Partitioning a graph into highly connected subgraphs

Valentin Borozan\textsuperscript{1,5}, Michael Ferrara\textsuperscript{2}, Shinya Fujita\textsuperscript{3},
Michitaka Furuya\textsuperscript{4}, Yannis Manoussakis\textsuperscript{5},
Narayanan N\textsuperscript{5,6} and Derrick Stolee\textsuperscript{7}

Abstract

Given \( k \geq 1 \), a \( k \)-proper partition of a graph \( G \) is a partition \( \mathcal{P} \) of \( V(G) \) such that each part \( P \in \mathcal{P} \) induces a \( k \)-connected subgraph of \( G \). We prove that if \( G \) is a graph of order \( n \) such that \( \delta(G) \geq \sqrt{n} \), then \( G \) has a 2-proper partition with at most \( n/\delta(G) \) parts. The bounds on the number of parts and the minimum degree are both best possible. We then prove that if \( G \) is a graph of order \( n \) with minimum degree

\[
\delta(G) \geq \sqrt{c(k-1)n},
\]

where \( c = \frac{2123}{180} \), then \( G \) has a \( k \)-proper partition into at most \( \frac{cn}{\delta(G)} \) parts. This improves a result of Ferrara, Magnant and Wenger [Conditions for Families of Disjoint \( k \)-connected Subgraphs in a Graph, Discrete Math. 313 (2013), 760–764] and both the degree condition and the number of parts are best possible up to the constant \( c \).

1 Introduction

A graph \( G \) is \( k \)-connected if the removal of any collection of fewer than \( k \) vertices from \( G \) results in a connected graph with at least two vertices. Highly connected

\textsuperscript{1}Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13-15, Hungary; valentin.borozan@gmail.com
\textsuperscript{2}Department of Mathematical and Statistical Sciences, University of Colorado Denver, Denver, CO, 80217-3364. michael.ferrara@ucdenver.edu.
\textsuperscript{3}International College of Arts and Sciences, Yokohama City University, 22-2 Seto, Kanazawa-ku, Yokohama 226-0027, Japan; shinya.fujita.ph.d@gmail.com
\textsuperscript{4}Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan; michitaka.furuya@gmail.com
\textsuperscript{5}L.R.I.,Bât.490, University Paris 11 Sud, 91405 Orsay Cedex, France; yannis.manoussakis@lri.fr
\textsuperscript{6}Department of Mathematics, Indian Institute of Technology Madras, Chennai, India.; naru@iitm.ac.in
\textsuperscript{7}Department of Mathematics, Department of Computer Science, Iowa State University, Ames, IA, 50011. dstolee@iastate.edu
graphs represent robust networks that are resistant to multiple node failures. When a graph is not highly connected, it is useful to partition the vertices of the graph so that every part induces a highly connected subgraph. For example, Hartuv and Shamir [9] designed a clustering algorithm where the vertices of a graph $G$ are partitioned into highly connected induced subgraphs. It is important in such applications that each part is highly connected, but also that there are not too many parts. Given a simple graph $G$ and an integer $k \geq 1$, we say a partition $\mathcal{P}$ of $V(G)$ is $k$-proper if for every part $P \in \mathcal{P}$, the induced subgraph $G[P]$ is $k$-connected. We study degree conditions on simple graphs that imply the existence of $k$-proper partitions into few parts.

Ferrara, Magnant, and Wenger [5] demonstrate that a minimum-degree condition on $G$ guarantees a $k$-proper partition.

**Theorem 1** (Ferrara, Magnant, Wenger [5]). Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$. If $\delta(G) \geq 2k\sqrt{n}$, then $G$ has a $k$-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq 2kn/\delta(G)$.

In addition, they present a graph $G$ with $\delta(G) = (1 + o(1))\sqrt{(k-1)n}$ that contains no $k$-proper partition. This example, which we make more precise below, leads us to make the following conjecture.

**Conjecture 2.** Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$. If $\delta(G) \geq \sqrt{(k-1)n}$, then $G$ has a $k$-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n-k+1}{\delta-k+2}$.

To see that the degree condition in Conjecture 2, if true, is approximately best possible, let $n, \ell$ and $p$ be integers such that $\ell = \sqrt{(k-1)(n-1)}$ and $p = \frac{\ell}{(k-1)} = \frac{n-1}{\ell}$. Starting from $H = pK_{\ell}$, so that $|H| = n-1$, construct the graph $G$ by adding a new vertex $v$ that is adjacent to exactly $k-1$ vertices in each component of $H$. Then $\delta(G) = \ell - 1$, but there is no $k$-connected subgraph of $G$ that contains $v$.

To see that the number of components in Conjecture 2 is best possible, let $r$ and $s$ be integers such that $r = \sqrt{(k-1)n} - k + 2$ and $s = \frac{n-k+1}{r}$. Consider then $G = sK_{r} \vee K_{k-1}$, which has minimum degree $r + k - 2 = \sqrt{(k-10)n}$, while every $k$-proper partition has at least $s = \frac{n-k+1}{\delta-k+2}$ parts.

As an interesting comparison, Nikiforov and Shelp [11] give an approximate version of Conjecture 2 with a slightly weaker degree condition. Specifically, they prove that if $\delta(G) \geq \sqrt{2(k-1)n}$, then there exists a partition of $V(G)$ such that $n - o(n)$ vertices are contained in parts that induce $k$-connected subgraphs.

