dp}. \]

In both cases we want the time step
\[ \Delta t < \min_k \epsilon_k. \]
since the stability conditions require that $\kappa = \frac{\Delta t}{\epsilon_k} < 1$. $Q$ is a factor that is dependent on the order of the scheme and the stability conditions to be presented in Section 3 provide the range of values $Q$ can take.

3. Stability analysis

In this section we show that the first and second order schemes are non-oscillatory in the sense of satisfying the maximum principle. Let $\Phi^n$ be a computed solution and $|\Phi^n|_\infty = \max_k \{|\Phi^n_k|\}$ define the standard $l^\infty$ norm. For simplicity we present the stability results only for the case $H(x,p) = H(p)$.

We first prove stability for global AE schemes. The proofs for local schemes are similar; only $\epsilon$ needs to be considered as defined locally instead of globally.

**Theorem 3.1.** Let $\Phi$ be computed from the first order AE scheme (2.4) for the Hamilton-Jacobi equation $\dot{\phi} + H(\phi_x) = 0$. If

$$\frac{\epsilon}{\Delta x} \max |H'(\cdot)| \leq 1 \quad \text{and} \quad \Delta t < \epsilon,$$

(3.1)

hold, then

$$\min_k \Phi^n_k \leq \Phi^{n+1}_k - H(0) \Delta t \leq \max_k \Phi^n_k, \quad n \in \mathbb{N}.$$  

(3.2)

**Proof.** The scheme (2.4) can be written as

$$\Phi^{n+1}_k = (1 - \kappa) \Phi^n_k + \kappa L_k[\Phi], \quad L_k[\Phi] = \frac{\Phi^n_k - \Phi^n_{k+1}}{2} - \epsilon H \left( \frac{\Phi^n_{k+1} - \Phi^n_{k-1}}{2\Delta x} \right).$$

When $\frac{\epsilon}{\Delta x} \max |H'| \leq 1$, $L_k[\Phi]$ becomes non-decreasing in both $\Phi_{k-1}$ and $\Phi_{k+1}$, thus we have

$$\min_k \Phi_k - \epsilon H(0) = L_k[\min_k \Phi_k] \leq L_k[\Phi] \leq L_k[\max_k \Phi_k] = \max_k \Phi_k - \epsilon H(0).$$

Note that for $\kappa < 1, \Phi^{n+1}_k$ is a convex combination of $\Phi^n_k$ and $L_k[\Phi^n]$, therefore the claimed estimate (3.2) follows.

Without loss of generality we can assume $H(0) = 0$, (this can be made so by a transform of $\phi - H(0)t$ in the original equation) so that the estimate (3.2) becomes

$$|\Phi^{n+1}|_\infty \leq |\Phi^n|_\infty.$$  

We now proceed to prove such an estimate for second order AE schemes. In order for the second order scheme to still preserve the maximum principle, we modify the coefficients of the reconstructed polynomials and define the second numerical derivative to be

$$s'_k = \frac{1}{2\Delta x} \min\{s_{k+2}, s_k, s_{k-2}\}.$$  

With such a choice, the stability result can be established as follows.

**Theorem 3.2.** Let $\Phi^n$ be computed from the second order AE scheme (2.5) for Hamilton-Jacobi equations with $H(0) = 0$ and with slopes $s'_k$ defined as in (3.3). If

$$\frac{\epsilon}{\Delta x} \max |H'(\cdot)| \leq \frac{3}{4} \quad \text{and} \quad \Delta t < \epsilon,$$

(3.4)

hold, then

$$|\Phi^{n+1}|_\infty \leq |\Phi^n|_\infty, \quad n \in \mathbb{N}.$$  

(3.5)

**Proof.** It suffices to show that

$$|L[\Phi^n]|_\infty \leq |\Phi^n|_\infty,$$

(3.6)

so that when $\kappa < 1$,

$$|\Phi^*|_\infty \leq (1 - \kappa)|\Phi^n|_\infty + \kappa|\Phi^n|_\infty \leq |\Phi^n|_\infty.$$
Then, from equation (2.6), it follows that

$$|\Phi^{n+1}|_\infty \leq \frac{1}{2}|\Phi^n|_\infty + \left(\frac{1 - \kappa}{2}\right)|\Phi^*|_\infty + \frac{\kappa}{2}|L[\Phi^*]|_\infty$$

from which the maximum principle as claimed in (3.5) follows.

We now prove (3.6). From (2.7) we have that

$$L_k[\Phi^n] = \frac{1}{2}(\Phi^n_{k+1} + \Phi^n_{k-1}) - \frac{s_k'}{2}(\Delta x)^2 - \epsilon H\left(\frac{\Phi^n_{k+1} - \Phi^n_{k-1}}{2\Delta x}\right).$$

Notice that $H(0) = 0$ and define

$$\beta^n_k := \begin{cases} -\frac{(\Delta x)^2 s_k'}{\Phi^n_{k+1} - \Phi^n_{k-1}} & \text{if } \Phi^n_{k+1} \neq \Phi^n_{k-1}, \\ 0 & \text{if } \Phi^n_{k+1} = \Phi^n_{k-1}. \end{cases}$$

