Problem 11. Let $m$ be a positive integer. Calculate the value of

$$
\sum_{i_m=0}^{2} \sum_{i_{m-1}=0}^{i_m} \sum_{i_{m-2}=0}^{i_{m-1}} \cdots \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2} \sum_{i_0=0}^{i_1} 1.
$$

Solution 11. The sum will be equal to the number of $m+1$-tuples 

$$(i_0, i_2, \ldots, i_m)$$

of integers with $0 \leq i_0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq 2$. To count the number of such $m+1$-tuples we use the standard “stars and bars” approach. In the following display, each number in an allowable $m+1$-tuple is denoted by a * and the spaces between these elements by −:

$$
- * - * - * - \ldots - * - *
$$

where we have included a space before the first and after the last *. We now insert two vertical bars | in the spaces. The two bars may be in different spaces or the two bars may be put in the same space. All * to the left of the left most | are 0’s, all those between the two bars are 1’s, and those to the right of the rightmost bar are 2’s. If a bar is in the first space, there will be no 0’s. If both bars are in the same space, there will be no 1’s. If both bars are in the same space, there will be no 1’s. Because there are $m+2$ spaces the bars can be put into different spaces in $\binom{m+2}{2}$ ways, and they can be put into the same space in $m+2$ ways. The total number of ways of placing the two bars (and the total number of $m+1$-tuples of the desired sort) is

$$
\binom{m+2}{2} + (m+2) = \binom{m+3}{2}.
$$