KDV DYNAMICS IN THE PLASMA-SHEATH TRANSITION

HAILIANG LIU AND MARSHALL SLEMROD

Contents

1. Introduction 1
2. Simplification of the Basic equations 2
3. Quantum Formulation 3
4. Boundary layers and transition layer 5
   4.1. The initial boundary value problem 5
   4.2. Sheath inner layer solution 7
5. Justification of the KdV dynamics 8
6. From Perturbed KdV to KdV 9
7. Conclusions 10
Acknowledgments 10
References 10

Abstract. A mathematical model is formulated to catch the dynamics hidden in the plasma-sheath transition layer and the inner sheath layer for planar motion of a plasma. It is shown that the rescaled potential in the plasma-sheath transition layer and inner layer is governed by a perturbed KdV equation, through which some of the complex interactions and couplings among physical mechanisms acting in the plasma-sheath formation process are elucidated. This model is analyzed and compared with the model used by Fokas in his study of nonlinear dispersive initial boundary value problems [1].

1. Introduction

Many problems in science involve structures on several distinct length scales. Usually the relevant length scales are not known a priori, but emerge from an attempt of the system to reach its equilibrium state. In plasma physics, for example, the typical length scale can be predicated by dimensional analysis, but the sheath transition and inner layer are determined by a complex interplay of the internal dynamics. The plasma-sheath transition is a fundamental problem in plasma physics and a good discussion is found in the book by Lieberman and Lichtenberg [5]. Mathematically it provides a challenge to the applied analyst in that there are multiple scales which must be resolved to obtain an adequate description of the physical process.

Matched asymptotic expansions provide a powerful method to predict limiting behavior. Using this method several authors [2, 4, 8, 9, 10, 11, 12, 13, 14] beginning with Franklin and Ockendon [2] have described the plasma-sheath transition by analysis of the balance laws on three relevant space scales for (A) the bulk quasi-neutral plasma, (B) the plasma-sheath transition layer, and (C) the sheath layer. Region (B) is governed by the Painlevé equation, governing the electrical potential in the transition layer joining quasi-neutral plasma to space charge sheath in a weakly ionized plasma.

Date: November 10, 2002
The intent of the present work is to provide a derivation of a new model, describing both the plasma-sheath transition layer and the sheath inner layer.

For an ionized plasma consisting of electrons and ions, one dimensional motion is described by a normalized Euler-Poisson system of the form

\[ \partial_t n + \partial_x (nu) = ze^{-\phi}, \]
\[ \partial_t u + u\partial_x u = \partial_x \phi - f(u) - ze^{-\phi}un^{-1}, \]
\[ \epsilon^2 \partial_x^2 \phi = n - e^{-\phi}, \]

where \( u \) represents ion velocity, and \( \phi \) is the electric potential (both suitably scaled), \( x \) is a dimensionless space variable, \( -L < x < x_w \), where \( x_w = -\epsilon^4 b(t \epsilon^{-2/5}) \) is the location of a possibly moving wall. The electron density is given by Boltzmann’s relation and has been set equal to \( e^{-\phi} \). At \( x = x_w \) we prescribe boundary conditions \( \phi = \phi_w(t, \epsilon) \) and \( u = u_w(t, \epsilon) \). In fact the main goal of this paper is to derive a KdV model for the plasma sheath transition and compare with the recent result of Fokas \[1\] for a similar yet subtly different problem.

2. Simplification of the Basic equations

For completeness, we provide the relevant balance laws for hydrodynamic models for plasma \[5\].

Let \( m_i \) denote the ion mass, \( n_i \) the ion density, \( u_i \) the ion velocity, \( m_e \) the electron mass, \( n_e \) the electron density, \( u_e \) the electron density, \( \Phi \) the electric potential, and \( Z \) denotes the rate of ionization. The balance laws of mass and momentum for ions are

\[ \partial_t n_i + \partial_X (n_i u_i) = Zn_e, \]
\[ \partial_t (n_i u_i) + \partial_X (n_i u_i^2) = -\frac{en_i}{m_i} \partial_X \Phi - n_i \frac{\tilde{f}(u_i)}{\lambda}, \]

where \( \tilde{f}/\lambda \) denotes the ion friction and \( \lambda > 0 \) the constant ion collision mean free path, and the balance laws of mass and momentum for electrons are