In Section 2, we verify Conjecture 2 in the case $k = 2$.

**Theorem 3.** Let $G$ be a graph of order $n$. If $\delta(G) \geq \sqrt{n}$, then $G$ has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.

Ore’s Theorem [12] states that if $G$ is a graph of order $n \geq 3$ such that $\sigma_2(G) = \min\{d(u) + d(v) \mid uv \notin E(G)\} \geq n$, then $G$ is hamiltonian, and therefore has a trivial 2-proper partition. As demonstrated by Theorem 3 however, the corresponding minimum degree threshold is considerably different. Note as well that if $G$ has a 2-factor $\mathcal{F}$, then $G$ has a 2-proper partition, as each component of $\mathcal{F}$ induces a hamiltonian, and therefore 2-connected graph. Consequently, the problem of determining if $G$ has a 2-proper partition can also be viewed as an extension of the 2-factor problem.
[1, 13], which is itself one of the most natural generalizations of the hamiltonian problem [6, 7, 8].

In Section 3, we improve the bound on the minimum degree to guarantee a $k$-proper partition for general $k$, as follows.

**Theorem 4.** If $G$ is a graph of order $n$ with
\[ \delta(G) \geq \sqrt{\frac{2123}{180}(k-1)n} \]
then $G$ has a $k$-proper partition into at most $\frac{2123n}{180}$ parts.

Conjecture 2 yields that both the degree condition and the number of parts in the partition in Theorem 4 are best possible up to the constant $\frac{2123}{180}$. Our proof of Theorem 4 has several connections to work of Mader [10] and Yuster [15], discussed in Section 3. One interesting aspect of our proof is that under the given conditions, the greedy method of building a partition by iteratively removing the largest $k$-connected subgraph will produce a $k$-proper partition.

**Definitions and Notation**

All graphs considered in this paper are finite and simple, and we refer the reader to [4] for terminology and notation not defined here. For $v \in V(G)$, we define the eccentricity $ecc_G(v)$ of $v$ in $G$ by $ecc_G(v) = \max\{d_G(v, u) \mid u \in V(G)\}$. The diameter of $G$ is defined to be the maximum of $ecc_G(v)$ as $v$ ranges over $V(G)$, and is denoted by $diam(G)$. Let $H$ be a subgraph of a graph $G$, and for a vertex $x \in V(H)$, let $N_H(x) = \{y \in V(H) \mid xy \in E(H)\}$.

A subgraph $B$ of a graph $G$ is a block if $B$ is either a bridge or a maximal 2-connected subgraph of $G$. It is well-known that any connected graph $G$ can be decomposed into blocks. A pair of blocks $B_1 \cap B_2$ are necessarily edge-disjoint, and if two blocks intersect, then their intersection is exactly some vertex $v$ that is necessarily a cut-vertex in $G$. The block-cut-vertex graph of $G$ is defined to be the bipartite graph $T$ with one partite set comprised of all cut-vertices of $G$ and the other partite comprised of all blocks of $G$. For a cut-vertex $v$ and a block $B$, $v$ and $B$ are adjacent in $T$ if and only if $v$ is a vertex of $B$ in $G$.

**2 2-Proper Partitions**

It is a well-known fact that the block-cut-vertex graph of a connected graph is a tree. This observation makes the block-cutpoint graph, and more generally the block structure of a graph, a useful tool, specifically when studying graphs with connectivity one. By definition, each block of a graph $G$ consists of at least two vertices. A block $B$ of $G$ is proper if $|B| \geq 3$. When studying a block decomposition of $G$, the structure of proper blocks is often of interest. In particular, at times one might hope that the proper blocks will be pairwise vertex-disjoint. In general, however, such an ideal structure is not possible. However, the general problem of
determining conditions that ensure a graph has a 2-proper partition, addressed in one of many possible ways by Theorem 3, can be viewed as a vertex analogue to that of determining when a graph has vertex-disjoint proper blocks.

A block of a graph $G$ is large if the order of the block is at least $2\delta(G) - 1$. A 2-proper partition $\mathcal{P}$ of a graph $G$ is strong if $|\mathcal{P}| \geq \delta(G)$ for every $P \in \mathcal{P}$. Note that if $G$ has a strong 2-proper partition $\mathcal{P}$, then $|V(G)| = \sum_{P \in \mathcal{P}} |P| \geq |\mathcal{P}| \cdot \delta(G)$, and so $|\mathcal{P}| \leq |V(G)|/\delta(G)$.

We prove the following stronger form of Theorem 3:

**Theorem 5.** Let $G$ be a graph of order $n$. Suppose that $\delta(G) \geq \sqrt{n}$. Then $G$ has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$. Furthermore, if $G$ has no large block, then $G$ has a strong 2-proper partition.

**Proof.** Sharpness follows from the sharpness of Conjecture 2, demonstrated above. Since every strong 2-proper partition of $G$ satisfies $|\mathcal{P}| \leq n/\delta(G)$, it suffices to show that

(i) if $G$ has no large block, then $G$ has a strong 2-proper partition, and

(ii) if $G$ has a large block, then $G$ has a (possibly strong) 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.

We proceed by induction on $n$, with the base cases $n \leq 4$ being trivial. Thus we may assume that $n \geq 5$.

