Critical Thresholds and Conditional Stability for Euler Equations and Related Models

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1 Introduction

We are concerned with time regularity of the velocity field for a class of so-called restricted flows, where the velocity field, u, is governed by the Newtonian law,

$$\partial_t u + u \cdot \nabla u = F, \quad x \in \mathrm{I\!R}^n, \tag{1.1}$$

which shows up in different contexts dictated by the different modeling of F's, where F stands for a general forcing acting on the flow. The examples range from Euler/Navier-Stokes equations to Euler-Poisson equations. There has been an enormous amount of papers related to the study of global behavior of solutions to (1.1) with possible associated laws (say, conservation of mass, energy etc). For the question of global behavior of strong solutions the choice of the initial data and/or damping forces is decisive. The classical methods of analysis include energy method (for small initial perturbations due to the nonlinearity) and the singularity prediction (the finite life span is often due to a *global* condition of large enough initial (generalized) energy, staying outside a critical threshold ball).

Indeed when dealing with the questions of time regularity for the above Euler-related equations, one encounters several limitations with the classical stability analysis. Among other issues, we mention that

(i) the stability analysis does not tell us how large perturbations are allowed before losing stability, say with the incompressible Navier-Stokes equations;

(ii) the steady solution may be only conditionally stable due to the weak dissipation in the system, say in certain Euler-Poisson models, [ELT].

In order to address these difficulties we propose a new notion of critical threshold (CT) in [ELT], which serves to describe the conditional stability for underlying physical problem. It is shown that the CT phenomena does reflect the delicate balance among various forcing mechanisms. Little or no attention has been paid to this remarkable phenomena, and our goal is to bridge the gap left in previous studies on the solution behavior of Euler-related equations in the small and in the large.

We first illustrate the CT phenomena by analyzing a 2×2 ODE system, which governs the time dynamics of 1-D Euler-Poisson equations along the particle path; The result on the CT phenomena for a 1-D convolution model [LT1] is also recorded.

We then discuss this remarkable CT phenomena associated with the Euler-Poisson equations [ELT], where the answer to questions of global smoothness vs. finite time breakdown depends on whether the initial configuration crosses an intrinsic, $O(1)$ critical threshold.

We investigate various one-dimensional problems with or without forcing mechanisms as well as multi-dimensional isotropic models with geometric symmetry. The critical thresholds for these essentially 1-D problems are shown to depend on the relative size of the initial velocity slope and the initial density.

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We then extend our discussion of the CT phenomena for multi-dimensional systems of the form (1.1). Here we utilize a novel description for the spectral dynamics of the (possibly complex) eigenvalues, $\lambda = \lambda(\nabla u)$ which are shown to be governed by the Ricatti-like equation $\lambda_t + u \cdot \nabla\lambda + \lambda^2 = \langle l, \nabla Fr\rangle$. Restricting attention to restricted Euler-Poisson equations driven by localized forcing we identify the set of their $[n/2]$ global invariants, which in turn yield (i) sufficient conditions for finite time breakdown, and (ii) characterization of a large class of 2-dimensional initial configurations leading to global smooth solutions.

Moreover, the critical thresholds for 2D REPs are shown to depend on the relative size of three quantities at initial time: density, divergence and the spectral gap $\lambda_2 - \lambda_1$.

The results on CT phenomena reviewed here are obtained in a series of papers [ELT, LT1, LT2, LT3, LT4], which provides an account on recent developments of conditional stability theory for a class of Eulerian flow models. The corresponding methods in obtaining these results will be briefly illustrated.

2 Critical Threshold Phenomena

It is known that a fundamental question for equation (1.1) is whether the smooth solution develops singularity in finite time, and a key issue in this line of research is the control of the velocity gradient tensor $\nabla_x u$. As pointed earlier the classical stability analysis often fails to detect the delicate balance between the convection and the underlying forcing. Our approach to address these difficulties is the use of a new notion of critical threshold (CT) in order to describe the conditional stability for the underlying problem.

In order to highlight the idea of CT notion we start with a 2×2 ODE system

$$u' = -uv, \tag{2.1}$$

$$v' = ku - v^2, \quad k = Const. > 0. \tag{2.2}$$

Observe that for the global solution behavior the linear part of the system is of no help since both eigenvalues of the coefficient matrix of linearized system are zero. Stability theory does NOT work even locally around the equilibrium point $(0, 0)$.