\[ \partial_t n_e + \partial_X (n_e u_e) = Zn_e, \]
\[ \partial_t (n_e u_e) + \partial_X (n_e u_e^2 + p_e) = \frac{en_e}{m_e} \partial_X \Phi, \]

where the pressure is given by \( p_e = km_e^{-1}n_e T_e \), here \( T_e \) denotes the electron temperature, the ion temperature is zero. In addition \( \Phi \) satisfies Poisson’s equation

\[ \frac{\epsilon_0}{e} \partial_X^2 \Phi = n_i - n_e, \]

where \( \epsilon_0 \) is the permittivity of free space. Usually, the ions are heavy compared to the electrons, i.e., \( m_i >> m_e \).Passing to the limit \( m_e \to 0 \) in the momentum equation for electron one can formally obtain

\[ \partial_X (kT_e n_e) = en_e \partial_X \Phi. \]

Integration in terms of \( X \) gives

\[ n_e = n_{e,0} \exp \left( \frac{e\Phi}{kT_e} \right). \]
This is the well-known Boltzmann relation, in which \( n_{ch} \) denotes the characteristic charged particle density, \( e \) the electron charge and \( k \) Boltzmann’s constant.

The above systems may be further simplified if we introduce quantities

\[
c_s = \sqrt{\frac{kT_e}{m_i}}, \quad \lambda_D = \sqrt{\frac{\epsilon_0 kT_e}{n_{ch} e^2}}
\]

representing the ion sound speed and Debye length. Indeed introducing the following dimensionless variables

\[
t = \frac{\tau c_s}{\lambda}, \quad x = \frac{X}{\lambda}, \quad \epsilon = \frac{\lambda_D}{\lambda},
\]

\[
\frac{n_i}{n_{ch}} \rightarrow n_i, \quad \frac{n_e}{n_{ch}} \rightarrow n_e, \quad \frac{u_i}{c_s} \rightarrow u_i, \quad \frac{u_e}{c_s} \rightarrow u_e, \quad m = \frac{m_e}{m_i}, \quad z = Z\lambda/c_s,
\]

the above coupled Euler-Poisson system may be rewritten as

\[
\partial_t n_i + \partial_t (n_i u_i) = zn_e,
\]

\[
\partial_t (n_i u_i) + \partial_x (n_i u_i^2) = n_i \partial_x \phi - n_i f(u_i), \quad f(u_i) := \frac{\tilde{f}(c_s u_i)}{c_s^2},
\]

and

\[
\partial_t n_e + \partial_x (n_e u_e) = zn_e,
\]

\[
\partial_t (n_e u_e) + \partial_x (n_e u_e^2) + \frac{1}{m} \partial_x n_e = -\frac{n_e}{m} \partial_x \phi,
\]

coupled with Poisson’s equation

\[
\epsilon^2 \partial_x^2 \phi = n_i - n_e.
\]

Recall that the limit \( m \to 0 \) in the above momentum equation (2.4) yields Boltzmann’s relation

\[
n_e = \exp(-\phi).
\]

Hence the limit system \((m \to 0)\) may be rewritten in non-dimensional form

\[
\partial_t n + \partial_x (nu) = ze^{-\phi},
\]

\[
\partial_t u + u \partial_x u = \partial_x \phi - f(u) - ze^{-\phi} \ln^{-1},
\]

\[
\epsilon^2 \partial_x^2 \phi = n - e^{-\phi}.
\]

3. Quantum Formulation

In this section we will derive the system of Schrödinger equations such that their semi-classical limit coincides with the Euler-Poisson system with linear damping \( f(u_i) = \alpha u_i \) in the momentum equation for ions.

Let the desired Schrödinger equation take the form

\[
i \hbar \partial_t \psi^\hbar = -\frac{\hbar^2}{2} \Delta_x \psi^\hbar + [V + \frac{i\hbar}{2} Q] \psi^\hbar
\]

with the potential \( V \) and nonhomogeneous term \( Q \) to be determined. We remark in passing that the connection between Schrödinger equations and the classical hydrodynamical
equations was already noted in 1927 by Madelung, in the context of semi-classical limit of
the nonlinear Schrödinger equation. To this end, one identifies two physical relevant
observable quantities—the fluid density $|\psi|^2$, and the fluid velocity $u^h := \hbar \nabla_x \text{arg} \psi^h$. For
the semi-classical regime it is customary to consider the following WKB(after Wentzel,
Kramers, and Brillouin) ansatz

