LECTURE 1
INTRODUCTION AND OUTLINE

JACOB T. SCHWARTZ
EDITED BY
KENNETH R. DRIESSEL

ABSTRACT. We describe a model of an economy by the Leontief model. Our model first is open and then be closed by introducing labor as an additional commodity. Our model gives rise to a theory of prices, a reflection of the real economy, and useful relationships among the various parameters of an economy.

1. What Will and What Will Not Be Treated

Mathematical economics currently includes, and perhaps is even dominated by, a number of branches with which we will have little to do. Thus, in order to define the subject of the present lectures, it is well to say something about these excluded branches. One topic that we shall not discuss to any great length is the subject that might be called efficiency economics in general, and is often called by the several names of its principal techniques—linear programming, operations research, perhaps also theory of games. In these subjects, the aim is to find the optimal adjustment, in one or another sense, to a given situation; they refer with greatest cogency and success to the profit-making possibilities of a single firm. As an omnibus reference to this area of thought let me cite Vajda’s Linear Programming and the Theory of Games, and also von Neumann and Morgenstern’s sparkling Theory of Games and Economic Behavior. Nor will we deal with econometrics, i.e. applied and theoretical economic statistics, except incidentally. Instead, we shall take economics as the cognitive study of a given object,
the economy, and ask in the sense of natural science: what is this object like, how does it behave, and why? For this reason, we find the term analytical prefacing economics in our title. In spirit, our economics will be theoretical or speculative rather than directly empirical, and thus close in its basic approach to what has been called classical economics. In form, however, we will be more systematically mathematical. The branch of mathematics of which we will make greatest use will be the theory of matrices; let me here make reference to D. T. Finkbeiner’s Introduction to Matrices and Linear Transformations, to Paul Halmos’ Finite Dimensional Vector Spaces, Gantmacher’s Theory of Matrices, and note the existence of numerous other introductory works on this subject. From time to time we will use a bit of calculus.

We will begin with a discussion of the theory of equilibrium prices—what has been traditionally called value theory—and go on to a discussion of business cycle theory, beginning with a model like that introduced by Lloyd Metzler, and developing the connection between this cycle theory and the equilibrium analysis that is more commonly called Keynesian. In the economic literature let me cite, in the first place, the famous General Theory of Keynes, which, as a pioneering work of science, is worth studying in spite of its numerous pedagogical and even theoretical mare’s nests. A stimulating companion volume for the admirer of Keynes is Henry Hazlitt’s The Failure of the New Economics. An Analysis of the Keynesian Fallacies. A superior mathematical exposition of the Keynesian theories is K. Kurihara’s Introduction to Keynesian Dynamics; another, particularly fine, work of a similar sort is H. J. Brems’s Output, Employment, Investment. Much of what we have to say will make reference to the “input-output” model of W. Leontief, on which there exists a vast literature. A good sample of this literature, full of references, is Activity Analysis of Production and Allocation, T. C. Koopmans, ed. Our attempts to compare speculative results with economic reality will be enormously facilitated by the extensive and painstaking work of the National Bureau of Economic Research, published in the form of a great many separate studies. A
very fresh and stimulating empirical account of business cycles is the

2. **A Bouquet of Warnings**

Mathematics may perhaps have a valuable role to play in economics—but its application brings several dangers. Mathematics necessarily works with exact models. In the course of investigating such a model, it is easy to forget that the mathematical exactness of one’s reasoning has nothing to do with the exactness with which the model reflects economic reality. For this reason, a few dampening admonitions are in order. I quote the first and most severe from Ludwig von Mises’ *Human Action*:

>The problems of prices and costs have been treated also with mathematical methods. There have even been economists who held that the only appropriate method of dealing with economic problems is the mathematical method and who derided the logical economists as “literary” economists.

> If this antagonism between the logical and the mathematical economists were merely a disagreement concerning the most adequate procedure to be applied in the study of economics, it would be superfluous to pay attention to it. The better method would prove its preeminence by bringing about better results. It may also be that different varieties of procedure are necessary for the solution of different problems and that for some of them one method is more useful than the other.