A manipulation like $[-v \times (2.1) + u \times (2.2)]/u^2$ leads to

$$\left(\frac{v}{u}\right)' = \frac{uv' - vu'}{u^2} = k, \quad \frac{v}{u} = kt + \frac{v(0)}{u(0)}.$$

Inserting back into (2.2) we obtain a decoupled system

$$v' + v^2 = \frac{kv}{kt + v(0)/u(0)},$$

from which we find that

$$v(t) = \frac{v(0) + ku(0)t}{1 + v(0)t + ku(0)t^2/2}.$$

Clearly this solution is globally bounded if and only if the initial data lie in the region

$$\{(u, v), \quad v > -\sqrt{2ku}, \quad u \geq 0\}.$$

The boundary of this region in the phase space is called the critical threshold (CT). The global behavior of the solution depends only on whether the initial data cross such critical threshold.

In [LT1] such CT phenomena is justified for a 1-D scalar PDE model:

$$u_t + uu_x = Q * u - u, \tag{2.3}$$

where $Q \geq 0$ is a regular symmetric kernel, monotonically decreasing on IR^+, subject to initial data

$$u(0, x) = u_0(x), \quad u_0 \in C^1_b(\mathsf{IR}). \tag{2.4}$$

Theorem 2.1. *[LT1] Consider the Cauchy problem* (2.3)–(2.4) *with initial data* $u_0 \in C_b^1(\mathbb{R})$. *Let the kernel* Q *be as stated above; then we have the following:*

- *If* $V(u_0) < \frac{1}{4Q(0)}$ *and*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) > -\frac{1}{2}\left[1+\sqrt{1-4Q(0)V(u_0)}\right], \quad V(u_0) := \max u_0 - \min u_0,$$

  then the smooth solution exists for all time.

- *If*

$$\inf_{x} \partial_x u_0(x) < -\frac{1}{2}\left[1+\sqrt{1+4Q(0)V(u_0)}\right],$$

  then the solution u *must break down at finite time* T.

The above upper and lower-threshold clearly confirm the existence of the critical threshold. From these results we see that if the magnitude of the initial profile is small, both thresholds given in Theorem 2.1 are close to $\inf_{x\in\mathbb{R}} \partial_x u_0(x) = -1$, which is exactly the critical threshold for the damped Burgers' equation:

$$u_t + uu_x = -u.$$

Indeed, along the particle path $x(\alpha, t)$ defined by

$$\frac{d}{dt}x(\alpha,t) = u(t, x(\alpha,t)), \quad x(\alpha, 0) = \alpha, \quad \alpha \in \mathbb{R},$$

the gradient of the solution to the damped Burgers' equation above can be written explicitly as

$$u_x(t,x) = [e^t(1 + (\partial_x u_0(\alpha))^{-1}) - 1]^{-1},$$

which is bounded from below for all time if and only if $\inf_{x\in\mathbb{R}} \partial_x u_0(x) \geq -1$. This remarkable critical threshold phenomenon explains why (2.3) admits narrower shock layers than those in the viscous Burgers' equation, see e.g. [KN, Ro, ST].

3 Isotropic Euler-Poisson Equations

In this section we present the results obtained in [ELT] on a new phenomena associated with the Euler-Poisson equations —the so called critical threshold phenomena. To this end we focus our attention on the n-dimensional isotropic model,

$$\begin{aligned} &r^\nu \rho_t + (\rho u r^\nu)_r = 0, \quad r > 0, \\ &\rho(u_t + uu_r) = k\rho\phi_r + \text{viscosity} + \text{relaxation}, \\ &(r^\nu \phi_r)_r = \rho r^\nu + \text{background}, \quad \nu = n-1. \end{aligned} \tag{3.1}$$

It is well known that finite time breakdown is a generic phenomena for nonlinear hyperbolic convection equations, which is realized by the formation of shock discontinuities. In the context of Euler-Poisson equation, however, there is a delicate balance between the forcing mechanism (governed by Poisson equation), and the nonlinear focusing (governed by Newton's second law), which supports a critical threshold phenomena.

