**COMPUTING MULTI-VALUED VELOCITY AND ELECTRIC FIELDS FOR 1D EULER-POISSON EQUATIONS**

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**Abstract.** We develop a level set method for the computation of multi-valued velocity and electric fields of one-dimensional Euler-Poisson equations. The system of these equations arises in the semiclassical approximation of Schrödinger-Poisson equations and semiconductor modeling. This method uses an implicit Eulerian formulation in an extended space — called field space, which incorporates both velocity and electric fields into the configuration space. Multi-valued velocity and electric fields are captured through common zeros of two level set functions, which solve a linear homogeneous transport equation in the field space. Numerical examples are presented to validate the proposed level set method.

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**1. Introduction**

This work together with a companion paper [25] is devoted to the design of an efficient numerical method for computing multi-valued solutions to one-dimensional Euler-Poisson Equations of the form

\[ \rho_t + (\rho u)_x = 0, \]

\[ u_t + uu_x = KE - \alpha u, \]

\[ E_x = \rho - c(x). \]

These are equations of conservation of mass, Newton’s second law, and the Poisson equation, respectively. Here \( K \) is a physical constant, which gives the property of forcing, i.e., repulsive when \( K > 0 \) and attractive when \( K < 0 \). And \( \rho(x,t) \) is the local density, \( u(x,t) \) is the velocity field, \( E(x,t) \) is usually the electric charge, \( c(x) \) is the background charge profile, and \( \alpha \) denotes the damping coefficient.

The Euler-Poisson system arises in many applications such as fluid dynamics, plasma dynamics, semiconductor modeling, and the semiclassical approximation of Schrödinger-Poisson equations. A remarkable feature of this system is the so called

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critical threshold phenomenon, which was observed and rigorously justified by Engle-
berg, Liu and Tadmor in [6]. It was shown there for a sub-critical set of initial data,
solutions of the system will develop singularity at a finite time. In some appli-
cations such as in fluid dynamics, a shock will develop after the singularity formation. But
in many other applications such as the semiclassical approximation of Schrödinger-
Poison equations and the wave breaking in Klystrons, one must allow multi-valued
solutions in order to capture physically relevant phenomena. Direct shock-capturing
methods can not be applied directly. We should mention that for one dimensional
the passage from the Schrödinger-Poisson Equation to the Euler-Poisson Equa-
tion was proved in [24] for a set of sub-critical initial data, and the passage from the
Schrödinger-Poisson equation to the Vlasov-Poisson equation was proved in [37] for
more general initial data.

Recently there has been a growing interest in developing efficient numerical meth-
ods for computing multi-valued solutions in the context of geometric optics [1, 2, 4,
7, 8, 9, 10, 11, 23, 29, 32, 33, 34], the semi-classical approximation of Schrödinger
equations [5, 13, 17, 18, 12, 14], and Euler-Poisson equations with applications to
wave breaking in klystrons [26], among others. The multi-valued solutions in physi-

cal space impose tremendous numerical challenges.

For numerical implementation, there are currently two categories of methods
available for computing multi-valued solutions, the Eulerian and Lagrangian method
(also called ray tracing). The former one based on nonlinear PDEs in physical space,
such as Hamilton-Jacobi equations, computes numerical solutions on a fixed grid,
and is generally preferred over the second one, which yields additional difficulties
in resolving wave front solutions in regions with inadequate rays. A relatively new
PDE based Eulerian method is to use kinetic formulation in phase space such as the
Liouville equation [1, 7, 8, 11, 13, 14, 17, 26, 34]. In order to reduce the computa-
tional cost, one often uses the moment method to compute moments directly or
computes some special solutions such as multi-phased wave fronts.

More recently, the level set method has been developed to capture multi-valued
solutions for first-order PDEs [5, 20, 22] in the entire domain and in particular for
the general WKB system

\[ \partial_x S + H(x, \nabla_x S) = 0, \]
\[ \partial_t \rho + \nabla_x \cdot (\rho \nabla_x H(x, \nabla_x S)) = 0, \]

with applications in the semi-classical approximation of the linear Schrödinger equa-
tions \((H = \frac{1}{2}|k|^2 + V(x))\) [5, 18], geometrical optics limit of the wave equation
\((H = c(x)|k|)\), see e.g. [19]. Note that the WKB system (1.4)-(1.5) is weakly nonlin-
ear, and the phase \(S\) can be solved independent of the density \(\rho\). In the Euler-Poisson
system the moment equation couples with the Poisson equation, hence the level set
methods mentioned above do not apply. The main goal of this paper is to introduce
a novel level set method to attack the difficulty caused by the nontrivial coupling
with the Poisson equation.

We still follow the methodology of the phase space based level set method [5,
20, 22] since the power of the level set method lies in that it automatically handles

topological changes [30, 35] such as multi-phases. This method has become a very
powerful numerical tool since it was introduced in [31]. As remarked above the
phase space based method does not apply to Euler-Poisson equations because it
fails to resolve the nontrivial coupling with the Poisson equation. The novelty of
our approach is to select an extended space, \((x, y, z) \in \mathbb{R}^3\) with \(y = u\) and \(z = E\), which we call the field space. In this field space the dynamics of full Euler-
Poisson equation can be recast into a closed ODE system along the particle path [6].
Then the level set equation is just a linear homogeneous transport equation with speed determined by the vector field of this ODE system. Multi-valued velocity and electric fields are thus resolved as common zeros of two level set functions initiated as \((y - u_0(x))\) and \((z - E_0(x))\) respectively. In a companion paper [25] we will consider the problem of computing the density \(\rho\) showing how to combine the level set formulations developed here with a post-processing step for the evaluation of density and other physical observables.