$$\psi^h = A^h(x,t) \exp \left( \frac{i S(x,t)}{\hbar} \right)$$

with $A^h \geq 0$ assuming that the phase and the amplitude are sufficiently smooth, and we
expand the amplitude in powers of $\hbar$:

$$A^h = A_0 + \hbar A_1 + \hbar^2 A_2 + \cdots$$

Insertion of this expression into (3.1) leads to the following relation between the wave
phase and its amplitude

$$-\hbar A^h \left( \partial_t S + \frac{1}{2} |\nabla S|^2 + V(x) \right) + \frac{i \hbar}{2} (2\partial_t A^h + A^h \Delta_x S + 2\nabla_x A^h \cdot \nabla_x S - Q A^h) + \frac{\hbar^2}{2} \Delta_x A^h = 0.$$  

Nullifying the expressions related to the first two powers of $\hbar$ we derive the WKB system

with corrector term $Q A_0^2$

$$\partial_t S + \frac{1}{2} |\nabla S|^2 + V = 0,$$

and the leading order of the amplitude $A_0$ solves the forced transport equation

$$\partial_t A_0^2 + \nabla \cdot (A_0^2 \nabla S) = Q A_0^2.$$  

When $\hbar$ is small the leading term $A_0$ becomes significant. Set $(\rho, U) := (A_0^2, \nabla S)$ one then
has the following system

$$\partial_t \rho + \nabla \cdot (\rho U) = Q \rho,$$

$$\partial_t U + U \cdot \nabla U = -\nabla V.$$  

In order to recover (2.1), (2.2), it suffice to take $(\rho, U) := (n_i, u_i)$, $Q := Q_i = z_n c_n^{-1}$
and $V := V_i$ such that $-\nabla_x V_i = \nabla \phi - \alpha \nabla x S_i - Q \nabla x S_i$, i.e.,

$$V_i = \alpha S_i + z \nabla x^{-1} \left[ \frac{n_e}{n_i} \nabla x S_i \right] - \phi.$$  

Assume the wave function for ions is $\psi_i$ and for electron is $\psi_e$, therefore

$$n_i = |\psi_i|^2, \quad n_e = |\psi_e|^2.$$  

Hence the Poisson equation (2.5) becomes

$$\epsilon^2 \Delta \phi = |\psi_i|^2 - |\psi_e|^2,$$

or $\phi = \epsilon^{-2} \Delta^{-1}(|\psi_i|^2 - |\psi_e|^2)$. Note that the phase $S_i = h \text{arg}(\psi_i)$. A combination of the
above facts gives

$$Q_i = z |\psi_e|^2 |\psi_i|^{-2},$$

$$V_i = -\epsilon^{-2} \Delta^{-1}(|\psi_i|^2 - |\psi_e|^2) + \alpha h \text{arg}(\psi_i) + \nabla x^{-1} [Q_i h \nabla x (\text{arg} \psi_i)].$$
To close the system we need to derive the equation for $\psi_e$. To this end, we take $(\rho, U) := (n_e, u_e)$, $Q := Q_e = z$ and

$$V := V_e = \frac{1}{m} [\phi + \ln |\psi_e|^2] + z \text{arg}(\psi_e)$$

such that $-\nabla V_e = -\frac{1}{m} \nabla_x [\phi + \ln n_e] - z u_e$.

From the above analysis we see that the scaled Euler-Poisson system (2.1)-(2.4) with (2.5) can be formally realized as a semi-classical limit of the following coupled Schrödinger-Poisson system for $\Psi = (\psi_i, \psi_e)$

\begin{equation}
\left(3.2\right) \quad i \hbar \partial_t \Psi = -\frac{\hbar^2}{2} \Delta \Psi + \left[ V + \frac{i \hbar}{2} Q \right] \Psi,
\end{equation}

with $V = (V_i, V_e)^\top$ and $Q = (Q_i, Q_e)^\top$ defined above.

Passing to the limit $m \to 0$ in the second equation of (3.2) one has

$$\phi + \ln |\psi_e|^2 = 0, \quad \text{i.e.} \quad |\psi_e|^2 = e^{-\phi}.$$ 

The limiting Schrödinger-Poisson equation for $\psi := \psi_i$ becomes

\begin{equation}
\left(3.3\right) \quad i \hbar \partial_t \psi = -\frac{\hbar^2}{2} \Delta \psi + \left[ \alpha \text{arg}(\psi) - \phi + \nabla^{-1}(z \hbar e^{-\phi}|\psi|^2 \nabla_x (\text{arg}(\psi))) \right] \psi,
\end{equation}

\begin{equation}
\left(3.4\right) \quad \epsilon^2 \Delta \phi = |\psi|^2 - e^{-\phi},
\end{equation}

which is the desired quantum description of the Euler-Poisson equation (1.1)-(1.3) for $f(u) = \alpha u$.