3.1 The basic model with zero background

We consider the 1D Euler-Poisson equation of the form

$$\rho_t + (\rho u)_x = 0, \tag{3.2}$$

$$u_t + uu_x = -k\phi_x = kE, \tag{3.3}$$

$$E_x = -\phi_{xx} = \rho. \tag{3.4}$$

Note that the density ρ and the velocity slope u_x satisfy the ODE (2.1)-(2.2) along the particle path $x = x(\alpha, t)$ governed by $\frac{dx}{dt} = u(x, t)$. The critical threshold observed for (2.1)-(2.2) enables us to conclude the following

Theorem 3.1. *[ELT] The system of Euler-Poisson equations (3.2)-(3.4) admits a global smooth solution if and only if*

$$u_0'(\alpha) > -\sqrt{2k\rho_0(\alpha)}, \quad \forall\alpha \in \mathrm{IR}. \tag{3.5}$$

This result reveals a phenomenon very much like the conditional breakdown of waves on the beach; only waves above certain initial critical threshold experience finite-time breakdown, but otherwise they propagate smoothly.

3.2 The basic model with constant background

We now consider the system

$$\rho_t + (\rho u)_x = 0, \tag{3.6}$$
$$u_t + uu_x = -k\phi_x = kE, \tag{3.7}$$
$$E_x = -\phi_{xx} = \rho - c, \tag{3.8}$$

with constant "background" state $c > 0$. Here we require that $\int_{-\infty}^{\infty} (\rho(\xi) - c)\, d\xi = 0$. The presence of a constant background c in the Poisson equation changes the "physical situation." However, the CT phenomena for this system are still observed and justified in [ELT].

Theorem 3.2. *Consider the system of Euler-Poisson equations (3.6)-(3.8) with constant background charge c. Then*

(i) for the repulsive force $k > 0$, it admits a global smooth solution if and only if

$$|u_0'(\alpha)| < \sqrt{k(2\rho_0(\alpha) - c)}, \quad \forall\alpha \in \mathrm{IR}. \tag{3.9}$$

In this case, the density oscillates around the nonzero background charge c, and the velocity gradient does not decay in time.

(ii) for attractive force, $k < 0$. Then, it admits a global smooth solution if and only if

$$u_0'(\alpha) \geq -\left(1 - \frac{\rho_0(\alpha)}{c}\right)\sqrt{-ck}, \quad \forall\alpha \in \mathrm{IR}. \tag{3.10}$$

In this case, the density approaches the zero exponentially in time, and the velocity gradient remains bounded uniformly in time.

The above results indicate that the presence of the background can balance the attractive forcing ($k < 0$). The fact that (3.9) when $c \downarrow 0$ does not yield (3.5) implies that the limit of $c \downarrow 0$ is a sort of singular limit.

3.3 Euler-Poisson system with relaxation

We consider a further modification of our problem (3.6), (3.8), where (3.7) is now augmented by a relaxation term

$$u_t + uu_x = -k\phi_x - \frac{u}{\epsilon} = kE - \frac{u}{\epsilon}, \quad \epsilon > 0. \tag{3.11}$$

We still require that $\int_{-\infty}^{\infty} (\rho(\xi) - c)\, d\xi = 0$.

Theorem 3.3. *Consider the system of Euler-Poisson equations (3.6), (3.11), (3.8) with a constant background charge $c > 0$, a repulsive force, $k > 0$ and a strong relaxation term, $\epsilon < \frac{1}{2\sqrt{ck}}$. If at all points $\alpha \in \mathsf{IR}$*

$$u_0'(\alpha) > \min\left\{0, -(1 - \frac{\rho_0(\alpha)}{c})\left(\sqrt{\frac{1}{4\epsilon^2} - ck} + \frac{1}{2\epsilon}\right)\right\}, \tag{3.12}$$

then the solution of (3.6), (3.11), (3.8) remains smooth for all time. In this case, $u(x,t) \to 0$ and $\rho(x,t) \to c$ exponentially fast as $t \to \infty$.