Hence,

$$L_k[\Phi^n] = \frac{1}{2}(\Phi^n_{k+1} + \Phi^n_{k-1}) + \frac{\beta^n_k}{2}(\Phi^n_{k+1} - \Phi^n_{k-1}) - \frac{\epsilon}{2\Delta x}H'(\cdot)(\Phi^n_{k+1} - \Phi^n_{k-1})$$

$$= \frac{1}{2}\left(1 + \beta^n_k - \frac{\epsilon}{\Delta x}H'(\cdot)\right)\Phi^n_{k+1} + \frac{1}{2}\left(1 - \beta^n_k + \frac{\epsilon}{\Delta x}H'(\cdot)\right)\Phi^n_{k-1},$$

where $(\cdot)$ is an unspecified intermediate value between 0 and $s_k$. The minmod property on the second numerical derivative $s_k'$ leads to

$$s_k' \leq \frac{|s_k|}{2\Delta x} = \frac{1}{4(\Delta x)^2} |\Phi^n_{k+1} - \Phi^n_{k}|,$$

and hence, $\beta^n_k \leq \frac{1}{4}$. This together with the condition $\frac{\epsilon}{\Delta x} \max |H'| \leq \frac{3}{4}$ ensures that all the coefficients on the right of (3.7) are non-negative and we can take the maximum norm to obtain

$$|L[\Phi^n]|_\infty \leq |\Phi^n|_\infty,$$

as claimed.

### 4. The Multi-Dimensional Case

By similar procedures we can construct AE schemes for multi-dimensional HJ equations:

$$\phi_t + H(x, \nabla x \phi) = 0, \quad x \in \mathbb{R}^d.$$

We start with the AE formulation

$$u_t + \frac{1}{\epsilon} u = \frac{1}{\epsilon} v - H(x, \nabla_x v).$$

Let $\{x_\alpha\}$ be uniformly distributed grids in $\mathbb{R}^d$. Consider $I_\alpha$ be a hypercube centered at $x_\alpha$ with vertices at $x_\alpha \pm 1$ where the number of vertices is $2^d$. Given grid values $\{\Phi_\alpha\}$, we construct a continuous, piecewise polynomial $p_\alpha[\Phi](x) \in P_r$ defined in $I_\alpha$ such that

$$p_\alpha[\Phi](x_\alpha \pm 1) = \Phi_\alpha \pm 1.$$

These polynomials are then put into the right hand side of (4.1). Here $P_r$ denotes a linear space of all polynomials of degree at most $r$ in all $x_i$:

$$P_r := \{p \mid p(x) = \sum_{0 \leq \beta \leq r} a_\beta x^\beta, \quad 1 \leq i \leq d, \quad a_\beta \in \mathbb{R}\}.$$
Note \( \text{dim}(P_r) = (r+1)^d \). Sampling the AE system (4.1) at \( x_\alpha \), which is the common vertex of \( I_{\alpha \pm 1} \), we obtain the semi-discrete AE scheme

\[
\frac{d}{dt}\Phi_{\alpha} + \frac{1}{\epsilon}\Phi_{\alpha} = \frac{1}{\epsilon}L_\alpha[\Phi],
\]

\[
L_\alpha[\Phi] = p_{\alpha}[\Phi](x_\alpha) - cH(x_k, \nabla_x p_{\alpha}[\Phi](x_\alpha)).
\]

The SSP Runge-Kutta method [10] can be used to achieve a time discretization with matching accuracy.

For simplicity, we now present our AE schemes in the two-dimensional case \( d = 2 \), and use \( x, y \) instead of \( x_1, x_2 \). For mesh sizes \( \Delta x, \Delta y, \Delta t \), \( \Phi_{j,k}^n \) will denote the numerical approximation to the viscosity solution \( \phi(x_j, y_k, t_n) = \phi(j \Delta x, k \Delta y, n \Delta t) \) of

\[
\phi_t + H(\phi_x, \phi_y) = 0.
\]

We shall use the following notations

\[
D_{x}^{0}\Phi_{j,k} = \frac{\Phi_{j+1,k} - \Phi_{j-1,k}}{2\Delta x}, \quad D_{y}^{0}\Phi_{j,k} = \frac{\Phi_{j,k+1} - \Phi_{j,k-1}}{2\Delta y},
\]

for central differences and

\[
A_{x}\Phi_{j,k} = \frac{\Phi_{j+1,k} + \Phi_{j-1,k}}{2}, \quad A_{y}\Phi_{j,k} = \frac{\Phi_{j,k+1} + \Phi_{j,k-1}}{2},
\]

for averages.

We begin with a continuous linear interpolation over each rectangular \( I_{j,k} \) centered at \( (x_j, y_k) = (j \Delta x, k \Delta y) \) with vertices at \( (x_{j \pm 1}, y_{k \pm 1}) \). In this case the polynomial is uniquely determined by grid values at four vertices of \( I_{j,k} \) satisfying

\[
p_{j,k}[\Phi](x_{j \pm 1}, y_{k \pm 1}) = \Phi_{j \pm 1, k \pm 1}.
\]

Such an interpolant in \( I_{j,k} \) is given by

\[
p_{j,k}^{1}[\Phi](x, y) = A_{x}A_{y}\Phi_{j,k} + D_{x}^{0}A_{y}\Phi_{j,k}(x - x_j) + D_{y}^{0}A_{x}\Phi_{j,k}(y - y_k) + D_{x}^{0}D_{y}^{0}\Phi_{j,k}(x - x_j)(y - y_k).
\]