> However, this is not a dispute about heuristic questions, but a controversy concerning the foundations of economics. The mathematical method must be rejected not only on account of its barrenness. It is an entirely vicious method, starting from false assumptions and leading to fallacious inferences. Its syllogisms are not only sterile; they divert the mind from the study of the real problems and distort the relations between the various phenomena.

> The deliberations which result in the formulation of an equation are necessarily of a nonmathematical character. The formulation of the equation is the consummation of our knowledge; it does not directly enlarge our knowledge. Yet, in mechanics the equation can render very important practical services. As there exist constant relations between various mechanical elements and as these relations can be ascertained by experiments, it becomes possible to use equations for the solution of definite technological problems. Our modern industrial civilization is mainly an accomplishment of this utilization of the differential equations of physics. No such
constant relations exist, however, between economic elements. The equations formulated by mathematical economics remain a useless piece of mental gymnastics and would remain so even if they were to express much more than they really do.

A corresponding sentiment is voiced by Keynes in his *General Theory*:

It is a great fault of symbolic pseudo-mathematical methods of formalizing a system of economic analysis, such as we shall set down in section VI of this chapter, that they expressly assume strict independence between the factors involved and lose all their cogency and authority if this hypothesis is disallowed; whereas, in ordinary discourse, where we are not blindly manipulating but know all the time what we are doing and what the words mean, we can keep “at the back of our heads” the necessary reserves and qualifications and the adjustments which we shall have to make later on, in a way in which we cannot keep complicated partial differentials “at the back” of several pages of algebra which assume that they all vanish. Too large a proportion of recent “mathematical” economics are mere concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentions and unhelpful symbols.

A more optimistic if still cautious opinion is stated by Professor F. H. Knight in the 1954 *Britannica*.

Any brief statement of principles is bound to make economic theories appear thinner and more remote from the concrete facts of economic life than they are. There is a place and a need for all degrees of generality. In recent decades this need has found increasing expression in the developing and spreading study of mathematical economics, in which exposition is made accurate and compact by the use of graphs and of algebraic formulae.

Only by the use of mathematics is it possible to bring together into a single comprehensible picture the variety, the complexity, and most of all the interdependence of the numerous factors which determine prices, costs, output and demand and the wages or hire of productive agents. . . . The principal value of such elaborate and abstract systems lies in forcibly reminding the enquirer that a change in practically any economic variable has direct or indirect effects on innumerable other magnitudes, and so preventing him from fatally oversimplifying conceptions of economic cause and effect. . . .
The more theoretical parts of economics cannot be taken to be a complete and adequate account of the mechanism of modern economic life. They afford serviceable approximations to partial, but important aspects of the truth.

The most striking and possibly the most important characteristic of recent work in economics, as contrasted with the older, is its greater realism. It does not attempt to do without abstract conceptions, but it does attempt to take these from the world of affairs, or bring them into line with facts.

Hoping to approach Professor Knight’s high goal, we may begin our investigations.

3. Introduction of a Model (Single Labor Sector)

By an economy we shall mean a complex of activities in which various commodities are produced and subsequently either consumed or utilized in the production of further commodities. If the economy absorbs commodities from outside itself, or if it supplies commodities to the outside, it is called open. On the other hand, if the economy is completely self-contained it is called closed. We wish to describe a model of an economy. Our model will in the first instance be open, in that labor must be supplied to the system by a “household sector” and products must be supplied to the household sector by the system. The model will then be “closed” by introducing labor as an additional commodity which is “used up” in the production of various other commodities, and for the production of which these various other commodities are required.

After formulating our model, we shall first indicate the manner in which it gives rise to a simple but interesting theory of prices; next give a brief discussion of the extent to which the model is a faithful reflection of the real economy; and subsequently pass to an extended mathematical analysis of the model, and to an investigation of the question of what additional and useful relationships among the various parameters of an economy can be elucidated by using the model.