If the relaxation is weak then the solution becomes oscillatory. However, a critical threshold can still be detected. We summarize by stating that

Theorem 3.4. *Consider the system of Euler-Poisson equations (3.6), (3.11), (3.8) with a constant background charge $c > 0$, a repulsive force $k > 0$ and weak relaxation $\epsilon > \frac{1}{2\sqrt{ck}}$. Then there exists a critical time $t^* > 0$ such that if at all points $\alpha \in \mathsf{IR}$,*

$$\left|u_0'(\alpha) + \frac{c - \rho_0(\alpha)}{2\epsilon c}\right| < \sqrt{k - \frac{1}{4c\epsilon^2}}\sqrt{\frac{\rho_0^2(\alpha)}{c}(e^{t^*/\epsilon} - 1) + 2\rho_0(\alpha) - c}, \tag{3.13}$$

the solution of (3.6), (3.11), (3.8) remains smooth for all time. In this case, $u(x,t) \to 0$ and $\rho(x,t) \to c$ exponentially as $t \to \infty$.

Remark 3.5. Note that as $\epsilon \to \infty$ we recover the local condition (3.10) for the case without relaxation.

In addition to the above mentioned example we also justify the CT phenomena for the 1-D Euler-Poisson equation with viscosity mechanism, we refer to [ELT] for further details.

3.4 Critical Thresholds–Multi-D model with Geometric Symmetry

Let us consider the Euler-Poisson equations governing the $\nu + 1$ dimensional isotropic ion expansion in the electrostatic fluid approximation for cold ions

$$r^\nu \rho_t + (\rho u r^\nu)_r = 0, \quad r > 0, \tag{3.14}$$
$$u_t + u u_r = -k\phi_r = kE, \tag{3.15}$$
$$(r^\nu \phi_r)_r = -\rho r^\nu, \tag{3.16}$$

subject to the initial data

$$(\rho, u)(r, 0) = (\rho_0, u_0)(r), \quad \rho_0(r) \geq 0. \tag{3.17}$$

We recall some results obtained in [ELT] for the multi- D case, $\nu \geq 1$, where we confirm the remarkable persistence of critical threshold phenomena in the multidimensional problem with geometric symmetry.

Theorem 3.6 (Global existence of smooth solutions for the cylindrical case $\nu = 1$). *A global smooth solution of Euler- Poisson equations (3.14)-(3.16) with $\nu = 1$ exists provided the initial data (u_0, ρ_0) with $E_0 = \alpha^{-1}\int_0^\alpha \rho_0(\xi)\xi d\xi$ satisfy*

$$u_0' > -\frac{k}{u_0}[\alpha\rho_0 h(\alpha) - E_0], \quad \forall \alpha \in \mathsf{IR}^+, \tag{3.18}$$

where $h(\alpha)$ is determined by

$$k\alpha^2\rho_0 u_0 \int_0^{h(\alpha)} \frac{h(\alpha) - \eta}{[u_0^2 + 2kE_0\alpha\eta]^{3/2}} e^\eta d\eta \equiv 1, \quad \forall \alpha \in \mathsf{IR}^+. \tag{3.19}$$

Condition (3.18) could be viewed as an upper threshold in the sense of providing a sufficient condition leading to global smooth solutions, though the permissible class of the initial data for global smooth solutions is clearly larger. However, the existence of the critical threshold can be ensured by combining this upper threshold with the following lower threshold for the finite time breakdown.

Theorem 3.7 (Breakdown of smooth solutions for cylindrical case $\nu = 1$). *The smooth solution to the Euler- Poisson equations (3.14)-(3.16) with $\nu = 1$ breaks down in finite time if the condition, $u_0'(\alpha) > -\sqrt{2k\rho_0(\alpha)}$, fails, i.e., if*

$$\exists \alpha \in \mathrm{IR}^+ \quad s.t. \quad u_0'(\alpha) \leq -\sqrt{2k\rho_0(\alpha)}. \tag{3.20}$$

For the 3-dimensional case we obtain the lower threshold for finite time breakdown.

Theorem 3.8 (Breakdown of smooth solutions for spherical case $\nu = 2$). *The solution of Euler-Poisson equations (3.14)-(3.16) for $\nu = 2$ blows up in finite time if the condition, $u_0' \geq -\frac{k}{u_0}[\alpha\rho_0 - E_0]$, fails, i.e.,*

$$\exists \alpha \in \mathrm{IR}^+ \quad s.t. \quad u_0' < -\frac{k}{u_0}[\alpha\rho_0 - E_0]. \tag{3.21}$$

Again an immediate critical threshold is confirmed by combining the above lower threshold and the upper threshold for the existence of global smooth solution below.