First suppose that $G$ is disconnected, and let $G_1, \ldots, G_m$ be the components of $G$. For each $1 \leq i \leq m$, since

$$\delta(G_i) \geq \delta(G) \geq \sqrt{n} > \sqrt{|V(G_i)|},$$

$G_i$ has a 2-proper partition $\mathcal{P}_i$ with at most $|V(G_i)|/\delta(G)$ parts, by induction. Therefore, $\mathcal{P} = \bigcup_{1 \leq i \leq m} \mathcal{P}_i$ is a 2-proper partition of $G$ with

$$|\mathcal{P}| = \sum_{1 \leq i \leq m} |\mathcal{P}_i| \leq \sum_{1 \leq i \leq m} |V(G_i)|/\delta(G) = n/\delta(G).$$

Note that since $\delta(G) \leq \delta(G_i)$, any large block $B$ of $G_i$ is also a large block of $G$. Thus, if $G$ contains no large block, then neither does any $G_i$. Hence by induction each $\mathcal{P}_i$ is a strong 2-partition and therefore so too is $\mathcal{P}$.

Hence $G$ is connected. If $G$ is 2-connected, then the trivial partition $\mathcal{P} = \{V(G)\}$ is a strong 2-proper partition of $G$, so we proceed by supposing that $G$ has at least one cut-vertex.

**Claim 1.** If $G$ has a large block, then $G$ has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.

**Proof.** Let $B$ be a large block of $G$. It follows that

$$|V(G - V(B))| \leq n - 2\delta(G) + 1 \leq n - 2\sqrt{n} + 1,$$

and

$$\delta(G - V(B)) \geq \delta(G) - 1 \geq \sqrt{n} - 1.$$
Let recall that $B$ be distinct blocks in $\mathcal{B}$. Therefore $\delta(G - V(B)) \geq \sqrt{n} - 1 \geq \sqrt{|V(G - V(B))|}$.

Applying the induction hypothesis, $G - V(B)$ has a 2-proper partition $\mathcal{P}$ with 

$$|\mathcal{P}| \leq (n - |V(B)|)/\delta(G - V(B)) \leq (n - (2\delta(G) - 1))/(\delta(G) - 1).$$

Since $n(\delta(G) - 1) - (n - \delta(G))\delta(G) = \delta(G)^2 - n \geq n - n = 0, n/\delta(G) \geq (n - \delta(G))/(\delta(G) - 1)$, and hence

$$|\mathcal{P} \cup \{V(B)\}| \leq \frac{n - (2\delta(G) - 1)}{\delta(G) - 1} + 1 = \frac{n}{\delta(G)} - 1 \leq \frac{n}{\delta(G)}.$$

Consequently $\mathcal{P} \cup \{V(B)\}$ is a 2-proper partition of $G$ with $|\mathcal{P} \cup \{V(B)\}| \leq n/\delta(G)$.

By Claim 1, we may assume that $G$ has no large block. Let $\mathcal{B}$ be the set of blocks of $G$. For each $B \in \mathcal{B}$, let $X_B = \{x \in V(B) \mid x \text{ is not a cut-vertex of } G\}$. Note that $N_G(x) \subseteq V(B)$ for every $x \in X_B$. Let $X = \bigcup_{B \in \mathcal{B}} X_B$. For each vertex $x$ of $G$, let $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in V(B)\}$. In particular, for each cut-vertex $x$ of $G$ we have that $|\mathcal{B}_x| \geq 2$.

**Claim 2.** Let $x$ be a cut-vertex of $G$, and let $C$ be a component of $G - x$. Then $|V(C)| \geq \delta(G)$. In particular, every end-block of $G$ has order at least $\delta(G) + 1$.

**Proof.** Let $y \in V(C)$. Note that $d_C(y) \geq d_G(y) - 1 \geq \delta(G) - 1$. Since $N_C(y) \cup \{y\} \subseteq V(C)$, $(\delta(G) - 1) + 1 \leq d_C(y) + 1 \leq |V(C)|$. \qed

**Claim 3.** Let $B$ be an end-block of $G$, and let $x \in V(B)$ be the unique cut-vertex of $G$ in $B$. Then $B - x$ is 2-connected.

**Proof.** Recall that $n \geq 5$. Since $|V(B)| \geq \delta(G) + 1 \geq \sqrt{n} + 1$ by Claim 2, we have $|V(B - x)| \geq 3$. Suppose that $B - x$ is not 2-connected and let $B_1$ and $B_2$ be distinct blocks in $B - x$. For each $i \in \{1, 2\}$, let $y_i \in V(B_1) - V(B_{2-i})$. Since $|(N_{B-x}(y_1) \cup \{y_1\}) \cap (N_{B-x}(y_2) \cup \{y_2\})| \leq |V(B_1) \cap V(B_2)| \leq 1,$

$$|V(B - x)| \geq |(N_{B-x}(y_1) \cup \{y_1\}) \cup (N_{B-x}(y_2) \cup \{y_2\})|$$

$$\geq |N_{B-x}(y_1) \cup \{y_1\}| + |N_{B-x}(y_2) \cup \{y_2\}| - 1$$

$$\geq 2\delta(G) - 1,$$

and hence $|V(B)| \geq 2\delta(G)$, which contradicts the assumption that $G$ has no large block. Therefore $B - x$ is 2-connected. \qed

**Claim 4.** For each $x \in V(G)$, $|N_G(x) \cap X| \geq 2$. In particular, for a block $B$ of $G$, if $X_B \neq \emptyset$, then $|X_B| \geq 3$. 