Substitution of this into right hand side of (4.2) and use of forward Euler time-discretization yields the first order AE scheme,

\[
\Phi_{j,k}^{n+1} = (1 - \kappa)\Phi_{j,k}^n + \kappa\Phi_{j,k}^n, \\
L_{j,k}[\Phi] = A_{x}A_{y}\Phi_{j,k} - cH(D_{x}^{0}A_{y}\Phi_{j,k}, D_{y}^{0}A_{x}\Phi_{j,k}).
\]

The scale parameter \( \epsilon \) is chosen to be

\[
\epsilon \cdot \left( \frac{\max |H_1(\cdot)|}{\Delta x} + \frac{\max |H_2(\cdot)|}{\Delta y} \right) \leq 1.
\]

Here, \( H_i(p, q) \) is the partial derivative of \( H \) with respect to the \( i \)th argument (dependence of \( H \) on \( x \) causes no difficulty). The scheme becomes a local AE scheme with \( \epsilon = \epsilon_j^{i,k} \) when maximum is taken over \( I_{j,k} \) for all \( \phi = p_{j,k}^{i}[\Phi^n](x, y) \). Indeed, under this CFL type restriction, \( L_{j,k}[\Phi] \) is non-decreasing in terms of its arguments. Therefore, for \( H(0, 0) = 0 \), we have a local bound for \( L_{j,k}[\Phi] \):

\[
\min_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \} \leq L_{j,k}[\Phi] \leq \max_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \}.
\]

This, when combined with \( \kappa \leq 1 \) leads to the local maximum principle

\[
\min_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \} \leq \Phi_{j,k}^{n+1} \leq \max_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \}, \quad n \in \mathbb{N}.
\]
4.1. Stability analysis of second-order AE schemes. We start again assuming that $\Phi_{n,j,k}$ has already been computed. We extend the linear reconstruction $p_{j,k}^1[\Phi]$ to a continuous, piecewise quadratic polynomial in $I_{j,k}$. Maintaining the interpolation at four vertices of $I_{j,k}$ such a polynomial is given by:

$$p_{j,k}^2[\Phi](x,y) = p_{j,k}^1[\Phi](x,y) + s_{j,k}'(x - x_{j-1})(x - x_{j+1}) + \frac{s_{j,k}'}{2}(y - y_{k-1})(y - y_{k+1}),$$

where $s_{j,k}'$ and $s_{j,k}$ are approximations to the corresponding exact derivatives $\phi_{xx}(x_j, y_k, t_n)$ and $\phi_{yy}(x_j, y_k, t_n)$, respectively. The polynomial is not unique, leaving a room to impose a nonlinear limiter on the numerical derivatives to ensure the non-oscillatory property. For instance, we compute

(4.3) \[ s_{j,k}' = D^0_x (\minmod\{\theta A_y D^0_y \Phi_j+1, k, A_y D^0_y \Phi_{j,k}, \theta A_y D^0_y \Phi_{j-1,k}\}), \]

(4.4) \[ s_{j,k} = D^0_y (\minmod\{\theta A_x D^0_x \Phi_{j,k+1}, A_x D^0_x \Phi_{j,k}, \theta A_x D^0_x \Phi_{j,k-1}\}). \]

Here, $\theta \in [1, 2]$ is a parameter that controls numerical dissipation where $\theta = 1$ is the most dissipative and $\theta = 2$ is the least dissipative. We now substitute $p_{j,k}^2[\Phi](x,y)$ into $L_{j,k}[\Phi]$ to obtain the second order semi-discrete AE scheme:

(4.5) \[ \frac{d}{dt}\Phi_{j,k} + \frac{1}{\epsilon} L_{j,k}[\Phi], \]

(4.6) \[ L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} - \frac{s_{j,k}'}{2}(\Delta x)^2 - \frac{s_{j,k}}{2}(\Delta y)^2 - \epsilon H(D^0_x A_y \Phi_{j,k}, D^0_y A_x \Phi_{j,k}). \]

An application of the 2nd order Runge-Kutta solver gives the fully discrete 2nd order AE scheme:

(4.7) \[ \Phi_{n+1,j,k} = (1 - \kappa)\Phi_{n,j,k} + \kappa L_{j,k}[\Phi^n], \]

(4.8) \[ \Phi_{n+1,j,k} = \frac{1}{2}\Phi_{n,j,k} + \frac{1 - \kappa}{2} \Phi_{n+1,j,k} + \frac{\kappa}{2} L_{j,k}[\Phi^n]. \]

Remark 4.1. Note that both $D_x^0$ and $D_y^0$ do not commute with the nonlinear minmod operator. In light of slope limiters for hyperbolic conservation laws, a better choice would be to impose a limiter directly upon second order derivatives. For instance, in our numerical experiments we will also use

$$s_{j,k}' = \minmod\{\theta A_y (D_x^0)^2 \Phi_{j+1,k}, A_y (D_x^0)^2 \Phi_{j,k}, \theta A_y (D_x^0)^2 \Phi_{j-1,k}\},$$

$$s_{j,k} = \minmod\{\theta A_x (D_y^0)^2 \Phi_{j,k+1}, A_x (D_y^0)^2 \Phi_{j,k}, \theta A_x (D_y^0)^2 \Phi_{j,k-1}\}. \]

However, those given in (4.3), (4.4) prove to be convenient for the stability analysis.

