Generating $p$-extremal graphs

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ISU MECS Seminar
1. Discussed generation of combinatorial objects.
2. “Defined” symmetry in terms of automorphism groups.
3. Presented **canonical deletion**, a method to remove isomorphic duplicates.
4. Discussed example for generating connected graphs by vertex augmentations.
Perfect Matchings

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**Question (Dudek, Schmitt, ’12)** What is the maximum number of edges in a graph with exactly $n$ vertices and $p$ perfect matchings?

**Definition** Let $n$ be an even number and fix $p \geq 1$.

$$f(n, p) = \max\{|E(G)| : |V(G)| = n, \Phi(G) = p\}.$$ 

Graphs attaining this number of edges are **$p$-extremal**.
Hetyei’s Theorem

Theorem (Hetyei’s Theorem, 1986) For all even $n \geq 2$,

$$f(n, 1) = \frac{n^2}{4}.$$
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The Form of $f(n, p)$

Theorem (Dudek & Schmitt)

For each $p$, there exist constants $n_p, c_p$ so that for all $n \geq n_p$,

$$f(n, p) = \frac{n^2}{4} + c_p.$$ 

Take $G$ with $\frac{n^2}{4} + c$ edges.
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Add edges to get $\frac{(n+2)^2}{4} + c$ edges.
The Excess of a Graph

Let $\Phi(G) > 0$. The excess $c(G)$ is

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In this sense, lower bounds on $c_p$ are “easy” (any $G$ with $\Phi(G) = p$, has $c(G) \leq c_p$).

Upper bounds are hard: must prove NO graph achieves a higher constant!
Edge Types

Let $\Phi(G) > 0$ and $e \in E(G)$.

- $e$ is **extendable** if there exists a perfect matching containing $e$.
- $e$ is **forbidden** otherwise.
Types of Graphs

Let $G$ be connected with $\Phi(G) > 0$. 

▶ $G$ is extendable if all edges are extendable.

▶ $G$ is a chamber if the set of extendable edges forms a connected (spanning) subgraph.

▶ $G$ is $p$-extremal if $\Phi(G) = p$ and $c(G) = c_p$. 

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![Diagram of graphs illustrating types of graphs](image-url)
Barsriers

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![Diagram of spires with chambers and barriers](image-url)
$p$-Extremal Graphs are Spires

**Theorem (Hartke, Stolee, West, Yancey ’12)** If $G$ is $p$-extremal, then $G$ is a spire of chambers $G_1, \ldots, G_k$, with barriers $X_i \subseteq V(G_i)$ of maximum size.
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4. $c_p = c(G) \leq \sum_{i=1}^{k} c(G_i)$ with equality if and only if $\frac{|X_i|}{|V(G_i)|} = \frac{1}{2}$ for all $i < k$. 
Order of Chambers

\[
\begin{align*}
X_4 & \; G_4 & \frac{|X_4|}{|V(G_4)|} \\
X_3 & \; G_3 & \frac{|X_3|}{|V(G_3)|} \\
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Finiteness

Characterizing $p$-extremal graphs becomes finite for each fixed $p$. 
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If $G$ has $p$ perfect matchings and $c = c(G)$, then

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is an upper bound on the maximum size of a $p$-extremal chamber.
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is an upper bound on the maximum size of a $p$-extremal chamber.

For $p \leq 10$, $N_p \leq 12$ and \texttt{geng} can enumerate all possible graphs.
Theorem (HSWY, ’12) For even $n$ with $n \geq 6$, the unique 7-extremal graph has $\frac{n^2}{4} + 3$ edges and is a spire with $k = n/2 - 2$ chambers $G_1, \dotsc, G_k$ are given by $G_i = K_2$ for $i < k$ and $G_k$ given below.
Let’s generate all graphs of order $n$ by adding vertices one-by-one.

**Augmentation:** Add a vertex adjacent to a set $S \subset V(G)$.

**IMPORTANT:** Only one augmentation per orbit!

**Deletion:** Select a vertex $v \in V(G)$ to delete, $G' = G - v$. 
Extremal Chambers for $p \leq 10$
Values of $c_p$ for $p \leq 10$

<table>
<thead>
<tr>
<th>$p$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_p$</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
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<tr>
<td>$c_p$</td>
<td>0</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
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<td>12</td>
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</tr>
</tbody>
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Table: Known values of $n_p$ and $c_p$. 
Φ Not Monotonic for Vertex Augmentations
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Focus on Chambers

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We break chambers into the extendable and forbidden edges.
Extendable Graphs are 2-connected

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The Structure of Extendable Edges

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**Theorem (Lovász Two-Ears Theorem)** If $H$ is a extendable graph, there is a graded ear decomposition

$$H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k$$

such that

1. $H_0 \sim C_{2\ell}$ for some $\ell$ and $H_k = H$.
2. Each $H_i$ is extendable.
3. Each ear augmentation $H_i \subset H_{i+1}$ uses one or two ears of even order.

Graphs which appear "between" two extendable graphs in a two-ear augmentation are **almost extendable** graphs.
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Example: Generating Graphs by Ear Augmentations

Let’s generate all graphs of order \( n \) by adding vertices one-by-one.

**Initialization:** Let \( G \) be a cycle.

**Augmentation:** Let \( x, y \in V(G) \) be distinct vertices and \( \ell \) a length. Add an ear of length \( \ell \) between \( x \) and \( y \).

**Deletion:** Select an ear to delete, such that \( G \) remains 2-connected.
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Canonical Deletion by Filtering

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3. Among ears in $S$, minimize their length.
4. Among ears in $S$, minimize the degrees of their endpoints.
5. Compute a canonical labeling $\ell$, and set $\varepsilon = \arg\min_{\varepsilon \in S} \{ n(G) \ell(\varepsilon_1) + \ell(\varepsilon_2) \}$.