We begin by establishing, in some definite but entirely arbitrary way, a certain standard physical unit for each commodity, as, e.g. 1 car, 1 ton of coal, 1 bushel of wheat, etc., hereinafter called one unit of
the commodity. The process of production of any commodity requires appropriate amounts both of circulating and of fixed capital. Thus, for instance, to produce one ton of pig iron it is required, in the first place, that certain amounts of coal—say, half a ton, and certain amounts of iron ore—say, one and a half tons, be used up; but, in addition it is required that a blast furnace be tied up, for a certain period, say for half a day. The blast furnace is tied up but not used up, and hence reckons only as fixed, but not as circulating capital.

These two aspects of production will be described in our general model as follows. Let the economy involve a total of \( n \) commodities, i.e. let \( C_1, \ldots, C_n \) be a total list of the commodities produced in an economy; cars, cigarettes, typewriters, etc. To produce one unit of any given commodity \( C_i \), it is (technologically) required that various amounts \( \pi_{ij} \) of other commodities \( C_j \) be used up; in addition, it is (technologically) required that \( \phi_{ij} \) units of \( C_j \) be tied up (and thus not available for the manufacture of some other product) for a standard production period, say one day, even though these \( \phi_{ij} \) units of \( C_j \) are not necessarily used up. Note that if the standard unit of \( C_1 \) is, say, a bushel, and that of \( C_2 \) is, say, one ton, \( \pi_{12} \) has the dimensions tons per bushel, and \( \phi_{12} \) the dimensions ton-days per bushel.

\( \phi_{ij} \) is said to be the amount of \( C_j \) utilized in the production of one unit of \( C_i \), while \( \pi_{ij} \) is said to be the amount of \( C_j \) consumed in the production of one unit of \( C_i \). When a commodity is consumed it is also utilized and therefore we shall assume

\[
(1.1) \quad \text{If } \pi_{ij} > 0 \text{ then } \phi_{ij} > 0 \quad i, j = 1, \cdots, n.
\]

The model as we have thus far defined it is called the open Leontief model; the matrix \( \pi_{ij} \) is often called the input-output matrix, and analysis of such a model is often called input-output analysis. The matrix \( \phi_{ij} \) may be called the fixed capital matrix. It is clear upon a moment’s reflection that this open model, as it has been defined, permits us to deduce, from a given “final demand” for a certain “bill of goods,” what inputs are required; and thus, for instance, by considering the desired output of military goods in a wartime situation, to predict where
“bottle-necks” are apt to develop. This sort of application has often
been stressed; reference may be made to the work Activity Analysis
of Production and Allocation cited above. Matrices $\pi_{ij}$ for the Ameri-
can economy divided into fifty and into two hundred sectors have been
computed by the Bureau of Labor Statistics. Our interest, however,
will not be in this direct sort of “bottleneck analysis,” but in the use
of the input-output model as a framework for more abstract economic
analysis. For this reason, we proceed at once to a description of a
corresponding closed model.

To close our model, we must introduce labor as an input and as an
output. Let $\pi_{j0}$ denote the amount of labor (measured say, in man-
hours) required for the production of $C_j$, and let $\pi_{0j}$ be the amount of
$C_j$ which is “consumed in order to produce a man-hour of labor,” i.e.,
the average real wages paid out per hour of labor. By the introduction
of these matrix elements the model economy is rendered closed, i.e.
the set of commodities produced is the same as the set of commodities
utilized in production.

In our simple linear economic model (often called the closed Leontief
model) we may readily set up a theory of prices. Let $p_0, p_1, \cdots, p_n$
be the prices of the various products produced; then $p_0, p_1, \cdots, p_n$
are also the prices of the commodities utilized and/or consumed in production.

We formulate the conditions that must be satisfied by the $p_i$.