This paper is organized as follows. §2 is devoted to the derivation of level set equations in field space for general Euler-Poisson equations, using both the particle path formulation and the PDE based formulation. The determination of the initial electric field \(E_0(x)\) in different cases is also discussed. In §3 we discuss numerical procedures for computing multi-valued velocity and electrical fields via the level set formulation. Both \(L^1\) and \(L^\infty\) stability are proved for the first-order upwind scheme applied to the level set equation. In §4 we present some numerical examples to validate the proposed level set method.

2. Level Set Formulation

2.1. Formulation Using Global Invariants. We start with the general one-dimensional Euler-Poisson system

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + uu_x &= KE - \alpha u, \\
E_x &= \rho - c(x),
\end{align*}
\]

as described in §1, subject to the following initial condition

\[
\rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x).
\]

In this model, \(c(x) \geq 0\) denotes the fixed positively charged background, i.e., the doping profile in semiconductor modeling [27]. For applications in plasma dynamics, the background charge is weak and ignorable, \(c(x) = 0\). The initial electric field needs to be determined differently for the two cases.

As shown in [6], for Euler-Poisson equations, only a subset of initial configurations leads to global smooth solutions. For subcrITICAL initial data the classical solution fails and the wave breakdown will occur in finite time. Beyond the breakdown time multi-valued solutions become physically relevant.

In order to capture multi-valued solutions, we propose a new method based on level set formulations in an extended space. The extended space we are taking is \((x, y, z) \in \mathbb{R}^3\) with \(y = u\) and \(z = E\), called the field space since it incorporates both velocity and electrical fields. Instead of looking for explicit solutions in the field space, we are seeking implicit solutions identified as a common zero of two implicit functions in the field space, in which multi-valued velocity and electrical field are implicitly represented.

We proceed to derive the level set formulation first by employing the particle path method. Let \(x = x(t)\) be a particle path determined by \(\frac{dx}{dt} = u(x(t), t)\), where \(u(x, t)\) is the associated velocity field, we then have

\[
\frac{du}{dt} = KE - \alpha u, \quad \frac{d}{dt} \phi + u \phi_x
\]

along this particle path. In order to have a closed system we need also derive a dynamic equation for the electric field \(E\). Note that \(E_x(x, t) = \rho(x, t) - c(x)\) implies that \(\rho_t + (\rho u)_x = 0\). Therefore, by \(\rho_t + (\rho u)_x = 0\), we have

\[
(E_t + \rho u)_x = 0.
\]
Integration gives
\[ E_t + \rho u = C(t), \quad \forall x \in R \]
for any function \( C(t) \) of \( t \). We are interested in the physical situation that both velocity and electric fields are zero at far field. Thus we simply take \( C(t) \) as zero and obtain
\[ E_t = -\rho u. \]
Then using \( \frac{dE}{dt} = E_t + E_x u \), we arrive at
\[
\frac{dE}{dt} = -\rho u + E_x u \\
= -\rho u + (\rho - c(x))u \\
= -c(x)u.
\]
Further we combine \( \frac{dx}{dt}, \frac{du}{dt}, \frac{dE}{dt} \) together to get the following closed system
\[
\begin{align*}
\frac{dx}{dt} &= u, \quad (2.5) \\
\frac{du}{dt} &= KE - \alpha u, \quad (2.6) \\
\frac{dE}{dt} &= -c(x)u. \quad (2.7)
\end{align*}
\]
From the above autonomous ODE system, we see that the variables, \( x = x(t), u = u(x(t), t) \) and \( E(t) = E(x(t), t) \) solve a closed ODE system. According to the ODE theory, there exists a 1-1 correspondence between the ODE solution and its trajectory in the field space \((x, u, E)\). A global invariant is just a level set of certain implicit functions
\[ \phi(t, x, u, E) = \text{Const.} \]
To recover \( u \) and \( E \) from implicit global invariants, we need two functions \( \Phi = (\phi_1, \phi_2)^T \). More precisely, where \( \frac{\partial(\phi_1, \phi_2)}{\partial(u, E)} \) is nonsingular, \( u(x, t) \) and \( E(x, t) \) can be determined by the zero level set of \( \Phi \), i.e.,
\[ \Phi(t, x(t), u(t), E(t)) = 0, \quad \forall t \in R^+. \quad (2.8) \]
Differentiation of (2.8) with respect to \( t \) leads to
\[
\frac{d\Phi}{dt} = \Phi_t + \frac{dx}{dt} \Phi_x + \frac{du}{dt} \Phi_y + \frac{dE}{dt} \Phi_z = 0, \quad \forall t > 0. \quad (2.9)
\]
Using the above ODE system and replacing \( u \) and \( E \) by \( y, z \) respectively, we have the following
\[
\Phi_t + y\Phi_x + (Kz - \alpha y)\Phi_y - c(x)y\Phi_z = 0 \quad \text{on} \quad \Phi = 0. \quad (2.10)
\]
This is a transport equation derived on \( \Phi = 0 \) in \((x, y, z)\) space. Following the main idea of the level set method [30, 31, 35], we shall solve the transport equation in the vicinity of \( \Phi = 0 \) or a larger computational domain, and then project back to \( \Phi = 0 \) when the solution is needed. Note that \( \Phi = 0 \) is a codimension-two curve in \( R^3 \), and interaction of zero level sets of two level set functions needs to be performed. Consult [3] for more details on handling codimension-two objects.
2.2. Alternative Derivation. We now give an alternative derivation of the level set equation. Let $\Phi(t, x, y, z) \in \mathbb{R}^2$ be a vector function and its Jacobian matrix 

