**Problem 13.** For a positive integer $n$, the numbers $2^n$ and $5^n$ are both expanded (in base ten.) Prove that if these two numbers have the same (non zero) leading digit, then this digit *must* be 3. (As an example $2^5 = 32$ and $5^5 = 3125$ both have leading digit of 3.)

**Solution.** Let $a$ be the common first digit for $2^n$ and $5^n$. It is easy to check that $n \geq 4$ so there are positive integers $k$ and $l$ so that

$$a \cdot 10^k < 2^n < (a + 1)10^k \quad \text{and} \quad a \cdot 10^l < 5^n < (a + 1)10^l.$$ 

Multiplying, we find

$$a^210^{k+l} < 10^n < (a + 1)^210^{k+l},$$

so

$$a^2 < 10^{n-k-l} < (a + 1)^2.$$ 

Because $a$ is a digit and a power of 10 is strictly between $a^2$ an $(a + 1)^2$, it follows that $a = 3$. 