4. Boundary layers and transition layer

4.1. The initial boundary value problem. We wish to show the initial boundary value problem for (1.1)-(1.3) with initial conditions

\begin{equation}
\left(4.1\right) \quad n = 1, \quad u = 1, \quad \phi = 0, \quad -\infty < x < 0, \quad t = 0,
\end{equation}

and boundary conditions

\begin{equation}
\left(4.2\right) \quad u = u_w(t, \epsilon), \quad \phi = \phi_w(t, \epsilon), \quad \text{at} \quad x = x_w(t, \epsilon), \quad t > 0.
\end{equation}

Consistency requires that $\phi_w(0) = 0$ and $u_w(0, \epsilon) = 1$.

In our formulation there will be two fundamental length scales:

$$\xi = \frac{x}{\epsilon} + \epsilon^{-1/5} b(\tau) \quad \text{the sheath scale (inner)},$$

$$y = \frac{x}{\epsilon^{4/5}} + b(\tau) \quad \text{the intermediate scale (outer)},$$

and the time scale

$$\tau = \frac{t}{\epsilon^{2/5}}.$$
For convenience set $\delta = \epsilon^{2/5}$. Let us rewrite the Euler-Poisson system at the intermediate scale by introducing the asymptotic expansions

$$u = 1 + \sum_{i=1}^{\infty} \delta^i u_i,$$

$$n = 1 + \sum_{i=1}^{\infty} \delta^i n_i,$$

$$\phi = \sum_{i=1}^{\infty} \delta^i \phi_i,$$

Substitution into the conservation of mass equation (1.1) with $D_\tau := \partial_y + b'(\tau) \partial_y$ gives

$$\delta^{-1} D_\tau [\delta n_1 + \delta^2 n_2] + \delta^{-2} \partial_y \delta(n_1 + u_1) + \delta^2 (n_1 u_1 + n_2 + u_2)] = z(1 - \delta \phi_1) + O(\delta^2),$$

and $O(\delta^{-1})$ terms are balanced if we take $n_1 = -u_1 + c(\tau)$, where $c(\tau)$ denotes an arbitrary function of $\tau$, so that

$$n = 1 - \delta(u_1 - c(\tau)) + \delta^2 n_2 + \cdots,$$

and $O(1)$ balance yields

$$D_\tau n_1 + \partial_y (n_1 u_1) = -\partial_y (n_2 + u_2) + z.$$  

Substitution of the expansion into the momentum equation (1.2) yields

$$\delta^{-1} D_\tau [\delta u_1 + \delta^2 u_2] + \delta^{-2} (1 + \delta u_1 + \delta^2 u_2) \partial_y [\delta u_1 + \delta^2 u_2]$$

$$= \delta^{-1} \partial_y [\phi_1 + \delta \phi_2] - f(1) - f'(1)(\delta u_1) - z \frac{(1 + \delta u_1 + \cdots)(1 - \delta \phi_1)}{1 + \delta n_1 + \delta^2 n_2 + \cdots} + O(\delta^2),$$

where Taylor’s expansion for $f(1 + \delta u_1 + \cdots) = f(1) + f'(1)(\delta u_1 + \cdots)$ has been used. Balance at levels of $O(\delta^{-1})$ and $O(1)$ gives

$$O(\delta^{-1}): \quad \partial_y (u_1 - \phi_1) = 0,$$

$$O(1): \quad D_\tau u_1 + u_1 \partial_y u_1 + \partial_y u_2 = \partial_y \phi_2 - f(1) - z,$$

respectively. Also Poisson’s equation (1.3) yields:

$$\epsilon^2 \delta^{-4} \partial_y^2 [\delta \phi_1 + \delta^2 \phi_2] = \delta(\phi_1 + n_1) + \delta^2 (n_2 + \phi_2 - \frac{\phi_1^2}{2}) + O(\delta^3).$$