$$\det \left( \frac{\partial \Phi(t, x, y, z)}{\partial (y, z)} \right) \neq 0,$$

the implicit function theorem suggests that $\Phi(t, x, y, z) = 0$ may determine two functions $y = y(x, t)$ and $z = z(x, t)$, at least locally where the Jacobian matrix is nonsingular. Let $y = u(x, t)$ and $z = E(x, t)$ be a solution of the Euler-Poisson system, we thus obtain

$$\Phi(t, x, u(x, t), E(x, t)) \equiv 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+. \quad (2.11)$$

Differentiation of (2.11) with respect to $t$ and $x$ respectively gives

$$\Phi_t + \Phi_y u_t + \Phi_z E_t = 0,$$

$$\Phi_x + \Phi_y u_x + \Phi_z E_x = 0.$$ 

Multiplying $u$ to the second equation and adding to the first one results in the following

$$\Phi_t + u \Phi_x + (u_t + uu_x) \Phi_y + (E_t + uE_x) \Phi_z = 0.$$ 

Applying $u = y$, $u_t + uu_x = KE - \alpha u$, and $E_t + E_x u = -c(x) u$ to the above equation, we get

$$\Phi_t + y \Phi_x + (Kz - \alpha y) \Phi_y - c(x) y \Phi_z = 0, \quad (2.12)$$

which is exactly what we derived in (2.10). Note that these two different approaches of deriving level set equations verify each other.

2.3. Initialization. Given the level set equation derived for general one-dimensional Euler-Poisson equations, we discuss different cases to be considered and how initial data are chosen in each case.

The level set formulation can be rewritten into a compact form

$$\Phi_t + \nabla \cdot \mathbf{V}(X) = 0, \quad t \in \mathbb{R}^+ \quad (2.13)$$

with $X = (x, y, z) \in \mathbb{R}^3$, and the transport speed is expressed as

$$\mathbf{V}(X) = (y, Kz - \alpha y, -c(x)y).$$

In many cases of semiclassical mechanics, no damping is considered, i.e., $\alpha = 0$. The transport speed becomes

$$\mathbf{V}(X) = (y, Kz, -c(x)y).$$

For the case with zero background $c(x) = 0$, we have

$$\mathbf{V}(X) = (y, Kz, 0).$$

In all cases the level set equation takes the same form (2.13), in order to solve it we need to prepare initial data

$$\Phi|_{t=0} = \Phi_0(X).$$

Note that for the level set method the choice of initial data is not unique. But their zero level sets should uniquely embed the given initial data for $u$ and $E$. Here we take

$$\phi_1(0, x, y, z) = y - u_0(x), \quad (2.14)$$

$$\phi_2(0, x, y, z) = z - E_0(x) \quad (2.15)$$

for smooth initial fields $u_0(x)$ and $E_0(x)$. Otherwise we need to use certain smoother reconstructions of the initial fields. We now discuss how to determine $E_0(x)$ from the given initial density $\rho_0(x)$ and the choice of $c(x)$. 
(1) Non-zero background charge \( c(x) \neq 0 \).

To be consistent with the Poisson equation, we assume that \( E(x, 0) \) satisfies
\[
E(x, 0) = \int_{-\infty}^{x} (\rho(\xi, 0) - c(\xi)) \, d\xi + C,
\]
which is denoted as \( E_0(x) \) for convenience. Using the conservation form, we have
\[
\int_{\mathbb{R}} (\rho(\xi, 0) - c(\xi)) \, d\xi = \int_{\mathbb{R}} (\rho(\xi, t) - c(\xi)) \, d\xi.
\]
Without loss of generality, we set
\[
\int_{\mathbb{R}} (\rho(\xi, 0) - c(\xi)) \, d\xi = 0.
\]
To be physically relevant, we should require \( E_0(\pm\infty) = 0 \). Thus we shall have \( C = 0 \) in (2.16), i.e.,
\[
E(x, 0) = \int_{-\infty}^{x} (\rho(\xi, 0) - c(\xi)) \, d\xi.
\]
(2) Zero background charge \( c(x) \equiv 0 \).

In this case, the physical situation becomes quite different. As derived in [6], the initial data for \( E(x, t) \) in this case is determined by
\[
E(x, 0) = \frac{1}{2} \left( \int_{-\infty}^{x} \rho(\xi, 0) \, d\xi - \int_{x}^{\infty} \rho(\xi, 0) \, d\xi \right),
\]
which satisfies \( E(+\infty, 0) = -E(-\infty, 0) \), that is \( E^2(+\infty, 0) = E^2(-\infty, 0) \). The later comes from the conservation of momentum, see [6].