5
Proof. Suppose that $|N_G(x) \cap X| \leq 1$. Each vertex $y \in N_G(x) - X$ is a cut-vertex of $G$, so there exists a component $C_y$ of $G - y$ such that $x \notin V(C_y)$. For distinct $y, y' \in N_G(x) - X$, we therefore have that $V(C_y) \cap (V(C_{y'}) \cup N_G(x)) = \emptyset$. By Claim 2, $|V(C_y)| \geq \delta(G)$ so

$$n \geq |N_G(x) \cup \{x\}| + \sum_{y \in N_G(x) - X} |V(C_y)|$$

$$\geq (\delta(G) + 1) + |N_G(x) - X|\delta(G)$$

$$\geq (\delta(G) + 1) + (\delta(G) - 1)\delta(G)$$

$$= \delta(G)^2 + 1$$

$$\geq n + 1,$$

which is a contradiction. 

A block $B$ of $G$ is a near end-block if $B$ is an inner-block of $G$ such that there exists a cut-vertex $u \in V(B)$ of $G$ such that, for every $x \in V(B) - (\{u\} \cup X_B)$, $B_x - \{B\}$ is a set of end-blocks of $G$ (see Figure 1). We call the cut vertex $u$ the internal cut vertex of $B$. Let $x$ be a cut vertex in $G$ and let $\{B_1, B_2, \ldots, B_t\} \subseteq B_x$. Define

$$\mathcal{P}(B_x; B_1, \ldots, B_t) = \bigcup_{B' \in B_x, B' \neq \{B_1, \ldots, B_t\}} (V(B') - \{x\}).$$

**Claim 5.** Let $B$ be a near end-block of $G$ and let $u$ be the internal cut vertex of $B$. If $C$ is the unique component of $G - u$ such that $V(B) - \{u\} \subseteq V(C)$, then $C$ has a 2-proper partition $\mathcal{P}$ such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}$.

Proof. For each cut vertex $x$ in $B$ distinct from $u$, fix a block $B_x \in B_x - \{B\}$. Since $B_x$ is an end-block of $G$, $|V(B_x)| \geq \delta(G) + 1$ by Claim 2.

Suppose that $X_B = \emptyset$ and let $\mathcal{C}$ denote the set of components of $C - \bigcup_{x \in V(B - u)} B_x$. Then each $B_x$ and each component of $\mathcal{C}$ is 2-connected by Claim 3, which implies that

$$\mathcal{P} = \left( \bigcup_{x \in V(B - u)} V(B_x) \right) \cup \left( \bigcup_{F \in \mathcal{C}} V(F) \right).$$
is a 2-proper partition of $G$. Furthermore, for every $P \in \mathcal{P}$, $|P| \geq \delta(G)$ by Claim 2. Thus we may assume that $X_B \neq \emptyset$.

We first show that there exists a block of $B - u$ which contains every vertex in $X_B$. Suppose otherwise, so that there are distinct vertices $x$ and $x'$ in $X_B$ that belong to different blocks of $B - u$. Then $|N_{B-u}(x) \cap N_{B-u}(x')| \leq 1$ and $xx' \notin E(G)$. Hence

$$|V(B) - \{u\}| \geq |N_{B-u}(x) \cup N_{B-u}(x') \cup \{x, x'\}|$$

$$= |N_{B-u}(x)| + |N_{B-u}(x')| - |N_{B-u}(x) \cap N_{B-u}(x')| + 2$$

$$\geq 2(\delta(G) - 1) - 1 + 2$$

$$= 2\delta(G) - 1,$$

which contradicts the assumption that $G$ has no large block. Thus there exists a block $B^*$ of $B - u$ such that $X_B \subseteq V(B^*)$. Since $|X_B| \geq 3$ by Claim 4, $B^*$ is 2-connected.

We next show that $|V(B^*)| \geq \delta(G)$. Suppose that $|V(B^*)| \leq \delta(G) - 1$. By the definition of a block, for any $x, x' \in X_B$ with $x \neq x'$, $N_{B-u}(x) \cap N_{B-u}(x') \subseteq V(B^*)$, and so $|(N_{B-u}(x) - (V(B^*) - \{x\})) \cup (N_{B-u}(x) - (V(B^*) - \{x\}))| = |N_{B-u}(x) - (V(B^*) - \{x\})| + |N_{B-u}(x') - (V(B^*) - \{x'\})|$ (see Figure 2). Hence

$$n \geq |V(C)|$$

$$\geq \left| (V(B) - \{u\}) \cup \left( \bigcup_{x \in V(B) \setminus (X_B \cup \{u\})} (V(B_x) - \{x\}) \right) \right|$$

$$= |V(B) - \{u\}| + \sum_{x \in V(B) \setminus (X_B \cup \{u\})} |V(B_x) - \{x\}|$$

$$\geq |V(B) - \{u\}| + \delta(G) (|V(B) - \{u\}| - |X_B|)$$

$$= (\delta(G) + 1)|V(B) - \{u\}| - \delta(G)|X_B|$$

$$\geq (\delta(G) + 1) \left( |V(B^*)| + \sum_{x \in X_B} |N_{B-u}(x) - (V(B^*) - \{x\})| \right) - \delta(G)|X_B|$$

$$= (\delta(G) + 1) \left( |V(B^*)| + \sum_{x \in X_B} (\delta(G) - 1 - (|V(B^*)| - 1)) \right) - \delta(G)|X_B|$$

$$= (\delta(G) + 1)(|V(B^*)| + |X_B|)(\delta(G) - |V(B^*)|) - \delta(G)|X_B|$$

$$= |X_B| \delta(G)^2 - |V(B^*)| (\delta(G) + 1)(|X_B| - 1)$$

$$\geq |X_B| \delta(G)^2 - (\delta(G) - 1)(\delta(G) + 1)(|X_B| - 1)$$

$$= \delta(G)^2 + |X_B| - 1$$

$$\geq n + 3 - 1,$$

which is a contradiction. Thus $|V(B^*)| \geq \delta(G)$. 