Theorem 3.9 (Global existence of smooth solutions for general $\nu \geq 2$ cases). *A global smooth solution of the Euler-Poisson equations (3.14)-(3.16) with $\nu \geq 2$ exists provided the initial data (u_0, ρ_0) with $E_0 = \alpha^{-\nu} \int_0^\alpha \rho_0(\xi)\xi^\nu d\xi$ is prescribed such that for all $\alpha \in \mathrm{IR}^+$*

$$u_0' > -\frac{k}{u_0}\left[\frac{\rho_0\alpha^\nu h_\nu}{\nu - 1} - E_0\right]. \tag{3.22}$$

Here, h_ν is determined by

$$\frac{ku_0\rho_0\alpha}{(\nu-1)^2}\int_0^{h_\nu} \frac{h_\nu(\alpha) - \eta}{[u_0^2 + \frac{2kE_0\alpha^\nu}{\nu-1}\eta]^{3/2}}(\alpha^{1-\nu} - \eta)^{\frac{\nu}{1-\nu}} d\eta \equiv 1. \tag{3.23}$$

The above results for various models indicate that the critical thresholds for these essentially 1-D problems are shown to depend on the relative size of the initial velocity slope and the initial density. For truly multi-D models the velocity gradient is a $n \times n$ square matrix, the key is to seek an appropriate characterization and control of its entries, which is the main topic of next section.

4 Spectral Dynamics—Multi-D Models

As noted earlier a key issue of studying the time regularity is the control of the velocity gradient $\nabla_x u$, and a classical approach in this context, is to consider linear combinations of the entries of ∇u, controlling physically relevant quantities like vorticity, divergence, etc. [BKM, CLM, M].

The novelty of the analysis taken in [LT2] is the use of the eigenvalues of the velocity gradient field. The eigenvalues, $\lambda = \lambda(\nabla u)$, exhibit of course a strong nonlinear dependence on the entries of ∇u, and are shown to play a crucial role in governing the behavior of the flow. Indeed, the dynamics of these eigenvalues $\lambda(\nabla u)$, is shown, in [LT2], to be governed by Ricatti-like equation

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = < l, \nabla F r >, \tag{4.1}$$

with $l(r)$ being the left (right) eigenvectors of ∇u. Equipped with this description for the spectral dynamics of ∇u, we are able to study the time dynamics for several physical models with different forcing.

In [LT2] we focus our investigation on four prototype models associated with different forcing F, ranging from simple linear damping and a viscous dusty medium models to the main thrust of the paper — the restricted models of Euler/Navier-Stokes equations and Euler-Poisson equations.

In particular, we address the question of the time regularity for these models, that is, whether they admit a finite time breakdown, a global smooth solution, or an intermediate scenario of critical threshold phenomena where global regularity depends on initial configurations.

Using the spectral dynamics as our essential tool in these investigations, we obtain a simple form of a critical threshold for the linear damping model and we identify the 2D vanishing viscosity limit for the viscous irrotational dusty medium model. Moreover, for the n-dimensional restricted Euler equations we obtain $[n/2]+1$ global invariants, interesting for their own sake, which enable us to precisely characterize the local topology at breakdown time, extending previous studies in the $n = 3$-dimensional case, see [Vi]. Finally, as a forth model we introduce the n-dimensional restricted Euler-Poisson (REP)system, identifying a set of $[n/2]$ global invariants, which in turn yield (i) sufficient conditions for finite time breakdown, and (ii) characterization of a large class of 2-dimensional initial configurations leading to global smooth solutions. Consequently, the 2D restricted Euler-Poisson equations are shown to admit a critical threshold.

Here we highlight the main idea by presenting the result for so-called Restricted Euler-Poisson dynamics introduced in [LT2], consult [LT2, LT3] for results about other related models.