Since for $\delta = \epsilon^{2/5}, \epsilon^2 \delta^{-4} = \delta$, the relations from balance of terms $O(\delta)$ and $O(\delta^2)$ give

$$O(\delta): \quad n_1 + \phi_1 = 0,$$

$$O(\delta^2): \quad \partial_y^2 \phi_1 = n_2 + \phi_2 - \frac{\phi_1^2}{2}.$$

Substitution of (4.6), (4.7) into (4.5) gives

$$D_\tau \phi_1 + c'(\tau) + c(\tau) \partial_y \phi_1 + \phi_1 \partial_y \phi_1 = \partial_y (\phi_2 - u_2) - f(1) - z.$$  

Note that (4.4) with $n_1 = -\phi_1$ and $u_1 = -n_1 + c(\tau)$ lead to

$$\partial_y (n_2 + u_2) = D_\tau \phi_1 + \partial_y \phi_1^2 + c(\tau) \partial_y \phi_1$$

and (4.9) gives

$$\partial_y (n_2 + \phi_2) = \frac{\delta^3}{2} \phi_1 + \phi_1 \partial_y \phi_1.$$
Both when inserted into (4.8) yield the following equation

\[ 2D_\tau \phi_1 + \partial_y \phi_1^2 + 2c(\tau) \partial_y \phi_1 - \partial_y^3 \phi_1 = -f(1) - z - c'(\tau). \]

In order to determine \( c(\tau) \) we need to impose an additional condition. Assume that the momentum \( nu \) approaches the state \( nu = 1 \) as \( y \to -\infty \) for all \( \tau \). Thus one has \((1 + \delta u_1 + \cdots)(1 + \delta u_1 + \cdots) \to 1\) as \( y \to -\infty \) and \( n_1 + u_1 \to 0 \) as \( y \to -\infty \). This combined with the fact that \( n_1 + u_1 = c(\tau) \) implies \( c(\tau) \equiv 0 \) and so the scaled potential \( \phi_1 \) is governed by the “KdV=-f(1)-z” equation

\[ (4.9) \quad 2D_\tau \phi_1 + \partial_y \phi_1^2 - \partial_y^3 \phi_1 = -f(1) - z. \]

4.2. Sheath inner layer solution. Set \( \xi = \frac{x}{\epsilon} + b(\tau) \epsilon^{1/5} \) and \( \tau = t/\epsilon^{2/5} \), we then have

\[ e^{3/5}[\partial_\tau + \epsilon^{-1/5} b'(\tau) \partial_\xi] n + \partial_\xi(nu) = \epsilon z e^{-\phi}, \]

\[ e^{3/5}[\partial_\tau + \epsilon^{-1/5} b'(\tau) \partial_\xi] u + u \partial_\xi u = \partial_\xi \phi - \epsilon f(u) - \epsilon z u n^{-1} e^{-\phi}, \]

\[ \partial_\xi^2 \phi = n - e^{-\phi}. \]

Thus to leading order in \( \epsilon \), the solution is a profile satisfying

\[ (4.13) \quad \partial_\xi(nu) = 0, \]

\[ (4.14) \quad u \partial_\xi u = \partial_\xi \phi, \]

\[ (4.15) \quad \partial_\xi^2 \phi = n - e^{-\phi}. \]

The original expansions for the transition layer give

\[ \lim_{\epsilon \to 0} \frac{u(y, \tau)}{y \to 0} = 1 + \epsilon^{2/5} u_1(y, \tau) + \cdots = 1, \]

\[ \lim_{\epsilon \to 0} \frac{n(y, \tau)}{y \to 0} = 1 + \epsilon^{2/5} n_1(y, \tau) + \cdots = 1, \]

\[ \lim_{\epsilon \to 0} \frac{\phi(y, \tau)}{y \to 0} = \epsilon^{2/5} \phi_1(y, \tau) + \cdots = 0, \]

and hence the matching condition for the steady inner sheath solution is

\[ (4.16) \quad \lim_{\xi \to -\infty} u(\xi, \tau) = \lim_{\xi \to -\infty} n(\xi, \tau) = 1, \quad \lim_{\xi \to -\infty} \phi(\xi, \tau) = 0. \]

Thus integration of (4.14) subject to the above boundary conditions yields

\[ (4.17) \quad u^2 = 1 + 2\phi. \]

Substitution of (4.17) into the Poisson equation (4.15) gives the classical sheath equation

\[ (4.18) \quad \partial_\xi^2 \phi = (1 + 2\phi)^{-1/2} - e^{-\phi}, \]

where \( nu = 1 \) derived from (4.13) has been used. Its energy integral is

\[ (4.19) \quad \frac{1}{2}(\partial_\xi \phi)^2 = \sqrt{1 + 2\phi + e^{-\phi}} - 2, \]

where again (4.16) is used. Consistency of the wall boundary implies that we must restrict ourselves to the case when \( u_w, \phi_w \) satisfy (4.17)-(4.19).
5. Justification of the KdV Dynamics

We are now in a position to modify the KdV model to match the dynamics hidden in the inner layer solution.