The union of components of $G - u$ which do not intersect $V(B) - \{u\}$.

Figure 2: Block $B$ of $G$ and block $B^*$ of $B - u$

By Claims 2 and 3,

$$V(B^*) \cup \left( \bigcup_{x \in V(B^*) - X_B} \mathcal{P}(B_x; B) \right) \cup \left( \bigcup_{x \in V(B) - (V(B^*) \cup \{u\})} (\{V(B_x)\} \cup \mathcal{P}(B_x; B_x)) \right)$$

is a 2-proper partition of $C$ such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}'$. \qed

We divide the remainder of the proof into three cases.

Case 1: $G$ has no inner-block.

Let $x$ be the unique cut-vertex of $G$ and note that $B_x = B$. Then for any $B_1 \in \mathcal{B}$, $\mathcal{P}(B; B_1)$ is a strong 2-proper partition of $G$ by Claims 2 and 3.

Case 2: $G$ has exactly one inner-block.

Let $B_0$ be the unique inner-block of $G$, so that necessarily $B_0$ is a near end-block and $V(B_0) - X_{B_0} \neq \emptyset$. Further, let $u \in V(B_0) - X_{B_0}$ and let $C$ be the unique component of $G - u$ such that $V(B_0) - \{u\} \subseteq V(C)$. Then by Claim 5, $C$ has a 2-proper partition $\mathcal{P}$ such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}$. Let $B_1 \in \mathcal{B}_u - \{B_0\}$. Then $\mathcal{P} \cup \{V(B_1)\} \cup \mathcal{P}(B_u; B_0, B_1)$ is a strong 2-proper partition of $G$ by Claims 2 and 3.

Case 3: $G$ has at least two inner-blocks.

Claim 6. There exists a cut-vertex $u \in V(G)$ of $G$ such that

(i) $\mathcal{B}_u$ contains at least two inner-blocks, and

(ii) there exists a block $B_0 \in \mathcal{B}_u$ such that each block in $\mathcal{B}_u - \{B_0\}$ is an end-block or a near end-block of $G$.

Proof. Let $T$ be the block-cut-vertex graph of $G$. Note that $T$ is a tree with $\text{diam}(T) \geq 6$. Choose a block $A$ of $G$ so that $\text{ecc}_T(A) = \text{diam}(T)$. Then $A$ is an end-block of $G$. Let $a \in V(A)$ be the unique cut-vertex of $G$. By the choice of
A, we see that \( B_u \) contains exactly one inner-block \( A' \) which is a near end-block of \( G \). Let \( u \) be the internal cut vertex of \( A' \). Since \( G \) has at least two inner-blocks, \( B_u \) contains at least two inner-blocks. By the choice of \( A, B_u \) contains at most one block which is neither an end-block nor a near end-block of \( G \). Therefore we get the desired result.

Let \( u \) and \( B_0 \) be as in Claim 6. Let \( \mathcal{C} \) be the set of components of \( G - u \) which do not intersect \( V(B_0) - \{u\} \). For each \( C \in \mathcal{C} \), \( C \) has a 2-proper partition \( \mathcal{P}_C \) such that \( |P| \geq \delta(G) \) for every \( P \in \mathcal{P}_C \) by Claims 2, 3 and 5. Going forward, fix \( C \) to be the element of \( \mathcal{C} \) with largest order. Since \( B_u - \{B_0\} \) contains at least two inner-blocks, \( |V(C)| \geq \delta(G) + 1 \) by Claim 2.

Let \( Z \) be a set of \( \delta(G) \) vertices, disjoint from \( V(G) \), that induces a clique, and let \( G' \) be the graph defined by \( V(G') = (V(G) - V(C)) \cup Z \) and \( E(G') = E(G - V(C)) \cup \{ux \mid x \in Z\} \cup E(Z) \). Then \( \delta(G') = \delta(G) \) and, since \( |V(C)| \geq \delta(G) + 1 \), \( |V(G')| < n \). Consequently, \( \delta(G') = \delta(G) \geq \sqrt{n} > \sqrt{|V(G')|} \). Furthermore, since \( G \) has no large block, \( G' \) also has no large block. Thus, by the induction hypothesis, \( G' \) has a strong 2-proper partition \( \mathcal{P} \). If possible, choose \( \mathcal{P} \) so that \( Z \in \mathcal{P} \). If \( Z \in \mathcal{P} \), then \((\mathcal{P} - \{Z\}) \cup \mathcal{P}_C \) is a strong 2-proper partition of \( G \), as desired. Thus we may assume that \( Z \notin \mathcal{P} \). By the construction of \( G' \), we then necessarily have that \( Z \cup \{u\} \in \mathcal{P} \).