Theorem 4.2. Let $\Phi^n$ be computed from the second order AE scheme (4.7), (4.8) and (4.6) for Hamilton-Jacobi equations with $H(0,0) = 0$ and slopes $s_{j,k}'$, $s_{j,k}$ defined in (4.3)-(4.4). If

(4.9) \[ \epsilon \left(\frac{\max |H_1(\nabla \phi)|}{\Delta x} + \frac{\max |H_2(\nabla \phi)|}{\Delta y}\right) \leq \frac{1}{2} \quad \text{and} \quad \Delta t < \epsilon \]

hold, then

$$|\Phi^{n+1}|_{\infty} \leq |\Phi^n|_{\infty}, \quad n \in \mathbb{N}. \]

Proof. It suffices to show

$$|L[\Phi^n]|_{\infty} \leq |\Phi^n|_{\infty},$$

so that when $\kappa < 1$, we have that

$$|\Phi^{n+1}|_{\infty} \leq \frac{1}{2} |\Phi^n|_{\infty} + \left(\frac{1}{2} - \frac{\kappa}{2}\right) |\Phi^n|_{\infty} + \frac{\kappa}{2} |L[\Phi^n]|_{\infty} \leq |\Phi^n|_{\infty}. \]

We now prove (4.2) as follows. Set

$$\beta_{j,k}^* := \left\{ \begin{array}{ll} -(\Delta x)^2 \frac{s_{j,k}'}{D_x A_y \Phi_{j,k}} & \text{if } D_x^0 A_y \Phi_{j,k} \neq 0, \\ 0 & \text{if } D_x^0 A_y \Phi_{j,k} = 0. \end{array} \right.$$
and

\[ \beta'_{j,k} := \begin{cases} 
-(\Delta y)^2 \frac{s'_{j,k}}{D_y^0 A_x \Phi_{j,k}} & \text{if } D_y^0 A_x \Phi_{j,k} \neq 0, \\
0 & \text{if } D_y^0 A_x \Phi_{j,k} = 0.
\end{cases} \]

Notice that \( H(0, 0) = 0 \), and by taking Taylor expansion of the Hamiltonian we have

\[ L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} + \left( \frac{\beta'_{j,k}}{2} - \epsilon H_1(\cdot) \right) D_y^0 A_y \Phi_{j,k} + \left( \frac{\beta''_{j,k}}{2} - \epsilon H_2(\cdot) \right) D_y^0 A_x \Phi_{j,k}, \]

where (\cdot) denotes some unspecified intermediate values. Regrouping terms we obtain

\[ (4.10) \]

\[ L_{j,k}[\Phi] = \frac{1}{4} \left( 1 + \frac{a_1}{\Delta x} + \frac{a_2}{\Delta y} \right) \Phi_{j+1,k+1} + \frac{1}{4} \left( 1 + \frac{a_1}{\Delta x} - \frac{a_2}{\Delta y} \right) \Phi_{j+1,k-1} + \frac{1}{4} \left( 1 - \frac{a_1}{\Delta x} + \frac{a_2}{\Delta y} \right) \Phi_{j-1,k+1} + \frac{1}{4} \left( 1 - \frac{a_1}{\Delta x} - \frac{a_2}{\Delta y} \right) \Phi_{j-1,k-1}, \]

where

\[ a_1 = \frac{\beta'_{j,k}}{2} - \epsilon H_1(\cdot), \quad a_2 = \frac{\beta''_{j,k}}{2} - \epsilon H_2(\cdot). \]

Due to the minmod limiters defined in (4.3),(4.4) it follows that

\[ |s'_{j,k}| \leq \frac{|A_y D_y^0 \Phi_{j,k}|}{2\Delta x}, \quad |s''_{j,k}| \leq \frac{|A_x D_y^0 \Phi_{j,k}|}{2\Delta y}. \]

Thereby,

\[ |\beta'_{j,k}| \leq \frac{\Delta x}{2}, \quad |\beta''_{j,k}| \leq \frac{\Delta y}{2}, \]

which when combined with the condition (4.9) yields

\[ \frac{|a_1|}{\Delta x} + \frac{|a_2|}{\Delta y} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1. \]

Thus all the coefficients on the right of (4.10) are non-negative, and \( L_{j,k}[\Phi] \) is a convex combination of four values \( \Phi_{j\pm 1,k\pm 1} \). This implies

\[ |L[\Phi^n]|_\infty \leq |\Phi^n|_\infty, \]

as claimed.

5. Numerical tests

In this section, we use some model problems to numerically test the first, second and third order global and local AE schemes. If \( \phi \) is the exact solution and \( \Phi \) is the computed solution, then the numerical \( L^1 \) and \( L^\infty \) errors in one-dimensional case are calculated as:

\[ L^1 \text{ error} = \sum_k |\phi_k - \Phi_k| \Delta x, \quad L^\infty \text{ error} = \max_k |\phi_k - \Phi_k|. \]

We call \( Q \) in equations (2.8) and (2.9) the CFL-type number for our numerical tests. In calculating the numerical derivatives in the approximation, ENO limiters with combined grid values \( \{ \Phi_k \} \) are used. In what follows we use \( N \) to denote the number of intervals the domain is divided into, i.e., \( \Delta x = (b-a)/N \) for domain \([a,b] \). Same notation will be also used in two dimensional case as long as the same partition number is used in both \( x \) and \( y \) direction.