Recall that the scaled quantities

\[ y = \epsilon^{-4/5}x + b(\tau), \quad \xi = \epsilon^{-1}x + \epsilon^{-1/5}b(\tau) \]

give

\[ y = \epsilon^{1/5}\xi. \]

If we write equation (4.9) in the independent and dependent variables \( \xi \) and \( \phi = \epsilon^{2/5}\phi_1 \) we obtain

\[ 2\epsilon^{3/5}D_{\tau}\phi = \partial_\xi[\partial_\xi^2\phi - \phi^2] - \epsilon(f(1) + z). \]

Note that the steady inner solution reads

\[ \partial_\xi^2\phi - \phi^2 = F(\phi), \]

where

\[ F(\phi) := (1 + 2\phi)^{-1/2} - e^{-\phi} - \phi^2 \sim O(\phi^3) \quad \text{for} \quad |\phi| < 1/2. \]

We thus introduce the model

\[ 2\epsilon^{3/5}D_{\tau}\phi = \partial_\xi[\partial_\xi^2\phi - \phi^2 - F(\phi)] - \epsilon(f(1) + z). \]

We now change back to the variables \( y \) and \( \phi_1 \) to obtain

\[ 2D_{\tau}\phi_1 = \partial_y^3\phi_1 - 2\phi_1\partial_y\phi_1 - \epsilon^{-2/5}F'(\epsilon^{2/5}\phi_1)\partial_y\phi_1 - (f(1) + z), \]

which is a perturbed KdV equation. In order to normalize the above equation we introduce

\[ \psi = \frac{\sqrt{2}}{6}\left(\phi_1 + \frac{f(1) + z}{2}\tau\right), \]

\[ \eta = \sqrt{2}\left(y + \frac{f(1) + z}{4}\tau^2\right). \]

Then consider \( \phi_1 = \frac{6}{\sqrt{2}}\psi - \frac{f(1)+z}{2}\tau : \)

\[ 2D_{\tau}\phi_1 = \frac{12}{\sqrt{2}}[D_{\tau}\psi + \frac{3\sqrt{2}}{2}(f(1) + z)\tau\partial_\eta\psi] - (f(1) + z), \]

\[ \{\partial_y, \partial_y^2, \partial_y^3\}\phi_1 = \frac{6}{\sqrt{2}}\{\sqrt{2}\partial_\eta, \sqrt{4}\partial_\eta^2, 2\partial_\eta^3\}\psi, \]

and so

(5.1) \[ D_{\tau}\psi + [6\psi + g(\epsilon, \tau, \psi)]\partial_\eta\psi - \partial_\eta^2\psi = 0, \]

where

\[ g := 3\epsilon^{-2/5}G\left(\epsilon^{2/5}\left(\frac{6}{\sqrt{2}}\psi - \frac{f(1) + z}{2}\tau\right)\right) \]

with \( G \) determined by

(5.2) \[ G(\phi) := F'(\phi) = e^{-\phi} - \phi - (1 + 2\phi)^{-3/2}. \]

As before we take \( \phi = \phi_w(\epsilon^{2/5}\tau, \epsilon) \) at the wall \( y = 0 \) and from the intermediate sheath layer we see \( \partial_\xi^2\phi = (1 + 2\phi_w)^{-1/2} - e^{-\phi_w} \) at \( y = 0 \) as well.
6. FROM PERTURBED KdV TO KdV

As is well known, the KdV equation of the form

\[ \partial_\tau q + \partial_\eta q + 6q \partial_\eta q - \partial^3_\eta q = 0 \]  

with periodic or decaying data on \((-\infty, \infty)\), is a completely integrable system and the solution of its corresponding initial value problem can be explicitly solved via the celebrated inverse scattering approach [3]. It is also believed that any perturbation imposed on the original KdV would easily render the failure of approach due to the loss of the integrability. In this section we wish to bridge between our proposed model (5.1) and the exact KdV equation (6.1).