**Claim 7.** \( |N_G(u) \cap X_{B_0}| \leq 1 \).

*Proof.* Suppose that \( |N_G(u) \cap X_{B_0}| \geq 2 \) and choose distinct vertices \( x \) and \( x' \) in \( N_G(u) \cap X_{B_0} \). Let \( P_x \) and \( P_{x'} \) be the unique sets in \( \mathcal{P} \) with that contain \( x \) and \( x' \), respectively. Then \( P_x \neq P_{x'} \subseteq V(B_0) \) and either \( P_x \cap P_{x'} = \emptyset \) or \( P_x = P_{x'} \). We claim that the latter holds. Indeed \( \mathcal{P} \) is a strong 2-proper partition of \( G' \), so \( |P_x| \geq \delta(G') \) and \( |P_{x'}| \geq \delta(G') \). However, \( B_0 \) is not a large block of \( G' \), \( |V(B_0)| \leq 2\delta(G' - 2 \). Hence \( P_x = P_{x'} \). Since \( ux, ux' \in E(G') \), \( P_x \cup \{u\} \) induces a 2-connected subgraph of \( G' \). Thus, by the construction of \( G' \), \((\mathcal{P} - \{P_x, Z \cup \{u\}\}) \cup \{P_x \cup \{u\}, Z \}) \) is a strong 2-proper partition of \( G' \) and contains \( Z \), which is a contradiction.

By Claims 4 and 7, there exists a block \( B_1 \in B_u - \{B_0\} \) such that \( N_G(u) \cap X_{B_1} \neq \emptyset \). Let \( C^* \) be the component of \( G - u \) such that \( V(B_1) - \{u\} \subseteq V(C^*) \). Then \( C^* \in \mathcal{C} \). Suppose that \( B_1 \) is an end-block of \( G \). Note that \( C^* = B_1 - u \). Since \( |V(B_1)| \geq \delta(G) + 1 \) by Claim 2, \((\mathcal{P} - \{Z \cup \{u\}\}) \cup (\cup_{C \in \mathcal{C} - \{C^*\}} \mathcal{P}_C) \cup (V(B_1)) \) is a strong 2-proper partition of \( G \), as desired. Thus we may assume that \( B_1 \) is a near end-block of \( G \). Since \( X_{B_1} \neq \emptyset \), \( \delta(G) + 1 \leq d_G(y) + 1 \leq |V(B_1)| \) where \( y \in X_{B_1} \). Hence

\[
\mathcal{P}^* = \{V(B_1)\} \cup \left( \bigcup_{x \in V(B_1) - (X_{B_1} \cup \{u\})} \mathcal{P}(B_x, B_1) \right)
\]

is a 2-proper partition of \( C^* \) such that \( |P| \geq \delta(G) \) for every \( P \in \mathcal{P}^* \). Therefore \((\mathcal{P} - \{Z \cup \{u\}\}) \cup (\cup_{C \in \mathcal{C} - \{C^*\}} \mathcal{P}_C) \cup \mathcal{P}^* \) is a strong 2-proper partition of \( G \). This completes the proof of Theorem 5. 

\[\square\]
3  k-Proper Partitions

Let $e(k, n)$ be the maximum number of edges in a graph of order $n$ with no $k$-connected subgraph. Define $d(k)$ to be

$$\sup \left\{ \frac{2e(k, n) + 2}{n} : n > k \right\}$$

and

$$\gamma = \sup \{ d(k)/(k - 1) : k \geq 2 \}.$$  

Recall that the average degree of a graph $G$ of order $n$ with $e(G)$ edges is $\frac{2e(G)}{n}$. This leads to the following useful observation.

**Observation 6.** If $G$ is a graph with average degree at least $\gamma(k-1)$, then $G$ contains an $k$-connected subgraph.

In [10], Mader proved that $3 \leq \gamma \leq 4$ and constructed a graph of order $n$ with no $k$-connected subgraph and $(\frac{3}{2}k - 2)(n - k + 1)$ edges. This led him to make the following conjecture.

**Conjecture 7.** If $k \geq 2$, then $e(k, n) \leq (\frac{3}{2}k - 2)(n - k + 1)$. Consequently, $d(k) \leq 3(k - 1)$ and $\gamma = 3$.

Note that Conjecture 7 holds when $k = 2$, as it is straightforward to show that $e(2, n) = n - 1$. The most significant progress towards Conjecture 7 is due to Yuster [15].

**Theorem 8.** If $n \geq \frac{9}{4}(k - 1)$, then $e(k, n) \leq \frac{193}{120}(k - 1)(n - k + 1)$.

Note that Theorem 8 requires $n \geq \frac{9}{4}(k - 1)$, which means that we cannot immediately obtain a bound on $\gamma$. The following corollary, however, shows that we can use this result in a manner similar to Observation 6.

**Corollary 9.** Let $G$ be a graph of order $n$ with average degree $\overline{d}$. Then $G$ contains a $\left\lfloor \frac{60\overline{d}}{193} \right\rfloor$-connected subgraph.