Example 5.1. We first test the numerical accuracy of the designed schemes by using the Hamilton-Jacobi equation with convex Hamiltonian and with smooth initial data. The equation is

\[ \Phi_t + \left( \frac{\Phi_x + 1}{2} \right)^2 = 0, \quad -1 \leq x \leq 1, \]

with initial data

\[ \Phi(x, 0) = -\cos \pi x. \]
For numerical accuracy results, the reference solution is computed using a fifth order WENO [14, 25], fourth order TVD Runge-Kutta [10, 9] and $N = 10240$ with periodic boundary conditions. In Tables 1-3, the numerical accuracy results for the first, second and third order global and local schemes are presented when the solution is smooth \( (\text{time } T = \frac{0.5}{\pi^2}) \). All the proposed methods give the desired order of accuracy.

The solution becomes discontinuous at time $\frac{1}{\pi^2}$ and in Figures 1 – 3, we plot the solutions at time $T = \frac{1.5}{\pi^2}$. We can see that local schemes give better numerical resolution than the global schemes since they have smaller numerical viscosity. For the numerical experiments the CFL number is taken to be 0.9 and $\Delta t = 0.9\epsilon$.

Table 1. The $L^1$ error for convex Hamiltonian, Example 5.1, using $N$ equally spaced cells for global and local AE 1st order schemes at $T = \frac{0.5}{\pi^2}$, when the solution is continuous.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>2.427554E-01</td>
<td>1.866756E-01</td>
<td>1.380253E-01</td>
<td>0.9211</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.658496E-01</td>
<td>1.380253E-01</td>
<td>0.9211</td>
<td>0.9211</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>6.191588E-02</td>
<td>6.012093E-02</td>
<td>0.9211</td>
<td>0.9211</td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>1.238305E-01</td>
<td>1.0410</td>
<td>1.180291E-01</td>
<td>0.7090</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>6.336444E-02</td>
<td>1.4880</td>
<td>2.604142E-02</td>
<td>1.2119</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>3.617763E-02</td>
<td>0.8310</td>
<td>3.248349E-02</td>
<td>0.9521</td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>6.009927E-02</td>
<td>1.0805</td>
<td>6.612020E-02</td>
<td>0.8661</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.439781E-02</td>
<td>1.4265</td>
<td>2.764345E-02</td>
<td>1.2322</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.737636E-02</td>
<td>1.0961</td>
<td>1.794381E-02</td>
<td>0.8871</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>2.964582E-02</td>
<td>1.0379</td>
<td>3.556733E-02</td>
<td>0.9107</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.042672E-02</td>
<td>1.2486</td>
<td>1.280366E-02</td>
<td>1.1201</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>8.684065E-03</td>
<td>1.0187</td>
<td>1.033219E-02</td>
<td>0.8107</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>1.467102E-02</td>
<td>1.0240</td>
<td>1.845097E-02</td>
<td>0.9526</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>4.711692E-03</td>
<td>1.1563</td>
<td>6.139405E-03</td>
<td>1.0801</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>4.278979E-03</td>
<td>1.0303</td>
<td>5.490336E-03</td>
<td>0.9204</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>7.293956E-03</td>
<td>1.0127</td>
<td>9.463572E-03</td>
<td>0.9703</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.228255E-03</td>
<td>1.0852</td>
<td>2.992475E-03</td>
<td>1.0414</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.120360E-03</td>
<td>1.0175</td>
<td>2.824118E-03</td>
<td>0.9634</td>
</tr>
<tr>
<td>640</td>
<td>AE</td>
<td>3.634536E-03</td>
<td>1.0072</td>
<td>4.788741E-03</td>
<td>0.9850</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.080075E-03</td>
<td>1.0471</td>
<td>1.479178E-03</td>
<td>1.0247</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.053280E-03</td>
<td>1.0117</td>
<td>1.430170E-03</td>
<td>0.9838</td>
</tr>
<tr>
<td>1280</td>
<td>AE</td>
<td>1.814853E-03</td>
<td>1.0030</td>
<td>2.490586E-03</td>
<td>0.9920</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>5.323055E-04</td>
<td>1.0220</td>
<td>7.316975E-04</td>
<td>1.0108</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>5.256089E-04</td>
<td>1.0040</td>
<td>7.208656E-04</td>
<td>0.9895</td>
</tr>
</tbody>
</table>

Example 5.2. The following example is with a non-convex Hamiltonian,

$$\dot{\Phi} + \frac{1}{4}(\Phi^2 - 1)(\Phi^2 - 4) = 0, \quad -1 \leq x \leq 1,$$