Clearly, the \( KdV = -f(1) - z \) equation is fundamentally different from the exact equation because the presence of the dissipation imposed by the damping and ionization. For the damping and ionization free case \( f(1) = 0, z = 0 \), we will show the perturbed equation can be linked to the exact KdV equation by a nontrivial transformation, see [6, 7].

First we replace the perturbed equation (5.1) by keeping only the leading perturbation term in \( g(\epsilon, \psi) \). It follows from (5.2) that

\[ G(\phi) = \left[ 1 - \phi + \frac{\phi^2}{2} + \cdots \right] - 2\phi - \left[ 1 - 3\phi + \frac{15}{2} \phi^2 + \cdots \right] = -7\phi^2 + O(\phi^3), \quad \text{as } |\phi| \to 0, \]

which upon substitution of its leading term into (5.1) gives a simplified perturbed equation

\[ \partial_\tau \psi + b'(\tau) \partial_\eta \psi + \left( 6\psi - \frac{1176}{\sqrt[4]{4}} \epsilon^{2/5} \psi^2 \right) \partial_\eta \psi - \partial^3_\eta \psi = 0. \]  

Let \( \psi \) be the solution of the equation (6.2), and introduce a transformation

\[ q := -\psi - \frac{14}{\sqrt[2]{2}} \epsilon^{1/5} \partial_\eta \psi + \frac{196}{\sqrt[4]{4}} \epsilon^{2/5} \psi^2. \]  

A simple calculation gives

\[ - [\partial_\tau q + b'(\tau) \partial_\eta q + 6q \partial_\eta q - \partial^3_\eta q] \]

\[ = \left( 1 + \frac{14}{\sqrt[2]{2}} \partial_\eta + \frac{392}{\sqrt[4]{4}} \epsilon^{2/5} \psi \right) \left[ \partial_\tau \psi + b'(\tau) \partial_\eta \psi + \left( 6\psi - \frac{1176}{\sqrt[4]{4}} \epsilon^{2/5} \psi^2 \right) \partial_\eta \psi - \partial^3_\eta \psi \right], \]

from which we can conclude that the new unknown \( q \) satisfies a KdV-type equation

\[ \partial_\tau q + b'(\tau) \partial_\eta q + 6q \partial_\eta q - \partial^3_\eta q = 0 \]  

if \( \psi \) satisfies the perturbed equation (6.2). Note that given \( q, \psi \) is not uniquely determined from the transformation (6.3). Nevertheless such transformation does lead us from perturbed KdV equation to the exact KdV equation with possibly time dependent linear convection (6.4). But now we are in a extremely interesting situation. If \( b'(\tau) = 1 \) i.e., \( b(\tau) = \tau \), then the initial boundary value problem for (6.1) is exactly the equation considered by Fokas [1] in the study of the boundary value problem for (6.1) on the right half line. But the case here is subtly different. In Fokas’s problem wave moves away from the boundary \( \eta = 0 \) while in our case on the negative half line waves move into the wall. In fact this can be seen as the source of the sheath formation. It thus seems very interesting to know if result such as Fokas’s can be obtained for the negative half line problem.
7. Conclusions

The intent of this investigation is to formulate a unified model to describe the dynamics hidden in the plasma-sheath transition layer and inner layer for weakly ionized plasma. The main observation in this work is that above mentioned dynamics is governed by a KdV equation, which reflects the dispersive mechanism hidden in the physical process.

The solution methodology is to use asymptotic methods to simplify the governing equations. The asymptotic expansions take advantage of the many different length and time scales in the problem, and the varying magnitudes of material parameters. In particular, the discrepancy in length scales allows us to isolate the sheath transition region from both pre-sheath region and the inner sheath region. Again this discrepancy allows us to combine the sheath transition and the inner layer into one model equation—a modified KdV equation.

Acknowledgments

Hailiang Liu’s research was supported in part by the National Science Foundation under Grant DMS01-07917. Marshall Slemrod’s research was supported in part by the National Science Foundation Grant Nos DMS-9803223 and DMS-00711463.

References

Mathematics Department, Iowa State University, Ames, IA 50011
E-mail address: hliu@iastate.edu

Mathematics Department, University of Wisconsin, Madison, WI 53706
E-mail address: slemrod@math.wisc.edu