3. Discretization

The level set equation for cases described in previous sections all take the form
\[
\Phi_t + \overrightarrow{V}(X) \cdot \nabla_X \Phi = 0, \quad t \in \mathbb{R}^+, \quad X \in \mathbb{R}^3
\]
schedule to the initial condition
\[
\Phi|_{t=0} = \Phi_0(X) \in \mathbb{R}^2,
\]
where \( X = (x, y, z) \), denoting the field space, in which \( y \) indicates the velocity and \( z \) indicates the electric field, \( \Phi_0(X) = (y - u_0(x), z - E_0(x))^T \), and \( \overrightarrow{V} = (V_1, V_2, V_3) \), depending only on \( X \). (3.1) is a transport equation, so we may use an upwind method to solve it.

3.1. First Order Upwind Scheme. We partition the field space \( (x, y, z) \in \mathbb{R}^3 \) into computation cells, centered at \( \{(x_i, y_j, z_k)\} \) for \( i, j, k \in \mathbb{Z} \). And, we use forward difference for time discretization. The choice of forward or backward difference of \( \Phi_x, \Phi_y \) and \( \Phi_z \) depends on their coefficient functions. During the computation, uniform mesh is used. We use standard notation that
\[
\Phi^+_x = \frac{\Phi_{i+1,j,k}^n - \Phi_{i,j,k}^n}{\Delta x}, \quad \Phi^-_x = \frac{\Phi_{i,j,k}^n - \Phi_{i-1,j,k}^n}{\Delta x},
\]
where \( \Phi_{i,j,k}^n \) denotes the value of the level set function at the cell center. The choice of forward or backward difference depends on the sign of the coefficient function.
and similar notations for \( \Phi^+_x, \Phi^-_y, \Phi^+_z, \Phi^-_x, \Phi^+_y, \Phi^-_z \). Then we may obtain the following first order upwind scheme,

\[
\frac{\Phi^{n+1}_{i,j,k} - \Phi^n_{i,j,k}}{\Delta t} + V_1(i,j,k) \Phi^+_x + V_2(i,j,k) \Phi^+_y + V_3(i,j,k) \Phi^+_z = 0,
\]

where \( \Phi^n_{i,j,k} = \Phi(\tau, x_i, y_j, z_k) \), and \( V_m(i,j,k) = V_m(x_i, y_j, z_k), m=1,2,3 \). If \( V_m(i,j,k) > 0 \), we use \( \Phi^- \); otherwise, \( \Phi^+ \) is applied. The CFL condition for this scheme is

\[
\Delta t \max \left( \frac{|V_1|}{\Delta x} + \frac{|V_2|}{\Delta y} + \frac{|V_3|}{\Delta z} \right) \leq 1.
\]

In implementation, it is necessary to require that \( V_1, V_2, V_3 \) be bounded in the computational domain in order to have finite \( \Delta t \). Moreover, this is only a first order accuracy method, which may require finer grid to achieve high resolution. For stability concern, implicit or semi-implicit methods may also improve the results. Another fact is about re-initialization of the level set function. It is well known that general level set method requires re-initialization during the computation, consult [30] for more details.

One important property for transport equations is the maximum principle. We now show that this property is well preserved in the first order upwind scheme.

**Theorem 3.1.** [Discrete Maximum Principle] Assume that \( V_i(x, y, z)(i = 1, 2, 3) \) are bounded functions in the computational domain. Let \( \Phi^n \) be a numerical solution produced by the first order upwind scheme (3.2) subject to the initial data \( \Phi^0 \), then

\[
||\Phi^n||_\infty \leq ||\Phi^0||_\infty.
\]

**Proof.** Denote

\[
V^+_m = \max(V_m(i,j,k),0) = \frac{|V_m(i,j,k)| + V_m(i,j,k)}{2},
\]

\[
V^-_m = \max(-V_m(i,j,k),0) = \frac{|V_m(i,j,k) - V_m(i,j,k)|}{2}
\]

for all \( i,j,k \) in computational domain with \( m=1,2,3 \).

Then the upwind scheme can be rewritten as

\[
\Phi^{n+1}_{i,j,k} = \left( 1 - (V^+_1 + V^-_1) \frac{\Delta t}{\Delta x} - (V^+_2 + V^-_2) \frac{\Delta t}{\Delta y} - (V^+_3 + V^-_3) \frac{\Delta t}{\Delta z} \right) \Phi^n_{i,j,k} + V^+_1 \frac{\Delta t}{\Delta x} \Phi^n_{i-1,j,k} + V^-_1 \frac{\Delta t}{\Delta x} \Phi^n_{i+1,j,k} + V^+_2 \frac{\Delta t}{\Delta y} \Phi^n_{i,j+1,k} + V^-_2 \frac{\Delta t}{\Delta y} \Phi^n_{i,j-1,k} + V^+_3 \frac{\Delta t}{\Delta z} \Phi^n_{i,j,k-1} + V^-_3 \frac{\Delta t}{\Delta z} \Phi^n_{i,j,k+1}.
\]

Further expanding all upwind partial derivatives, we obtain the following

\[
\Phi^{n+1}_{i,j,k} = \left( 1 - (V^+_1 + V^-_1) \frac{\Delta t}{\Delta x} - (V^+_2 + V^-_2) \frac{\Delta t}{\Delta y} - (V^+_3 + V^-_3) \frac{\Delta t}{\Delta z} \right) \Phi^n_{i,j,k} + V^+_1 \frac{\Delta t}{\Delta x} \Phi^n_{i-1,j,k} + V^-_1 \frac{\Delta t}{\Delta x} \Phi^n_{i+1,j,k} + V^+_2 \frac{\Delta t}{\Delta y} \Phi^n_{i,j+1,k} + V^-_2 \frac{\Delta t}{\Delta y} \Phi^n_{i,j-1,k} + V^+_3 \frac{\Delta t}{\Delta z} \Phi^n_{i,j,k-1} + V^-_3 \frac{\Delta t}{\Delta z} \Phi^n_{i,j,k+1}.
\]

