Problem 3. Let $x = a$ be a solution of the equation

$$x^{2012} - 7x + 6 = 0.$$ 

Find all possible values for

$$1 + a + a^2 + \cdots + a^{2011}.$$ 

Solution. If $a \neq 1$ then

$$1 + a + a^2 + \cdots + a^{2011} = \frac{a^{2012} - 1}{a - 1}. \quad (1)$$

If $x = a$ is a solution to $x^{2012} - 7x + 6 = 0$, then $a^{2012} = 7a - 6$. Substituting in (1) we find

$$1 + a + a^2 + \cdots + a^{2011} = \frac{(7a - 6) - 1}{a - 1} = 7.$$ 

Finally, note that $a = 1$ is also a solution to $x^{2012} - 7x + 6 = 0$. For this value of $a$ equation (1) is not valid, however in this case it is easy to calculate

$$1 + a + a^2 + \cdots + a^{2011} = 2012.$$ 

Thus the two possible values for the sum are 7 and 2012.

Note. The argument leading to the answer 7 is true for all zeros different from 1, in particular, for each of the non-real roots. Some students noted that the polynomial has two real zeros, $\alpha$ and 1 with $0 < \alpha < 1$. Reasoning this way they went on to note that $\alpha^{2012}$ is (likely?) small so then can be ignored. This leads to

$$0 = \alpha^{2012} - 7\alpha + 6 \approx -7\alpha + 6 \quad \text{so that} \quad \alpha \approx \frac{6}{7}.$$ 

However, this value is not a zero of the polynomial and the sum

$$\sum_{k=0}^{2011} \left( \frac{6}{7} \right)^k = \frac{1 - \left( \frac{6}{7} \right)^{2012}}{1 - \frac{6}{7}}$$

is close to 7 but is not equal to 7.