The Euler-Poisson equations

$$\rho_t + \nabla \cdot (\rho u) = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \tag{4.2}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = k\rho \nabla \phi, \tag{4.3}$$

$$\Delta \phi = \rho, \quad x \in \mathbb{R}^n, \tag{4.4}$$

are the usual statements of the conservation of mass, Newton's second law, and the Poisson equation defining, say, the electric field in terms of the charge.

Passing to Lagrangian coordinates, that is, using the change of variables $\alpha \mapsto x(\alpha, t)$ with $x(\alpha, t)$ solving

$$\frac{dx}{dt} = u(x,t), \quad x(\alpha, 0) = \alpha,$$

then Euler-Poisson equations have the equivalent formulation in terms of velocity gradient field $M := \nabla_x u$ and the density ρ

$$\frac{d}{dt}M + M^2 = kR[\rho], \quad \frac{d}{dt} := \partial_t + u \cdot \nabla, \tag{4.5}$$

$$\frac{d}{dt}\rho + \rho tr M = 0. \tag{4.6}$$

Here $R[w]$ denotes the $n \times n$ matrix whose entries are given by $(R[w])_{ij} := R_i R_j(w)$ where R_j denote the Risez transforms, $R_j = -(-\Delta)^{-1/2}\partial_j$, i.e.,

$$\widehat{[R_j w]}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{w}(\xi) \quad \text{for} \quad 1 \le j \le n.$$

It is the *global* term in the above equations, $R[\rho]$, which makes the problem rather intricate to solve, both analytically and numerically. A localized alternative to (4.5)-(4.6) was introduced in [LT2] as a restricted Euler-Poisson dynamics. Specifically, we consider a gradient flow governed by

$$\partial_t M + u \cdot \nabla M + M^2 = \frac{k}{n}\rho I_{n\times n}, \tag{4.7}$$

$$\partial_t \rho + u \cdot \nabla \rho + \rho tr M = 0. \tag{4.8}$$

If we let $\lambda_i(x,t)$ denote the eigenvalues of velocity gradient tensor ∇u, then by the spectral dynamics relation (4.1), the eigenvalues and the density ρ are coupled through

$$\partial_t \rho + u \cdot \nabla \rho + \rho \sum_{j=1}^{n} \lambda_j = 0, \tag{4.9}$$

$$\partial_t \lambda_i + u \cdot \nabla \lambda_i + \lambda_i^2 = \frac{k\rho}{n}, \quad i = 1, \cdots, n. \tag{4.10}$$

This is a closed system governing the restricted Euler-Poisson (REP) equations, which serves as a simple approximation for the evolution of the full Euler-Poisson system (4.2)-(4.4).

Theorem 4.1. *[LT2][Global material invariants for $n \geq 2$] Consider the restricted Euler-Poisson dynamics (4.9)-(4.10) with real initial data $(\rho_0, \lambda_1(0), \cdots, \lambda_n(0))$. Then, there exist $\left[\frac{n}{2}\right]$ global invariants*

$$\frac{\Pi_{(i,j)\in\mathcal{I}}(\lambda_i - \lambda_j)}{\rho^N} = \text{Const.}, \quad N = \begin{cases} 1, & n \text{ even}, \\ 2, & n \text{ odd}. \end{cases} \tag{4.11}$$

The general solution may break down at finite time in the orthant $\{+, -, \cdots, -\}$.

This result indicates that the spectral gap $\lambda_i - \lambda_j$ play important roles in the evolution of the flow. Also in [LT2] we identify a class of 2D initial configurations for global smoothness.

Theorem 4.2 (Global existence). *The solutions of the 2-D restricted Euler-Poisson equations (4.9),(4.10) ($n = 2$) remains smooth for all time $t > 0$ if both $\lambda_i(0)$, $i = 1, 2$ are complex, i.e., $Im(\lambda_i(\alpha, 0)) \neq 0$.*

As a consequence, it follows that the 2D restricted Euler-Poisson equations admit a critical threshold which distinguishes between initial configurations leading to finite time breakdown and global smooth solutions. A detailed study of this 2-dimensional critical threshold phenomena in this context is provided in [LT3] for the 2-D REP with zero and nonzero background. This complements the study of critical threshold phenomena for isotropic configurations in the general (global) Euler-Poisson equations presented in [ELT].

We recall the result for zero background case, and refer to [LT3] for nonzero background case.