Note that \( V^+_m + V^-_m = |V_m| \) for \( m=1,2,3 \). By the CFL condition (3.3), the coefficient of \( \Phi^n_{i\pm1,j,k} \), \( \Phi^n_{i,j\pm1,k} \), \( \Phi^n_{i,j,k\pm1} \) are all nonnegative too. Due to the positivity of all coefficients, when we take absolute value both sides and relax on the right-hand-side, we apply absolute sign only on \( \Phi^n_{i,j,k}, \Phi^n_{i\pm1,j,k} \), \( \Phi^n_{i,j\pm1,k}, \Phi^n_{i,j,k\pm1} \), which are all bounded by \( |\Phi^0| \). Then, we obtain the following equation
\[ \|\Phi^{n+1}\|_\infty \leq (1 - |V_1| \frac{\Delta t}{\Delta x} - |V_2| \frac{\Delta t}{\Delta y} - |V_3| \frac{\Delta t}{\Delta z}) \|\Phi^n\|_\infty \\
+ |V_1| \frac{\Delta t}{\Delta x} \|\Phi^n\|_\infty + |V_2| \frac{\Delta t}{\Delta y} \|\Phi^n\|_\infty + |V_3| \frac{\Delta t}{\Delta z} \|\Phi^n\|_\infty \]
\[= \|\Phi^n\|_\infty. \]

Therefore, we obtain the stability estimate (3.4) as asserted. \( \square \)

**Remark 3.1.** This conclusion holds for all bounded \( V_m \) with \( m = 1, 2, 3 \). Hence it is applicable to the level set equation for Euler-Poisson equations derived in previous sections.

We now turn to the \( L^1 \) stability. Define the numerical \( L^1 \) norm by

\[ \|\Phi^n\|_1 = \sum_{i,j,k} |\Phi^n_{(i,j,k)}| \Delta x \Delta y \Delta z. \]

The \( L^1 \)-stability can be stated in

**Theorem 3.2.** Assume that \( V_i(x, y, z)(i = 1, 2, 3) \) are bounded and Lipschitz continuous in its \( i \)-th argument in the computational domain. Let \( \Phi^n \) be a numerical solution produced by the first order upwind scheme (3.2) subject to the initial data \( \Phi^0 \), then for finite time \( T \), there exists a constant \( M \), such that

\[ \|\Phi^n\|_1 \leq e^{MT} \|\Phi^0\|_1. \]  

(3.6)

**Proof.** Summation of equation (3.5) over all \( i, j, k \), shifting the index of terms, \( \Phi^n_{(\pm 1, j, k)} \), \( \Phi^n_{(i, \pm 1, k)} \), and \( \Phi^n_{(i, j, \pm 1)} \) leads to

\[ \sum_{i,j,k} |\Phi^{n+1}_{(i,j,k)}| \leq \sum_{i,j,k} (1 - (V_1^+ - V_1^- (i + 1, j, k) + V_1^- (i - 1, j, k)) \frac{\Delta t}{\Delta x} \\
-(V_2^+ - V_2^- (i, j + 1, k) + V_2^- (i, j - 1, k)) \frac{\Delta t}{\Delta y} \\
-(V_3^+ - V_3^- (i, j, k + 1) + V_3^- (i, j, k - 1)) \frac{\Delta t}{\Delta z}) |\Phi^n_{(i,j,k)}|. \]

By assumption, there exist Lipschitz constants \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) such that

\[ |V_1^\pm - V_1^\pm (i \pm 1, j, k)| \leq \alpha_1 \Delta x, \]
\[ |V_2^\pm - V_2^\pm (i, j \pm 1, k)| \leq \alpha_2 \Delta y, \]
\[ |V_3^\pm - V_3^\pm (i, j, k \pm 1)| \leq \alpha_3 \Delta z, \]

then

\[ \sum_{i,j,k} |\Phi^{n+1}_{(i,j,k)}| \leq \sum_{i,j,k} (1 + 2(\alpha_1 + \alpha_2 + \alpha_3) \Delta t) |\Phi^n_{(i,j,k)}|. \]

Multiplying \( \Delta x \Delta y \Delta z \) both sides, we obtain

\[ \sum_{i,j,k} |\Phi^{n+1}_{(i,j,k)}| \Delta x \Delta y \Delta z \leq \sum_{i,j,k} (1 + 2(\alpha_1 + \alpha_2 + \alpha_3) \Delta t) |\Phi^n_{(i,j,k)}| \Delta x \Delta y \Delta z, \]

\[ \|\Phi^n\|_1 \leq (1 + M \Delta t)^n \|\Phi^0\|_1, \]  

(3.7)

with \( M = 2(\alpha_1 + \alpha_2 + \alpha_3) \). Moreover, in finite time \( T, n = T/\Delta t \). We can simplify equation (3.7) further

\[ \|\Phi^n\|_1 \leq e^{MT} \|\Phi^0\|_1, \]  

(3.8)

as \( n \) tends to \( \infty \). The proof is complete. \( \square \)
Remark 3.2. This conclusion holds for all bounded and Lipschitz continuous $V_m$ with $m = 1, 2, 3$. Hence it is applicable to level set equations for the Euler-Poisson system derived in previous sections.