The ear $\varepsilon$ is the canonical deletion.
Let $S$ be the set of ears of $G$. Filter $S$ until $|S| = 1$ by the following conditions:

1. Remove ears $e \in S$ such that $G - e$ is not 2-connected.
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If $H$ is extendable, then let $\mathcal{E}(H)$ be the collection of supergraphs $G$ where all edges in $E(G) \setminus E(H)$ are forbidden.
Lemma. Let $H$ be a 1-extendable graph on $n$ vertices with $\Phi(H) = q$. Let $H'$ be a 1-extendable supergraph of $H$ built from $H$ by a graded ear decomposition. Let $\Phi(H') = p > q$ and $N = n(H')$. Choose $G \in \mathcal{E}(H)$ and $G' \in \mathcal{E}(H')$ with the maximum number of edges in each set. Then,

$$c(G') \leq c(G) + 2(p - q) - \frac{1}{4}(N - n)(n - 2).$$
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The Full Search Algorithm (In Parts)

Begin with $p$, $c$, $N$. Generate all chambers $G$ with $p$ perfect matchings, $c(G) \geq c$, and $n(G) \leq N$. 
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3. If \( \Phi(H) > p \), then backtrack.
4. Maintain and update list of barriers on \( H \).
5. Find maximum chambers \( G \) by adding forbidden edges to \( H \).
6. If \( c(G) + 2(p - \Phi(H)) < c \), then backtrack.
7. If \( \Phi(H) = p \), then output all maximum \( G \) (with \( c(G) \geq c \)).
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6. If \( c(G) + 2(p - \Phi(H)) < c \), then backtrack.
The Full Search Algorithm (In Parts)

Begin with $p$, $c$, $N$. Generate all chambers $G$ with $p$ perfect matchings, $c(G) \geq c$, and $n(G) \leq N$.

1. Start with an even cycle $H$ of order at most $N$.
2. Add ear augmentations to $H$ that match canonical deletions.
3. If $\Phi(H) > p$, then backtrack.
5. Find maximum chambers $G$ by adding forbidden edges to $H$.
6. If $c(G) + 2(p - \Phi(H)) < c$, then backtrack.
7. If $\Phi(H) = p$, then output all maximum $G$ (with $c(G) \geq c$).
## Timing

<table>
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<tr>
<th>( p )</th>
<th>( N_p )</th>
<th>( c_p )</th>
<th>CPU Time</th>
<th>( p )</th>
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<th>( c_p )</th>
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## Results

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|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| $p$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $c_p$ | 5 | 5 | 5 | 6 | 5 | 5 | 6 |
| $n_p$ | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
Extremal Chambers for $11 \leq p \leq 27$
$p$-Extremal Configurations for $p \in \{2, 4\}$

- $p = 2$
  - $c_2 = 1$

- $p = 4$
  - $c_4 = 2$

- $p = 4$
  - $c_4 = 2$
$p$-Extremal Configurations for $p = 8$

$p = 8$

$c_8 = 3$
$p$-Extremal Configurations for $p = 16$

$p = 16$

$c_{16} = 4$
Open Problems!

1. Compute more values of $c_p$. 

Conjecture (Hartke, Stolee, West, Yancey, '12)

Let $p, k, t$ be integers so that $k \in \{1, \ldots, 2^t\}$ and 

$$k(2^t - 1)!! \leq p < (k + 1)(2^t - 1)!!$$

and set 

$$C_p = t2^t - t + k - 1.$$ 

Always $c_p \leq C_p$.

If the conjecture holds, then $c_p \leq O\left((\log p \log \log p)^2\right)$. 

Open Problems!

1. Compute more values of $c_p$.
2. Show a growing lower bound on $c_p$. 

Conjecture (Hartke, Stolee, West, Yancey, '12)

Let $p$, $k$, $t$ be integers so that

$$k \in \{1, \ldots, 2^t\} \quad \text{and} \quad k \cdot (2^t - 1)!! \leq p < (k + 1) \cdot (2^t - 1)!!$$

set $C_p = \frac{t}{2} - t + k - 1$.

Always $c_p \leq C_p$.

If the conjecture holds, then $c_p \leq O((\log p \log \log p)^2)$. 

Open Problems!

1. Compute more values of $c_p$.
2. Show a growing lower bound on $c_p$.
3. Show a logarithmic(?) upper bound on $N_p^*$.
Open Problems!

1. Compute more values of $c_p$.
2. Show a growing lower bound on $c_p$.
3. Show a logarithmic(?) upper bound on $N_p^*$.

**Conjecture (Hartke, Stolee, West, Yancey, ’12)** Let $p, k, t$ be integers so that $k \in \{1, \ldots, 2t\}$ and

$$k(2t - 1)!! \leq p < (k + 1)(2t - 1)!!$$

and set $C_p = t^2 - t + k - 1$. **Always** $c_p \leq C_p$. 
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and set $C_p = t^2 - t + k - 1$. **Always** $c_p \leq C_p$.

If the conjecture holds, then $c_p \leq O \left( \left( \frac{\log p}{\log \log p} \right)^2 \right)$. 
If you learned ANYTHING...
If you learned ANYTHING...

...then it should be that

pairing structural theorems with specialized algorithms can be very effective!
Generating \( p \)-extremal graphs

Derrick Stolee

Iowa State University
dstolee@iastate.edu
http://www.math.iastate.edu/dstolee/

December 9, 2013
ISU MECS Seminar