To state the main result, we introduce two quantities with which we characterize the behavior of the velocity gradient tensor $M = \nabla U$, namely, the divergence

$$d := trM = \lambda_1 + \lambda_2$$

and the nonlinear quantity

$$\Gamma := (trM)^2 - 4detM = (\lambda_2 - \lambda_1)^2,$$

which serves as an index for the spectral gap.

Theorem 4.3. *[LT3][2D REP with zero background]. Consider the 2D repulsive REP system (4.9),(4.10)($n = 2$) with $k > 0$ and with zero background $c = 0$. Then the solution of 2D REP remains smooth for all time if and only if the initial data (ρ_0, M_0) lies in one of the following two regions, $(\rho_0, d_0, \Gamma_0) \in S_1 \cap S_2$:*

(i) Either

$$(\rho_0, d_0, \Gamma_0) \in S_1, \quad S_1 := \left\{ (\rho, d, \Gamma) \quad | \quad \Gamma \leq 0 \quad \textit{and} \quad \begin{cases} d \geq 0 & \text{if} \quad \rho = 0 \\ d \quad \text{arbitrary} & \text{if} \quad \rho > 0 \end{cases} \right\};$$

(ii) or

$$(\rho_0, d_0, \Gamma_0) \in S_2, \quad S_2 := \{(\rho, d, \Gamma) \quad | \quad \rho > 0, \quad \Gamma > 0, \quad \text{and} \quad d \geq g(\rho, \Gamma)\}$$

where

$$g(\rho, \Gamma) := \text{sgn}(\Gamma - 2k\rho)\sqrt{\Gamma - 2k\rho + 2k\rho ln\left(\frac{2k\rho}{\Gamma}\right)}.$$

Remarks:

1. The above result shows that the global smooth solution is always ensured if the velocity gradient tensor has complex eigenvalues, as stated in Theorem 4.2, which applies, for example, for a class of initial configurations with sufficiently large vorticity $|u_{0y} - v_{0x}| >> 1$. With other initial configurations, however, the finite time breakdown of solutions may – and actually does occur, unless the initial divergence is above a critical threshold, expressed in terms of the initial density and initial spectral gap. Hence global regularity depends on whether the initial configuration crosses an intrinsic, $O(1)$ critical threshold.

2. The critical threshold in the 1D Euler-Poisson equations depends on the relative size of the initial velocity slope and the initial density, consult [ELT]. In contrast the critical threshold presented here depends on three initial quantities: density ρ_0, divergence $\nabla \cdot U_0$ and initial spectral gap $\Gamma_0 = (u_{0x} - v_{0y})^2 + 4u_{0y}v_{0x}$.

3. Theorem 4.3 tells us that the size of initial sub-critical range which gives rise to regular solution is decreasing as the initial ratio Γ_0/ρ_0 is increasing. In particular when this ratio is larger than $2k$, then the initial divergence must stay above a positive critical threshold to avoid the finite time breakdown. □

5 And there's more ...

In the previous sections we show how the persistence of the global features of the solutions for various models hinges on a delicate balance between the nonlinear convection and the underlying forcing. This remarkable CT phenomena is expected in various contexts. Also the CT as a new notion precisely characterizes the conditional stability of the corresponding physical problems.

As another application we recall that in Geophysical flows the rotational forcing plays an important role. Indeed in [LT4] we show that the rotational forcing also supports the CT phenomena, therefore serves as a candidate forcing to prevent the finite time breakdown.

Once a critical threshold is detected for a given system, the scaling limit in the different regime can be further justified. We recall here a result obtained in [LT5], where the link of the Euler-Poisson equation and the quantum description of the physical process–Schödinger-Poisson equation is discussed. The semiclassical limit is justified in the regime where the corresponding Euler-Poisson equation has global smooth solution.