The price of a commodity $C_i$ is made up of the sum of two terms.
The first is the sum of the values of all of the products consumed in
the manufacture of $C_i$, i.e.

$$\sum_{j=0}^{n} \pi_{ij} p_j. \quad (1.2)$$

The second is the “return to capital,” “markup,” or “profit” propor-
tional to the sum of the values of the products which are utilized but
not necessarily consumed in the manufacture of $C_i$, i.e.

$$\rho \sum_{j=0}^{n} \phi_{ij} p_j. \quad (1.3)$$
(We take $\phi_{i0} = 0$).

We have here taken an essential step in assuming the rate of profit, $\rho$, to be the same for all types of production. This corresponds to the ordinary assumption, in the theory of prices, of “free competition”; it can be justified in the usual way by arguing that a situation in which the production of different commodities yields different rates of profit cannot be stable, since investments would be made only in the industry yielding the highest rate of profit to the exclusion of other commodities yielding lower rates of profit. Long-term equilibrium, of which our simple theory is alone descriptive, would be reached only when all such rates of profit became equal. The proportionality constant $\rho$ has the dimensions per cent per day (or year).

Forming the price of $C_i$ additively out of the two expressions (1.2) and (1.3), we have the set of equations

$$(1.4) \quad p_i = \sum_{j=0}^{n} \pi_{ij} p_j + \rho \sum_{j=0}^{n} \phi_{ij} p_j, \quad i = 1, \cdots, n.$$ 

Thus far we have only an “open” system of equations for the prices $p_i$. We can obtain a closed system by recalling that $\pi_{0j}$ denotes the collection of commodities which form real wages for an hour’s labor; thus the price $p_0$ of an hour’s labor must be given by the equation

$$(1.5) \quad p_0 = \sum_{i=0}^{n} \pi_{0i} p_i.$$ 

If we introduce additional matrix elements $\phi_{0j}$ by putting $\phi_{0j} = 0$, we may write (1.5) in the same form as the equation (1.4), and hence may write (1.5) and (1.4) together in the simple form

$$(1.6) \quad p_i = \sum_{j=0}^{n} \pi_{ij} p_i + \rho \sum_{j=0}^{n} \phi_{ij} p_j, \quad i = 0, \cdots, n.$$ 

This set of $n + 1$ equations is homogeneous in the $n + 1$ variables $p_j$, but contains the additional unknown $\rho$. Thus, we would expect that the system (1.6) determines the quantity $\rho$ and the set of $n$ ratios of the $n + 1$ quantities $p_i$. We will show in the next lecture that this is
rigorously correct. Before going over to the necessary detailed and general mathematical investigation, however, let us examine some simple transformations and special cases of the system (1.6).

In the first place, we may make use of the particularly simple form of equation (1.5) to eliminate $p_0$ from the system (1.6). This gives

$$(1.7) \quad p_i = \sum_{j=1}^{n} [\pi_{ij} + (1 - \pi_{00})^{-1} \pi_{i0} \pi_{0j}] p_j + \rho \sum_{j=1}^{n} \phi_{ij} p_j, \quad i = 1, \ldots, n.$$  

If we define a modified input-output and fixed capital matrix by

$$(1.8) \quad \tilde{\pi}_{ij} = \pi_{ij} + (1 - \pi_{00})^{-1} \pi_{i0} \pi_{0j}; \quad \tilde{\phi}_{ij} = \phi_{ij}, \quad i, j = 1, \ldots, n,$$

the system (1.7) takes on the form

$$(1.9) \quad p_i = \sum_{j=1}^{n} \tilde{\pi}_{ij} p_j + \rho \sum_{j=1}^{n} \tilde{\phi}_{ij} p_j,$$

i.e. takes on a form exactly analogous to that of the system (1.6) but with the price $p_0$ eliminated. The system (1.9) may in consequence be called the labor-eliminated form of our price equations, and the matrices $\tilde{\pi}_{ij}$ and $\tilde{\phi}_{ij}$ the labor-eliminated input-output matrix and the labor-eliminated fixed capital matrix respectively. The transformation which leads from (1.6) to (1.9) may be given the following heuristic interpretation. If each time labor appears as an input in production we replace this input by the corresponding real wage bill, we come to a hypothetical situation in which the only inputs required for the production of commodities are other (non-labor) commodities. Thus we may, if it is convenient for one or another theoretical purpose, consider our model to refer to a self-enclosed world of material commodities, produced out of each other with no additional input. We will make use of this modified description of our model economy (which is, of course, entirely equivalent to our initial description) at a number of points in what follows.