3.2. Numerical Procedures. Now we highlight our algorithms for numerical simulations.

Step 1: Initialization and Discretization
First we give the initial data $\Phi$ at each grid point according to (2.14) and (2.15) with $E_0$ specified in §2.3. Then we discretize the computation domain $[a, b] \times [c, d] \times [e, f] \in R^3$ into uniform grid with fixed $\Delta x, \Delta y, \Delta z$.

Step 2: Solving Transport Equation (2.13)
During computation, the first order upwind scheme (3.2) is used. Meanwhile, both $L_{\infty}$ and $L_1$ norms are computed. However, in reality second or higher order accurate method is generally preferred.

For second or higher order accuracy method, TVD Runge-Kutta methods may be employed for discretizing time, while Lax-Wendroff or ENO [15, 16, 36](WENO [21, 28, 36])-type methods may be applied to variable $\{x, y, z\}$.

Step 3: Retrieving the Solution in Field Space
After we obtained $\Phi$ at desired time $T$, we may retrieve the solution through common zeros of two level set functions, $\phi_1$ and $\phi_2$. The projection of common zeros onto $x - u$ and $x - E$ spaces gives the visualizations of multi-valued $u(x, T)$ and $E(x, T)$.

4. Computational Experiments
In this section, we validate our level set methods with several numerical examples and compute both $L_{\infty}$ and $L_1$ norms to demonstrate the stability. In the following experiments, the first order upwind scheme is employed.

4.1. Numerical test 1. We test the damping free model (2.1)-(2.3) with $\alpha = 0$, $c(x) = 0$, $K = 0.01$, and subject to the condition,

\[
\begin{align*}
  u(x, 0) &= \sin(x) |\sin(x)|, \\
  \rho(x, 0) &= e^{-(x-\pi)^2/2\pi}.
\end{align*}
\]

This data is used in [14], where the semiclassical approximation of the Schrödinger-Poisson equation is studied.

As stated in §2, for the case $c(x) = 0$, the initial value for $E(x, 0)$ shall be given as

\[
E(x, 0) = \frac{1}{2} \left( \int_{-\infty}^{x} \rho(s, 0) ds - \int_{x}^{\infty} \rho(s, 0) ds \right).
\]

The necessary and sufficient condition for existence of global smooth solution to the Euler-Poisson system (2.1)-(2.3) is $\partial_x u(x, 0) > -\sqrt{2K} \rho(x, 0)$, $\forall x \in R$, which is given in [6]. This condition is clearly violated, say for example at $x = \frac{3\pi}{4}$. Thus, the classical solution fails in finite time and develops into a multi-valued solution. Here we compute multi-valued $u(x, t)$ and $E(x, t)$ using the first order scheme (3.2) with $\Phi_0 = (y - u(x, 0), z - E(x, 0))^T$. The numerical results are shown in Fig.1 for multi-valued $u$ and $E$ at time $t = 2.5$.

To check the $L_{\infty}$ and $L_1$ stability for the scheme (3.2), we have calculated the $L_{\infty}$ and $L_1$ norm for both $\phi_1$ and $\phi_2$ at $t = 0$ and $t = 2.5$ in Table 1. It is not hard to see in the table that the $L_{\infty}$ and $L_1$ of $\phi_1$ and $\phi_2$ are both decreasing as time evolves. Thus the results of numerical experiments are consistent with the stability properties that we proved in Theorem 3.1 and 3.2 in Section 3.1.
Figure 1. $u(x,t)$ and $E(x,t)$ for Euler-Poisson equations from the semiclassical approximation of Schrödinger-Poisson equation at time $t = 0.5, 1, 2.5$ from top to bottom respectively.
COMPUTING MULTI-VALUED VELOCITY AND ELECTRIC FIELDS

Table 1. The initial and end-time $L^\infty$ and $L^1$ norms of $\Phi$ of semiclassical approximation of Schrödinger-Poisson equation

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$\Phi$</th>
<th>$L^\infty$ at t=0</th>
<th>$L^\infty$ at t=2.5</th>
<th>$L^1$ at t=0</th>
<th>$L^1$ at t=2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$41\times41\times41$</td>
<td>$\phi_1$</td>
<td>2.00</td>
<td>1.83</td>
<td>39.26</td>
<td>37.29</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>4.15</td>
<td>4.14</td>
<td>87.77</td>
<td>79.87</td>
</tr>
<tr>
<td>$61\times61\times61$</td>
<td>$\phi_1$</td>
<td>2.00</td>
<td>1.88</td>
<td>38.25</td>
<td>36.67</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>4.15</td>
<td>4.14</td>
<td>85.14</td>
<td>78.79</td>
</tr>
<tr>
<td>$81\times81\times81$</td>
<td>$\phi_1$</td>
<td>2.00</td>
<td>1.91</td>
<td>37.75</td>
<td>36.41</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>4.15</td>
<td>4.14</td>
<td>83.85</td>
<td>78.39</td>
</tr>
<tr>
<td>$101\times101\times101$</td>
<td>$\phi_1$</td>
<td>2.00</td>
<td>1.92</td>
<td>37.45</td>
<td>36.29</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>4.15</td>
<td>4.14</td>
<td>83.08</td>
<td>78.22</td>
</tr>
</tbody>
</table>

4.2. Numerical test 2. Here, we test the model (2.1)-(2.3) with $\alpha = 0$, $c(x) = 1$ and $K = 0.01$. The boundary condition is given as

\begin{align}
u(0, t) &= 1 + 0.2 \sin(5\pi t), \\
E(0, t) &= 0.
\end{align}

This data was used in [26] in applications to electron beam propagation in Klystrons.