Consider the nonlinear Schrödinger-Poisson (NLSP) equation,

$$i\epsilon\psi^\epsilon_t = -\frac{\epsilon^2}{2}\Delta_x\psi^\epsilon - k\Big(\Delta_x^{-1}(|\psi^\epsilon|^2 - c)\Big)\psi^\epsilon, \quad x \in \mathrm{IR}^n, \quad t \in \mathrm{IR}^+, \tag{5.1}$$

subject to the initial condition

$$\psi^\epsilon_0(x) = \sqrt{\rho_0(x)}e^{iS_0(x)/\epsilon}, \quad x \in \mathrm{IR}^n. \tag{5.2}$$

Here, $\psi^\epsilon(\cdot, t)$ is a complex-valued wavefunction depending on the scaled Planck constant $\epsilon \sim \hbar$, with k and $c > 0$ being scaled physical constants. The NLSP equation (5.1) has been studied in different contexts, and in particular, as the fundamental equation in semiconductors applications. We refer the reader to the recent review [GLM] and references therein.

Indeed, introducing S^ϵ as the phase of the wave function

$$\psi^\epsilon = \sqrt{\rho^\epsilon(x,t)}e^{iS^\epsilon(x,t)/\epsilon}$$

and separating real and imaginary parts in the NLSP equation (5.1), one obtains the following irrotational flow equations for the density-velocity pair, $(\rho^\epsilon, u^\epsilon := \nabla_x S^\epsilon)$, consult e.g., [LL]

$$\rho^\epsilon_t + \nabla\cdot(\rho^\epsilon u^\epsilon) = 0, \tag{5.3}$$

$$u^\epsilon_t + u^\epsilon\cdot\nabla u^\epsilon = k\nabla\Delta_x^{-1}(\rho^\epsilon - c) + \frac{\epsilon^2}{2}\nabla\left[\frac{\Delta\sqrt{\rho^\epsilon}}{\sqrt{\rho^\epsilon}}\right]. \tag{5.4}$$

Ignoring $O(\epsilon^2)$ term on the right of (5.4) for small ϵ one then obtains the Euler-Poisson equations with nonzero background, whose 1D version is exactly (3.6)-(3.8) we met before.

The above argument is, of course, only formal. The challenging issue here is to justify such a dispersive limit. The asymptotics of observable quantities as $\epsilon \to 0$ is known as 'semi-classical', expressing the passage from quantum to Newton mechanics, where the time and distance scales become large enough relative to Planck's constant. The 'semi-classical' limit problem associated with the NLSP equation (5.1) can be summarized as follows. We consider the family of solutions for the NLSP Cauchy problem (5.1), $\{\psi^\epsilon, \epsilon > 0\}$, subject to the WKB-type initial data (5.2), where ρ_0 and S_0 are given smooth functions. The purpose is to determine the behavior of the wavefunction ψ^ϵ as $\epsilon \to 0$, through the dynamics governing semi-classical limits of its observables, the density, $\rho = \lim|\psi^\epsilon|^2$, and the velocity $u = \lim \epsilon\nabla_x \arg\psi^\epsilon$.

Theorem 5.1. *[LT5][Global convergence] Consider the 1-dimensional NLSP equation (5.1) ($n = 1$) subject to WKB initial data (5.2), $\psi^\epsilon_0(x) = \sqrt{\rho_0(x)}exp\{iS_0(x)/\epsilon\}$, with $(\rho_0(x), S_0(x))$ bounded in $(H^2(\mathrm{IR}), H^4(\mathrm{IR}))$. Assume that the initial data satisfy the sub-critical condition*

$$|\partial_x^2 S_0(\alpha)| < \sqrt{k\big(2\rho_0(\alpha) - c\big)}, \quad \alpha \in \mathrm{IR}.$$

Then, for ϵ small enough, (5.1) admits a global solution of the form

$$\psi^\epsilon = \sqrt{\rho^\epsilon(x,t)} exp\{iS^\epsilon(x,t)/\epsilon\}$$

for all $t < Const. ln\frac{1}{\epsilon}$, with the following semi-classical limit, $(\rho^\epsilon, u^\epsilon := S^\epsilon_x) \longrightarrow (\rho, u)$ where (ρ, u) is the solution of the Euler-Poisson equations (3.6)-(3.8). Moreover, the following error estimate holds

$$\|\rho^\epsilon(\cdot,t) - \rho(\cdot,t)\|_{H_2} + \|u^\epsilon(\cdot,t) - u(\cdot,t)\|_{H_1} \le Const_T \cdot \epsilon, \qquad t \le T \sim \ln\frac{1}{\epsilon}.$$

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