It is instructive to study the system (1.9) in its most trivial special case, the case $n = 1$, i.e. the case in which there is only one commodity.
If there is but one commodity equations (1.9) become

\[ p = \pi_n p + \rho \phi_n p \]  

with the solution

\[ \rho = \frac{(1 - \pi_n)}{\phi_n}. \]

The “price” in this case has, of course, no significance, since our theory is one of relative prices, i.e., price ratios, only; the “absolute price level” being meaningless in a theory like the one before us. The rate of profit, however, is given by (1.11). The most significant observation we can make about this equation is that the rate of profit, \( \rho \), is positive if and only if \( \pi_n < 1 \), i.e., if and only if we are able to produce one unit of commodity without consuming an equivalent or larger amount of commodity. It is clear that only in this case is production capable of yielding a physical surplus. The case \( \pi_n > 1 \) describes an economy inevitably fated to extinction through starvation, and may be typified by the unenviable situation of an invalid on a desert island, who to gain strength to gather one coconut must eat two. It is clear that in such a situation any initial supply of commodity will diminish to zero as time progresses; this situation may then be described as a “starvation economy.” The phenomenon which we meet here in its most primitive form can occur also for economies in which there are many commodities; the present paragraph may serve as forewarning of this fact.

Equation (1.11) also shows that \( \rho \) is inversely proportional to the amount of capital which must be tied up to produce a unit of the single commodity occurring in our model. Thus equation (1.11) shows that, in this simple case, the rate \( \rho \) of profit behaves in an entirely reasonable manner.

4. Critical Discussion of the Model

We have assumed a model in which all relationships are linear and homogeneous, that is, we have assumed “constant returns to scale.” Thus we have neglected all possible economies and diseconomies of scale, and have as well neglected the fact that significant pieces of
equipment are not continuously divisible: one cannot build 0.78 of a blast furnace. The force of these objections, however, should be substantially diminished by the prevalence of mass-production, which, by taking systematic advantage of economies of scale, will tend to operate largely in the linear range of production. Many processes may still deviate to a considerable degree from linearity; even so, we remember that calculus tells us that more general functional relationships can often be represented by linear approximations, useful at least over small intervals.

A more substantial difficulty of the above sort arises in those situations in which production makes use of unique natural resources without duplicates; the circumstances in which economic rents arise. These resources might be mines, oil wells, agricultural land especially well adapted (as, e.g. by climate) for the production of one or another commodity, etc. These phenomena also are ignored by our model, which implicitly assumes that no producer can have a clear technological advantage over any other producer.

We have also assumed, in writing production coefficients $\pi_{ij}$ and $\phi_{ij}$, that both input and fixed capital costs can be assigned unambiguously to a given output; so that we neglect the common fact that many costs are joint costs rather than costs assignable to a particular output, and that many commodities are produced jointly, some being “by-products” of the production of others. Administrative and auxiliary costs, in particular, would fit only grudgingly into a model like the one before us. Generally speaking, our input-output model portrays the conditions of industrial production rather better than the economic circumstances either of the trade or the service sectors. Since industry amounts only to some 30% of total economic effort, we must be on guard against distortions which may arise from the overly “industrial” view of the economy as a whole which the use of a strict input-output model imposes on us.

We have, of course, neglected all indication of the role of taxes; something will be said on this score later on, however.
Our model is certainly not very explicit as to how transportation costs shall enter, though such costs can be fitted into our model by distinguishing as separate commodities the same physical object in different locations—“ingots f.o.b. mill,” and “ingots delivered,” and taking “trucking” or “rail transport” as an input required to transform “ingots f.o.b. mill” into “ingots delivered.” The same remark might in principle apply to other categories of auxiliary costs, so that one might in principle distinguish between “ingots delivered” and “ingots checked, counted, and recorded,” for instance.