Before computation, we analyze this boundary value problem to see whether global smooth solution exists. Along the particle path $t = t(x)$, we define $\frac{d}{dx} \triangleq \frac{1}{u} \frac{\partial}{\partial t}$. From (2.5)-(2.7), we have

\begin{align}
\frac{du}{dx} &= KE, \\
\frac{dE}{dx} &= -1.
\end{align}

By using (4.1), (4.2), we derive the following solution along characteristics,

\begin{align}
u^2(x, t) &= u^2(0, t) - Kx^2, \\
E(x, t) &= -x.
\end{align}

Clearly there will be no smooth solution beyond

\begin{align}x > \sqrt{u^2(0, t)/K}.
\end{align}

Thus this boundary value problem shall develop a multi-valued solution in finite space marching.

Note that though this is a boundary value problem, a variation of the level set formulation above can still be applied. If we restrict our computation domain $(t, y, z) = (0, T) \times (a, b) \times (c, d)$ with $a > 0$, we can divide the level set equation (2.10) both sides by $y$ to reach

\begin{align}\Phi_x + \frac{1}{y} \Phi_t + \frac{Kz}{y} \Phi_y - \Phi_z = 0.
\end{align}

Hence, we may apply our level set formulation regarding space variable $x$ as a marching parameter with computation domain for variables $(t, y, z)$.

Now we discuss how to determine the computation domain by using characteristic curves. By (4.5) and (4.6), we have the relation that

\begin{align}u^2 + KE^2 = u^2(0, t).
\end{align}

Along with the initial condition (4.1), (4.2), we can roughly determine the range of $u$ and $E$ as $(0.7, 1.3) \times (-2, 2)$.
Table 2. The initial and end-time $L^\infty$ and $L^1$ norms of $\Phi$ for modulated electron beam in a klystron

<table>
<thead>
<tr>
<th>$Mesh$</th>
<th>$\Phi$</th>
<th>$L^\infty$ at $x=0$</th>
<th>$L^\infty$ at $x=1$</th>
<th>$L^1$ at $x=0$</th>
<th>$L^1$ at $x=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$41 \times 41 \times 41$</td>
<td>$\phi_1$</td>
<td>0.50</td>
<td>0.44</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>2.00</td>
<td>1.33</td>
<td>1.06</td>
<td>0.79</td>
</tr>
<tr>
<td>$61 \times 61 \times 61$</td>
<td>$\phi_1$</td>
<td>0.50</td>
<td>0.45</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>2.00</td>
<td>1.42</td>
<td>1.03</td>
<td>0.81</td>
</tr>
<tr>
<td>$81 \times 81 \times 81$</td>
<td>$\phi_1$</td>
<td>0.50</td>
<td>0.46</td>
<td>0.18</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
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<td>0.83</td>
</tr>
<tr>
<td>$101 \times 101 \times 201$</td>
<td>$\phi_1$</td>
<td>0.50</td>
<td>0.46</td>
<td>0.18</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>2.00</td>
<td>1.65</td>
<td>0.99</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Table 3. The initial and end-time $L^\infty$ and $L^1$ norms of $\Phi$ of semiclassical approximation of Schrödinger-Poisson equation

<table>
<thead>
<tr>
<th>$Mesh$</th>
<th>$\Phi$</th>
<th>$L^\infty$ at $t=0$</th>
<th>$L^\infty$ at $t=5$</th>
<th>$L^1$ at $t=0$</th>
<th>$L^1$ at $t=5$</th>
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<tr>
<td>$61 \times 41 \times 41$</td>
<td>$\phi_1$</td>
<td>0.70</td>
<td>0.68</td>
<td>1.39</td>
<td>1.34</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>0.97</td>
<td>0.96</td>
<td>2.03</td>
<td>1.90</td>
</tr>
<tr>
<td>$81 \times 61 \times 61$</td>
<td>$\phi_1$</td>
<td>0.70</td>
<td>0.69</td>
<td>1.36</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>0.97</td>
<td>0.96</td>
<td>1.98</td>
<td>1.87</td>
</tr>
<tr>
<td>$101 \times 81 \times 81$</td>
<td>$\phi_1$</td>
<td>0.70</td>
<td>0.69</td>
<td>1.34</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>0.97</td>
<td>0.96</td>
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<td>1.86</td>
</tr>
<tr>
<td>$121 \times 101 \times 101$</td>
<td>$\phi_1$</td>
<td>0.70</td>
<td>0.69</td>
<td>1.33</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>0.97</td>
<td>0.96</td>
<td>1.94</td>
<td>1.85</td>
</tr>
</tbody>
</table>

Fig. 2 presents the overturn beyond the critical location at $x = 1$ as predicted in (4.7). On the other hand, we know from section 3.1, our scheme has both $L^\infty$ and $L^1$ stability, which we can observe numerically in Table 2.