**Proof.** Let $k = \left\lfloor \frac{60\overline{d}}{193} \right\rfloor$ and suppose that $G$ does not contain a $k$-connected subgraph. If $n \geq \frac{9}{4}(k - 1)$, then Theorem 8 implies

$$\frac{1}{2} \overline{d}n = e(G) \leq \frac{193}{120}(k - 1)(n - k + 1) < \frac{193}{120} \left( \frac{60\overline{d}}{193} \right) n = \frac{1}{2} \overline{d}n.$$  

Thus, assume that $n < \frac{9}{4}(k - 1)$. This implies that

$$n < \frac{9}{4}(k - 1) < \frac{9}{4} \frac{60\overline{d}}{193} \overline{d} < \frac{7}{10} \Delta(G),$$  

a contradiction. $\square$

Finally, prior to proving our main result, we require the following simple lemma, which we present without proof.
Lemma 10. If $G$ is a graph of order $n \geq k + 1$ such that $\delta(G) \geq \frac{n+k-2}{2}$, then $G$ is $k$-connected.

We prove the following general result, and then show that we may adapt the proof to improve Theorem 4.

Theorem 11. Let $k \geq 2$ and $c \geq \frac{11}{3}$. If $G$ is a graph of order $n$ with minimum degree $\delta$ with $\delta \geq \sqrt{c\gamma(k-1)n}$, then $G$ has a $k$-proper partition into at most $\lceil \frac{c\gamma n}{\delta} \rceil$ parts.

Proof. Since $n > \delta \geq \sqrt{c\gamma(k-1)n}$, we have $n^2 > c\gamma(k-1)n$ and hence $n > c\gamma(k-1) \geq 11(k-1)$. Therefore, by Lemma 10, it follows that

$$
\delta < \frac{n + k - 2}{2} < \frac{n + (k - 1)}{2} \leq n + \frac{1}{11}n \leq \frac{6}{11}n.
$$

Let $G_0 = G$, $\delta_0 = \delta$, and $n_0 = |V(G)|$. We will build a sequence of graphs $G_i$ of order $n_i$ and minimum degree $\delta_i$ by selecting a $k$-connected subgraph $H_i$ of largest order from $G_i$ and assigning $G_{i+1} = G_i - V(H_i)$. This process terminates when either $G_i$ is $k$-connected or $G_i$ does not contain a $k$-connected subgraph. We claim the process terminates when $G_i$ is $k$-connected and $H_i = G_i$.

By Observation 6, $G_i$ contains a $\lfloor \frac{\delta_i}{3} \rfloor$-connected subgraph $H_i$. If $\frac{\delta_i}{3} \geq k - 1$, then $H_i$ is $k$-connected and has order at least $\lfloor \frac{\delta_i}{3} \rfloor + 1 > \frac{\delta_i}{3}$. Since $H_i$ is a maximal $k$-connected subgraph in $G_i$, every vertex $v \in V(G_i) \setminus V(H_i)$ has at most $k-1$ edges to $H_i$ by a simple consequence of Menger’s Theorem. Therefore, we have

$$
\delta_{i+1} \geq \delta_i - (k - 1)
$$

and

$$
n_{i+1} = n_i - |H_i| < n_i - \delta_i/\gamma.
$$

This gives us the estimates on $\delta_i$ and $n_i$ of

$$
\delta_i \geq \delta - i(k - 1),
$$

and

$$
n_i \leq n - \sum_{j=0}^{i-1} \delta_j / \gamma \leq n - \frac{1}{\gamma} \left( \delta - j(k - 1) \right) = n - \frac{1}{\gamma} \left( i\delta - (k - 1) \binom{i}{2} \right).
$$

Let $t = \lceil \frac{c\gamma n}{\delta} - 4 \rceil = \frac{c\gamma n}{\delta} - (4 - x)$, where $x \in [0, 1)$. We claim that the process terminates with a $k$-proper partition at or before the $(t+1)^{\text{st}}$ iteration (that is, at or before the point of selecting a $k$-connected subgraph from $G_t$). First, we have that

$$
\delta_{t-1} \geq \delta - (t - 1)(k - 1) > \delta - \left( \frac{c\gamma n}{\delta} - 4 \right) (k - 1) = \delta - \frac{c\gamma(k-1)n}{\delta} + 4(k-1).
$$

Note that $\delta^2 \geq c\gamma(k-1)n$ and hence $\delta - \frac{c\gamma(k-1)n}{\delta} \geq 0$. Therefore,

$$
\delta_{t-1} > 4(k - 1) \geq \gamma(k - 1) \quad \text{and} \quad \delta_t \geq 3(k - 1).
$$
As $\delta_i$ is a decreasing function of $i$, we have that $\delta_i > 4(k-1)$ for all $0 \leq i \leq t-1$. Thus each $G_i$ with $i < t$ contains a $k$-connected subgraph. Next, consider $n_t$.

\[ n_t \leq n - \frac{1}{\gamma} \left[ t\delta - (k-1) \left( \frac{t}{2} \right) \right] \]

\[ = n - \frac{1}{\gamma} \left[ c\gamma n - (4-x)\delta - \frac{1}{2}(k-1) \left( \frac{c\gamma n}{\delta} - (4-x) \right) \left( \frac{c\gamma n}{\delta} - (5-x) \right) \right] \]

\[ = n - \frac{1}{\gamma} \left[ c\gamma n - (4-x)\delta - \frac{c^2\gamma^2(k-1)\delta^2}{2} + \frac{c\gamma(9-2x)(k-1)}{2} n - \frac{1}{2}(k-1)(4-x)(5-x) \right] \]

\[ = \frac{1}{\delta^2} \left[ \frac{4-x}{\gamma} \delta^3 + \frac{c^2\gamma(k-1)}{2} n^2 - (c-1)n\delta^2 \right] \]

\[ + \frac{(4-x)(5-x)}{2\gamma}(k-1) - \frac{c(9-2x)(k-1)}{2} n. \]