$$\Phi(x, 0) = -2|x|.$$  

This test problem is used to show resolution of discontinuities when the initial data has discontinuous derivatives. As in the previous problem, the reference solution is computed using a fifth order WENO
Table 2. The $L^1$ error for convex Hamiltonian, Example 5.1 using $N$ equally spaced cells for global and local AE 2nd order schemes at $T = \frac{0.5}{\pi^2}$, when the solution is continuous.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^\infty$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>1.808044E-01</td>
<td>1.230779E-01</td>
<td>1.230779E-01</td>
<td>1.230779E-01</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.605897E-01</td>
<td>1.205706E-01</td>
<td>1.205706E-01</td>
<td>1.205706E-01</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>7.158793E-02</td>
<td>7.280369E-02</td>
<td>7.280369E-02</td>
<td>7.280369E-02</td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>5.035859E-02</td>
<td>1.9768</td>
<td>4.249643E-02</td>
<td>1.6445</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>3.793765E-02</td>
<td>2.2314</td>
<td>3.704884E-02</td>
<td>1.8249</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.226361E-02</td>
<td>1.8063</td>
<td>2.548232E-02</td>
<td>1.6235</td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>1.227800E-02</td>
<td>2.1095</td>
<td>1.52047E-02</td>
<td>1.9510</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>8.356335E-03</td>
<td>2.2613</td>
<td>1.039131E-02</td>
<td>1.9001</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>6.406106E-03</td>
<td>1.8619</td>
<td>8.868757E-03</td>
<td>1.5775</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>3.046794E-03</td>
<td>2.0469</td>
<td>2.759160E-03</td>
<td>2.0991</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.931040E-03</td>
<td>2.1516</td>
<td>2.648734E-03</td>
<td>2.0075</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.698534E-03</td>
<td>1.9497</td>
<td>2.546340E-03</td>
<td>1.8328</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>7.525114E-04</td>
<td>2.0357</td>
<td>7.102240E-04</td>
<td>1.9755</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>4.586449E-04</td>
<td>2.0926</td>
<td>6.910419E-04</td>
<td>1.9559</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>4.309275E-04</td>
<td>2.1516</td>
<td>6.800969E-04</td>
<td>1.9092</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>1.870988E-04</td>
<td>2.0170</td>
<td>1.829021E-04</td>
<td>1.9660</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.119356E-04</td>
<td>2.0439</td>
<td>1.815522E-04</td>
<td>1.9371</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.085639E-04</td>
<td>1.9979</td>
<td>1.810838E-04</td>
<td>1.9302</td>
</tr>
<tr>
<td>640</td>
<td>AE</td>
<td>4.662925E-05</td>
<td>2.0090</td>
<td>4.765022E-05</td>
<td>1.9449</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.763822E-05</td>
<td>2.0225</td>
<td>4.714968E-05</td>
<td>1.9494</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.722453E-05</td>
<td>2.0001</td>
<td>4.711239E-05</td>
<td>1.9469</td>
</tr>
<tr>
<td>1280</td>
<td>AE</td>
<td>1.164294E-05</td>
<td>2.0040</td>
<td>1.224287E-05</td>
<td>1.9628</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>6.869442E-06</td>
<td>2.0107</td>
<td>1.212996E-05</td>
<td>1.9609</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>6.818169E-06</td>
<td>1.9997</td>
<td>1.212727E-05</td>
<td>1.9601</td>
</tr>
</tbody>
</table>

[14, 25], fourth order TVD Runge-Kutta [10, 9] and $N = 10240$. In testing our AE methods, the CFL number used for all order schemes was 0.9. Linear extension boundary conditions are used for the numerical tests. In Figures 4-6, we can clearly see that the local AE2 scheme gives the best numerical accuracy, followed by local AE1 and the global AE for first, second and third order methods.

Example 5.3. One dimensional Eikonal equation

$$\Phi_t + |\Phi_x| = 0, \quad 0 \leq x \leq 2\pi,$$

with initial data

$$\Phi(x, 0) = \sin x.$$

The exact solution is given as in [5]

- If $0 \leq t \leq \pi/2$,

$$\Phi(x, t) = \begin{cases} 
\sin(x - t) & \text{if } 0 \leq x \leq \pi/2 \\
\sin(x + t) & \text{if } \pi/2 \leq x \leq 3\pi/2 - t \\
-1 & \text{if } 3\pi/2 - t \leq x \leq 3\pi/2 + t \\
\sin(x - t) & \text{if } 3\pi/2 + t \leq x \leq 2\pi
\end{cases}$$
Table 3. The $L^1$ error for convex Hamiltonian, Example 5.1 using $N$ equally spaced cells for global and local AE 3rd order schemes at $T = \frac{0.5}{\pi^2}$, when the solution is continuous.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>9.217579E-02</td>
<td>9.436562E-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>8.879066E-02</td>
<td>9.366517E-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>3.882247E-02</td>
<td>3.899054E-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>1.452062E-02</td>
<td>2.8581</td>
<td>2.009461E-02</td>
<td>2.3920</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.200976E-02</td>
<td>3.0938</td>
<td>1.889613E-02</td>
<td>2.4756</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>5.699280E-03</td>
<td>2.9672</td>
<td>7.723670E-03</td>
<td>2.5041</td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>1.997324E-03</td>
<td>2.9650</td>
<td>3.885110E-03</td>
<td>2.4562</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.374387E-03</td>
<td>3.2400</td>
<td>2.847301E-03</td>
<td>2.8288</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>7.962369E-04</td>
<td>2.9418</td>
<td>1.256969E-03</td>
<td>2.7134</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>2.501019E-04</td>
<td>3.0515</td>
<td>5.805090E-04</td>
<td>2.7920</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.454294E-04</td>
<td>3.2988</td>
<td>3.220735E-04</td>
<td>3.2008</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.064920E-04</td>
<td>2.9548</td>
<td>1.256969E-03</td>
<td>2.7134</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>3.087791E-05</td>
<td>3.0451</td>
<td>7.616378E-05</td>
<td>2.9566</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.586493E-05</td>
<td>3.2252</td>
<td>3.980785E-05</td>
<td>3.2738</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.342152E-05</td>
<td>3.0151</td>
<td>2.507739E-05</td>
<td>2.8592</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>3.820356E-06</td>
<td>3.0284</td>
<td>9.628130E-06</td>
<td>2.9972</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>1.827268E-06</td>
<td>3.1321</td>
<td>3.763506E-06</td>
<td>3.1889</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>1.675592E-06</td>
<td>3.0153</td>
<td>2.150767E-06</td>
<td>2.9768</td>
</tr>
<tr>
<td>640</td>
<td>AE</td>
<td>4.747847E-07</td>
<td>3.0151</td>
<td>1.206421E-06</td>
<td>3.0033</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.183270E-07</td>
<td>3.0729</td>
<td>4.391123E-07</td>
<td>3.1064</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.089154E-07</td>
<td>3.0105</td>
<td>4.054263E-07</td>
<td>2.9941</td>
</tr>
<tr>
<td>1280</td>
<td>AE</td>
<td>5.916589E-08</td>
<td>3.0078</td>
<td>1.509158E-07</td>
<td>3.0023</td>
</tr>
<tr>
<td></td>
<td>AE1</td>
<td>2.665971E-08</td>
<td>3.0372</td>
<td>5.295197E-08</td>
<td>3.0553</td>
</tr>
<tr>
<td></td>
<td>AE2</td>
<td>2.607256E-08</td>
<td>3.0057</td>
<td>5.086366E-08</td>
<td>2.9981</td>
</tr>
</tbody>
</table>