A more serious objection is the following. Our model does not take into account the fact that the same commodity can very often be produced by any one of a number of schemes, each requiring the consumption and utilization as fixed capital of different amounts of the “input” commodities, or of the same commodities in differing proportions. On this score, we shall adopt the following point of view. The objective of “efficiency economics” (theory of games, linear programming, and the extensions of these doctrines) is that of determining, according to some given criterion, methods for selecting the “best” production plan, i.e. the plan best according to a criterion of profitability. The efficiency economists usually confine their attention to parts of an economy that are so small that the price structure of the total economy cannot sensibly be affected by the choice of the production scheme within the portion of the economy under study, and under such conditions, the choice of maximum profit under the existing (invariant) prices seems a reasonable one as the criterion for “best.” The point of view taken here is that the production coefficients are to be thought of as those obtained by such “local” optimizations, which then becomes the “given” production coefficients in our model. This, of course, neglects the mutual interaction of the “local optimizations” through the price mechanism; nevertheless, we may hope that these neglected interactions cannot affect the general configuration of prices so severely as to introduce grave errors into our more approximate procedure. A more elaborate analysis of this point is reserved for a later lecture (cf. Lecture 17).
In assuming a homogeneous “labor” as input to all sorts of production we are, of course, ignoring the existence of a variety of distinct labor specialties and skills, convertible into each other only with some difficulty, and paid at different rates. We will see subsequently that by generalizing our model to include several labor sectors some of the force of this objection can be removed. A more serious objection along the same lines comes from our inclusion of the real wage bill as the row $\pi_{0j}$ in the input matrix, as if input to labor was also technologically determined. In the first place, we have ignored the fact that different people will choose to spend their money in different ways. Some of the force of this objection may be met by remarking that the elements $\pi_{0j}$ may be regarded as statistically average consumption indices. But what is more to the point, the total wage rate is determined not technologically but socially; thus, say, it is presumably possible technologically if not socially to reduce the American wage-rate to wage-rate which supports the urban population of China, i.e., by a factor of twenty. The elements $\pi_{0j}$ cannot be regarded as technological inputs necessary for the minimal sustenance of labor, but as a socially determined “living wage.” Machines will never quarrel with their employers about what inputs they require, but labor may! On the other hand, if the elements $\pi_{0j}$ are taken to represent “household demand” for the various commodities $C_j$, their legitimacy as elements of an input-output matrix seems doubtful, since these demands depend upon price, income distribution, etc. We will unravel these difficulties as we go along; one method is to study price theory in an open rather than in a closed model. Neoclassical “Walrasian” economics has had something to say on this score; we shall examine its contributions when we come to study Walras’ notion of general equilibrium.

The elements $\pi_{i0}$ are themselves dubious, if not as dubious as the elements $\pi_{0j}$. To know these necessary labor inputs we must distinguish the boundary between necessary labor input and “feather-bedding”; a boundary which is in continuing dispute. (At the moment these lines are written, a number of railroads are shut down in consequence of a
A number of criticisms may be directed against the manner in which a uniform rate $\rho$ of profit has been introduced into our model. In the first place, the rate of profit is dependent on the rate of industrial turnover, and will rise, for example, if two shifts rather than one shift are worked with given capital equipment. Thus the number of days for which capital equipment must be tied up in the production of a given output is determined as much socially as technologically. In assuming a rate of profit invariant from industry to industry, and in basing this assumption on an argument involving considerations of long-term equilibrium, we are open to criticism on the grounds that the time required for an economy to come to equilibrium is not short compared with the effect of the perturbing factors, e.g. technological advances which affect the production coefficients $\pi_{ij}$ and $\phi_{ij}$, nor, for that matter, short relative to the apparent frequency of social disturbances and disasters. Nor is the actual economy entirely free of monopoly! But, before involving ourselves with any of these complications, let us study the mathematical features of the model at hand.