We have that $\delta^2 \geq c\gamma(k-1)n$ and $(c-1)^2 \geq c^2/2$, so

\[ \frac{(c-1)}{c}((c-1)n\delta^2) \geq (c-1)^2\gamma(k-1)n^2 \geq \frac{c^2}{2}\gamma(k-1)n^2. \]

Also, we have $n > \frac{11\delta}{6}$, and $\frac{c-1}{c} \geq \frac{8}{11}$, hence

\[ \frac{1}{c}((c-1)n\delta^2) > \frac{8}{11} \cdot \frac{11}{6} \delta^3 = \frac{4}{3} \delta^3 \geq \frac{4-x}{\gamma} \delta^3. \]

Summing these inequalities, we get that

\[ \left[ \frac{4-x}{\gamma} \delta^3 + \frac{c^2\gamma(k-1)}{2} n^2 - (c-1)n\delta^2 \right] < 0 \]

and hence $n_t < \frac{(4-x)(5-x)}{2\gamma}(k-1) \leq \frac{20}{2\gamma}(k-1) \leq \frac{10}{2}(k-1)$. However, $\delta_i \geq 3(k-1)$, so if the process has not terminated prior to the $(t+1)^{\text{st}}$ iteration, $G_t$ is $k$-connected by Lemma 10.

*Theorem 11 immediately yields the following.*

**Corollary 12.** Suppose Conjecture 7 holds. We then have that if $G$ is a graph with minimum degree $\delta$ where $\delta \geq \sqrt{11(k-1)n}$, then $G$ has a $k$-proper partition into at most $\frac{11n}{\delta}$ parts.

*We are now ready to prove Theorem 4.*

**Proof.** Observe that the proof of Theorem 11 holds at every step when substituting $\gamma = \frac{193}{60}$ by using Corollary 9 to imply that $G_i$ contains a $\left\lfloor \frac{606}{193} \right\rfloor$-connected subgraph. Finally, note that $\left( \frac{11}{3} \right) \frac{193}{60} = \frac{2123}{180}$. ☐

### 4 Application: Edit Distance to the Family of $k$-connected Graphs

Define the *edit distance* between two graphs $G$ and $H$ to be the number of edges one must add or remove to obtain $H$ from $G$ (edit distance was introduced independently...
in [2, 3, 14]). More generally, the edit distance between a graph \(G\) and a set of graphs \(\mathcal{G}\) is the minimum edit distance between \(G\) and some graph in \(\mathcal{G}\).

Utilizing Theorem 4 and observing that \(2123/180 = 11.794 < 11.8\) we obtain the following corollary, which is a refinement of Corollary 11 in [5] for large enough \(k\).

**Corollary 13.** Let \(k \geq 2\) and let \(G\) be a graph of order \(n\). If \(\delta(G) \geq \sqrt{11.8(k-1)n}\), then the edit distance between \(G\) and the family of \(k\)-connected graphs of order \(n\) is at most \(\frac{11.8kn}{\delta(G)} - k < k(4\sqrt{n} - 1)\).

**Proof.** Let \(H_1, \ldots, H_l\) be the \(k\)-connected subgraphs of the \(k\)-proper partition of \(G\) guaranteed by Theorem 4; note that \(l \leq \frac{11.8n}{\delta(G)}\). For each \(i \in \{1, \ldots, l-1\}\), it is possible to produce a matching of size \(k\) between \(H_i\) and \(H_{i+1}\) by adding at most \(k\) edges between \(H_i\) and \(H_{i+1}\). Thus, adding at most \(k\left(\frac{11.8n}{\delta(G)}\right)\) edges yields a \(k\)-connected graph. \(\square\)

## 5 Conclusion

We note here that it is possible to slightly improve the degree conditions in Theorems 4 and 5 at the expense of the number of parts in the partition. In particular, a greedy approach identical to that used to prove Theorem 5 can be used to prove the following.

**Theorem 14.** Let \(k \geq 2\), \(c_k \geq \frac{k-1}{k} \cdot 2\gamma\), and \(p = \frac{\sqrt{c_k}}{k}\). If \(G\) is a graph of order \(n\) with \(\delta(G) \geq kp = \sqrt{c_k}kn\), then \(G\) has a \(k\)-proper partition into at most \(\frac{k}{k-1}p\) parts.

This gives rise to the following, which improves on the degree condition in Theorem 4.

**Theorem 15.** If \(G\) is a graph of order \(n\) with minimum degree

\[
\delta(G) \geq kp = \sqrt{\frac{193}{30}(k-1)n},
\]

then \(G\) has a \(k\)-proper partition into at most \(\frac{k}{k-1}p\) parts.

**Acknowledgments**

The second author would like to acknowledge the generous support of the Simons Foundation. His research is supported in part by Simons Foundation Collaboration Grant #206692. The third author would like to thank the laboratory LRI of the University Paris South and Digiteo foundation for their generous hospitality. He was able to carry out part of this research during his visit there. Also, the third author’s research is supported by the Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B) (20740095).
References