- If $\pi/2 \leq t \leq \pi$,

$$
\Phi(x, t) = \begin{cases} 
-1 & \text{if } 0 \leq x \leq t - \pi/2 \\
\sin(x - t) & \text{if } t - \pi/2 \leq x \leq \pi/2 \\
\sin(x + t) & \text{if } \pi/2 \leq x \leq 3\pi/2 - t \\
-1 & \text{if } 3\pi/2 - t \leq x \leq 2\pi 
\end{cases}
$$

- If $t \geq \pi$,

$$
\Phi(x, t) = -1.
$$

Note that local and global schemes are all the same for this example since $|H_p| = 1$. The numerical solution is compared for first, second and third order schemes at time $T = 1$ in Figure 7. For the viscosity solution, there is a shock wave in $\phi_x$ at $x = \pi/2$ and a rarefaction wave at $x = 3\pi/2$. The CFL number used on the numerical tests is 0.7.

Example 5.4. Two-dimensional linear HJ equation with variable coefficients

$$
\phi_t - y\phi_x + x\phi_y = 0
$$
with initial condition
\[
\phi_0(x, y) = \begin{cases} 
0 & \text{if } 0.3 \leq r, \\
0.3 - r & \text{if } 0.1 \leq r \leq 0.3, \\
0.2 & \text{if } r \leq 0.1,
\end{cases}
\]
where \( r = \sqrt{(x - 0.4)^2 + (y - 0.4)^2} \). The computational domain is \([-1, 1]^2\) and we impose periodic boundary conditions. The exact solution can be expressed as
\[
\phi(x, y, t) = \phi_0(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t)).
\]

The results for the second order schemes at time \( t = 2\pi \) are shown in Figure 8. We choose \( \theta = 2 \) in the minmod computation for the approximations \( s_{j,k}, s'_{j,k} \) and let \( \Delta t = .95\epsilon \). Here, we look along the line \( x = y \) and compare the global AE scheme to the local AE1 scheme for various mesh sizes. We can see that the local AE1 scheme provides better resolution to the exact solution for the same number of grid points as the global AE scheme.
Example 5.5. We solve the two-dimensional Burgers’ equation using first and second order global schemes:

\[ \phi_t + \frac{\left( \phi_x + \phi_y + 1 \right)^2}{2} = 0, \]

with initial data

\[ \phi(x, y, 0) = -\cos(x + y). \]

The computational domain is \([0, 2\pi]^2\) and we impose periodic boundary conditions. The solution is still smooth at time \(t = 0.1\) and we test the order of accuracy at this time in Tables 4 and 5. For the first order scheme, we get the expected order in all norms. However, for the second order scheme, we get the expected second order only for the \(L^1\) and \(L^2\) norms.

Example 5.6.

\[ \phi_t + \sqrt{\phi_x^2 + \phi_y^2} + 1 = 0, \]
Figure 3. Comparison of plots for HJ equation with convex Hamiltonian, Example 5.1, at discontinuity on $[-1, 1]$, $T = 1.5/\pi^2$, $N = 40$, $\Delta t = 0.8\epsilon$, 3rd order scheme.

Table 4. $L^1$, $L^2$, and $L^\infty$ comparison for Example 5.5 at time $t = 0.1$ using 1st Order global AE scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
<th>$L^2$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>10.49449215</td>
<td>1.675046220</td>
<td>0.366482628</td>
<td>0.197594956</td>
<td>0.8911986916</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>5.152680283</td>
<td>1.026237371</td>
<td>0.943190511</td>
<td>0.871160468</td>
<td>0.9244851809</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>2.592637495</td>
<td>0.990902576</td>
<td>0.448658593</td>
<td>0.957320444</td>
<td>0.9781821864</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>1.283476615</td>
<td>1.014363490</td>
<td>0.224778473</td>
<td>0.997114164</td>
<td>1.0403723587</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>0.607976748</td>
<td>1.077968954</td>
<td>0.107154337</td>
<td>1.068813633</td>
<td>1.1816649737</td>
<td></td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.262862679</td>
<td>1.209706825</td>
<td>0.046475460</td>
<td>1.205149189</td>
<td>1.1816649737</td>
<td></td>
</tr>
</tbody>
</table>

with smooth initial data

$$\phi(x, y, 0) = \frac{1}{4} (\cos(2\pi x) - 1) (\cos(2\pi y) - 1) - 1,$$
Figure 4. Comparison of plots for HJ equation with Riemann initial data, Example 5.2, at discontinuity on $[-1, 1]$, $T = 1$, $N = 80$, $\Delta t = 0.8\epsilon$, 1st order scheme.