4.3. Numerical test 3. Now, we test the model (2.1)-(2.3) with $\alpha = 0$, $K = 0.01$, and subject to condition,

\[
\begin{align*}
  u(x, 0) &= 0.3 \sin(x), \\
  \rho(x, 0) &= 0.3e^{-(x-\pi)^2/\pi}.
\end{align*}
\]

The initial data is used in [14], where the author studied the semiclassical approximation of the Schrödinger-Poisson equation of self-consistent electron cloud within Mathieu’s potential.

Since $c(x) = 0$, the initial value for $E(x, 0)$ is determined by

\[
E(x, 0) = \frac{1}{2} \left( \int_{-\infty}^{x} \rho(s, 0) ds - \int_{x}^{\infty} \rho(s, 0) ds \right).
\]

As predicted in [6], this kind of system supports critical threshold phenomena, and subcritical initial data will develop singularity in finite time. And in this example, the classical solution does fail in finite time and hence multi-valued solutions should be computed to achieve the physically relevant solution. In Fig. 3, we observe that the velocity develops overturns at around $x = 2.2$ and $x = 3.8$. Once again, we have included the $L^\infty$ and $L^1$ norm in Table 3 to show the $L^\infty$ and $L^1$ stability numerically.
Figure 2. $u(x,t)$ and $E(x,t)$ for Euler-Poisson equations of model for modulated electron beam in a klystron at position $x = 0.2, 0.5, 1$ from top to bottom respectively.
Figure 3. $u(x,t)$ and $E(x,t)$ for Euler-Poisson equations from the semi-classical approximation of Schrödinger-Poisson equation at time $t = 1$, $3$, $5$ from top to bottom respectively.
Table 4. The initial and end-time $L^\infty$ and $L^1$ norms of $\Phi$ of semiclassical approximation of Schrödinger-Poisson equation

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$\Phi$</th>
<th>$L^\infty$ at t=0</th>
<th>$L^\infty$ at t=2</th>
<th>$L^1$ at t=0</th>
<th>$L^1$ at t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$81 \times 51 \times 51$</td>
<td>$\phi_1$</td>
<td>2.50</td>
<td>2.37</td>
<td>36.94</td>
<td>35.42</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>1.39</td>
<td>1.39</td>
<td>21.11</td>
<td>18.92</td>
</tr>
<tr>
<td>$101 \times 51 \times 51$</td>
<td>$\phi_1$</td>
<td>2.50</td>
<td>2.38</td>
<td>36.84</td>
<td>35.37</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>1.39</td>
<td>1.39</td>
<td>21.06</td>
<td>18.96</td>
</tr>
<tr>
<td>$101 \times 8 \times 61$</td>
<td>$\phi_1$</td>
<td>2.50</td>
<td>2.38</td>
<td>36.27</td>
<td>34.93</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>1.39</td>
<td>1.39</td>
<td>20.78</td>
<td>18.83</td>
</tr>
<tr>
<td>$301 \times 151 \times 101$</td>
<td>$\phi_1$</td>
<td>2.50</td>
<td>2.44</td>
<td>35.42</td>
<td>34.53</td>
</tr>
<tr>
<td></td>
<td>$\phi_2$</td>
<td>1.39</td>
<td>1.39</td>
<td>20.27</td>
<td>18.87</td>
</tr>
</tbody>
</table>

4.4. Numerical test 4. Now, we test the model (2.1)-(2.3) with $\alpha = 0.01$, $K = 0.01$, and

$$c(x) = \begin{cases} 0.5, & x \in [-1, 1] \\ 0, & \text{otherwise}. \end{cases}$$

The initial condition is

$$u(x, 0) = \cos(x + 0.15),$$
$$\rho(x, 0) = \frac{1}{2\sqrt{\pi}} \left( e^{-(x+\pi/2)^2} + e^{-(x-\pi/2)^2} \right).$$

Since $c(x) \neq 0$, the initial value for $E(x, 0)$ of Euler-Poisson equations with background shall be given as

$$E(x, 0) = \int_{-\infty}^{x} (\rho(s, 0)ds - c(s)) ds.$$

From the Fig.4, we observe that the solution develops overturn before $t = 1.4$, which shows that our method captures the multi-valued solution. The sharp corner of $E$ at time $t = 1.4$ is caused by sharp corner of initial value of $E$. And in Table 4, we show the $L^\infty$ and $L^1$ stability numerically.

5. Conclusion

We introduce a new level set method for computing multi-valued velocity and electric fields for 1D Euler-Poisson equations. The proposed method is built upon a new level set formulation in an extended space—field space. The multi-valued fields are computed by evolving the same linear transport equation with smooth initial data $(y - u_0(x), z - E_0(x))$. The projection of common zeros of two computed level set functions enables us to obtain the sharp result efficiently. Moreover, both the $L^\infty$ and $L^1$ stability for first-order upwind schemes of the level set equation are established. Compared to moment methods based on the Vlasov-Poisson equation introduced in [26], our approach automatically computes multi-valued fields that occur in the system, and the method in [26] gives, instead, averaged density and other moments. The computation of density and other physical quantities based on the level set method introduced here is under our current study and will appear in [25].
Figure 4. $u(x,t)$ and $E(x,t)$ for Euler-Poisson equations from the semi-classical approximation of Schrödinger-Poisson equation at time $t = 0.2, 0.5, 1.4$ from top to bottom respectively.
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