Table 5. $L^1$, $L^2$, and $L^\infty$ comparison for Example 5.5 at time $t = 0.1$ using 2nd Order global AE scheme.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
<th>$L^2$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>4.766867637</td>
<td>0.776275572</td>
<td>0.163573520</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>1.011207150</td>
<td>2.236962995</td>
<td>0.17033517</td>
<td>2.186346857</td>
<td>0.047659626</td>
<td>1.7790996866</td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>0.207120787</td>
<td>2.287534306</td>
<td>0.037043836</td>
<td>2.201056933</td>
<td>0.012777997</td>
<td>1.8991058639</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>0.048087210</td>
<td>2.106747221</td>
<td>0.008953910</td>
<td>2.048643748</td>
<td>0.004385925</td>
<td>1.5427084915</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>0.013264592</td>
<td>1.858072902</td>
<td>0.002412646</td>
<td>1.8910901191</td>
<td>0.002312778</td>
<td>0.9232543366</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.003371353</td>
<td>1.976180556</td>
<td>0.000631069</td>
<td>1.934746966</td>
<td>0.000934469</td>
<td>1.3074076586</td>
</tr>
</tbody>
</table>

and computational domain $[0, 1]^2$ and periodic boundary conditions. The results at time $t = 0.6$ are shown in Figure 9. The AE scheme provides high resolution in the formation of the singularity.
Figure 5. Comparison of plots for HJ equation with Riemann initial data, Example 5.2, at discontinuity on $[-1,1]$, $T = 1$, $N = 80$, $\Delta t = 0.8\epsilon$, 2nd order scheme.

Example 5.7.

$$\phi_t - \cos(\phi_x + \phi_y + 1) = 0,$$

with initial data

$$\phi(x, y, 0) = -\cos\left(\frac{\pi(x + y)}{2}\right),$$

on the computation domain $[-2, 2]^2$ with periodic boundary conditions. At time $t = \frac{0.5}{\pi^2}$, the solution is still smooth and we test the order of accuracy in Tables 6 and 7. At time $t = \frac{1}{\pi^2}$, the solution develops discontinuous derivative and results at time $t = \frac{1.5}{\pi^2}$ are shown in Figure 10.

6. Concluding remarks

In this work, we have developed several AE schemes based on the alternating evolution approximation to Hamilton-Jacobi equations, motivated by our prior work on AE schemes for hyperbolic conservation laws [19, 24]. The AE system captures the exact solution when initially both components
Figure 6. Comparison of plots for HJ equation with Riemann initial data, Example 5.2, at discontinuity on $[-1, 1]$, $T = 1$, $N = 80$, $\Delta t = 0.8\epsilon$, 3rd order scheme.

Table 6. $L^1$, $L^2$, and $L^\infty$ comparison for Example 5.7 at time $t = \frac{0.5}{\pi^2}$ using 1st Order global AE scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
<th>$L^2$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>AE</td>
<td>1.382921081</td>
<td>0.353570436</td>
<td>0.133803519</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>AE</td>
<td>0.710038098</td>
<td>0.961750488</td>
<td>0.193017974</td>
<td>0.873262416</td>
<td>0.082485059</td>
<td>0.697911336</td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>0.349334625</td>
<td>1.023286789</td>
<td>0.097352565</td>
<td>0.987444300</td>
<td>0.040216620</td>
<td>1.036340993</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>0.170525564</td>
<td>1.034621609</td>
<td>0.048205074</td>
<td>1.014033984</td>
<td>0.02168029</td>
<td>0.925904877</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>0.082157355</td>
<td>1.053526391</td>
<td>0.023392243</td>
<td>1.043154809</td>
<td>0.011609669</td>
<td>0.866560069</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.038313568</td>
<td>1.100534346</td>
<td>0.010955990</td>
<td>1.094310317</td>
<td>0.006511828</td>
<td>0.834192278</td>
</tr>
</tbody>
</table>

are chosen as the given initial data for the Hamilton-Jacobi equation. We numerically approximate the Hamilton-Jacobi equation by sampling two components of the AE system over alternating grids. High order accuracy is achieved by a combination of high-order non-oscillatory polynomial reconstruction.
from the obtained grid values and an ODE solver in time discretization with matching accuracy. Local AE schemes are made possible by choosing the scale parameter $\epsilon$ to reflect the local distribution of nonlinear waves. The AE schemes have the advantage of easy formulation and implementation and efficient computation of the solution. For the first and second order local AE schemes, we have proved the maximum principle. Numerical tests for both one and two dimensional Hamilton-Jacobi equations presented in this work have demonstrated the high order accuracy and capacity of the AE schemes.

**ACKNOWLEDGMENTS**

This research was partially supported by the National Science Foundation under Grant DMS 09-07963.

**REFERENCES**


Figure 8. Plots of second order global and local AE scheme along the line $x = y$ for Example 5.4 at time $t = 2\pi$ where $\Delta t = .95\epsilon$ and $\theta = 2$.


Figure 9. Surface and contour plots at time $t = 0.6$ for Example 5.6.

Figure 10. Surface plot for Example 5.7 at time $t = \frac{1.5}{\pi^2}$, with uniform mesh $\Delta x = \Delta y = \frac{1}{10}$.