1 INTRODUCTION

In this chapter we study another important discipline in the mathematical sciences in which the theory of nonnegative matrices finds elegant applications. In particular, we will see that the primary mathematical tools in the study of linear economic models involve nonnegativity and, especially, the theory of M-matrices developed in Chapter 6. In the present chapter we show how many of the results on M-matrices given earlier can be used to greatly simplify the construction and the analysis of Leontief's input–output models in economics.

It has been said by Miernyk [1965] that: “When Wassily Leontief published his ‘Quantitative input–output relations in the economic system of the United States’ in The Review of Economics and Statistics [1936], he launched a quiet revolution in economic analysis that has steadily gained momentum.” It was only a matter of timing that the article, which represents a turning point in the development of economic thought, did not at first attract wide acclaim. The nations of the noncommunist world were in the midst of the Great Depression. Moreover, John Maynard Keynes had just published his General Theory of Employment, Interest, and Money [1936], a treatise that immediately attracted worldwide attention since it was focused on the problems of chronic unemployment in the capitalist economics of that day. It turns out that, unlike Keynes, Leontief was not concerned with the causes of disequilibrium in a particular type of economic system during a particular phase of its development; he was primarily interested in the structure of economic systems. In particular, he was interested in the way the component parts of an economy fit together and influence one another. He developed an analytical model that can be applied to any kind of economic system during any phase of its development. As he noted himself, input–output is above all a mathematical tool. It can be used in the analysis of a wide variety of economic
problems and as a guide for the implementation of various kinds of economic policies.

Leontief’s input–output analysis deals with this particular question: *What level of output should each of n industries in a particular economic situation produce, in order that it will just be sufficient to satisfy the total demand of the economy for that product?* With this in mind we now give an overview of Leontief’s models. These concepts will be made more precise in later sections of this chapter.

In Leontief’s approach, production activities of an economy are disaggregated into n sectors of industries, though not necessarily to individual firms in a microscopic sense, and the transaction of goods among the sectors is analyzed. His basic assumptions are as follows:

1. Each of the n sectors produces a single kind of commodity. Broadly interpreted, this means that the n sectors and n kinds of commodities are in one-to-one correspondence. The sector producing the ith good is denoted by i.

2. In each sector, production means the transformation of several kinds of goods in some quantities into a single kind of good in some amount. Moreover this pattern of input–output transformation is assumed to be stable.

Intuitively, in a Leontief system this pattern assumes the following form. To produce one unit of the jth good, $t_{ij}$ units of the ith good are needed as inputs for $i = 1, \ldots, n$ in sector $j$, and $\lambda$ units of output of the jth good require $\lambda t_{ij}$ units of the ith good. The magnitudes $t_{ij}$ are called *input coefficients* and are usually assumed to be constant. In the economist’s terminology, the ratios of inputs are constant, and constant returns to scale prevail.

Let $x_i$ denote the *output of the ith good* per fixed unit of time. Part of this *gross output* is consumed as the input needed for production activities of the n sectors. Thus

$$
\sum_{j=1}^{n} t_{ij} x_j
$$

units of the ith good is consumed in production activities, leaving

$$
d_i = x_i - \sum_{j=1}^{n} t_{ij} x_j
$$

units of the ith good as the net output. This net output $d_i$ is normally called the *final demand of the ith good*. Alternatively, $d_i$ can be thought of as the contribution of the open sector of the economy, in which labor costs, consumer purchases leading to profits, etc., are taken into account.
Thus letting $x$ and $d$ denote the $n$-vectors with components $x_i$ and $d_i$, respectively, we obtain the system of linear equations

$$ (I - T)x = d. $$

The coefficient matrix

$$ A := I - T $$

of this system of linear equations is obviously in $\mathbb{Z}^{n \times n}$. It will be seen later that the economic situation is “feasible” if and only if $A$ is a nonsingular M-matrix; in which case the system can be solved for the gross output vector $x = A^{-1}d$, which is necessarily nonnegative. Thus the system (1.1) has the characteristic feature that for the obvious economic reason, the relevant constants $t_{ij}$ and $d_j$, as well as the solutions $x_i$, should satisfy the nonnegativity constraint. From the economic point of view, the *solvability* of (1.1) *in the nonnegative unknowns* $x_i \geq 0$ *means the feasibility of the Leontief model*, as previously mentioned.

The model just described is called the *open Leontief model*, since the open sector lies outside the system. If this open sector is absorbed into the system as just another industry, the model is called a *closed Leontief model*. In this situation, final demand does not appear; in its place will be the input requirements and the output of the newly conceived industry. All goods will now be intermediate in nature, for everything that is produced is produced only for the sake of satisfying the input requirements of the industries or sectors of the model. Mathematically, the disappearance of the final demands means that we now have a homogeneous system of linear equations, where the coefficient matrix is again in $\mathbb{Z}^{n \times n}$. The problem here is to determine when this matrix is a (singular) M-matrix and when the system has nonnegative solutions. This will be discussed later in this chapter.

Thus far we have considered only the *static Leontief models*; that is, models in which the input coefficients and the demands from the open sector are held constant. However, a dynamic version of, say, the open Leontief model can be constructed as follows. Let $x_t^k$ denote the output of the $i$th good at time $k$, let $t_{ij}$ denote the amount of output of industry $i$ per unit of input of industry $j$ at the next time stage, and let $\alpha$, $0 < \alpha < 1$, be the proportion of output, which is the same for each industry, that is available for internal use in the economy. Then if the final demands $d_i$ are held constant, we have the difference equation

$$ x^{k+1} = \alpha Tx^k + d, $$

which can then be studied in terms of the framework of Chapter 7 on iterative methods. However, we shall be concerned only with the static models in the present chapter.
The input–output models just described have wide ranging important applications. One of the original purposes was to study national economies. Consider the following example. The American economy is known for its high wage rates and intensive use of capital equipment in production. Labor is commonly regarded as a scarce factor and capital as an abundant factor. Therefore it was generally believed that America's foreign trade is based on exchanging capital-intensive goods for labor-intensive goods. Thus it was quite a surprise when Leontief [1953] published his finding, by use of input–output analysis, that the pattern of American trade is just opposite to the common view. In his study on American exports and imports, Leontief calculated the total requirements of labor and capital for the production of both exports and imports. It turned out that the United States actually exported labor-intensive goods and imported capital-intensive goods, a remarkable discovery made possible by input–output analysis. More recently Almon et al. [1974] have used such methods to produce an interindustry forecast of the American economy for the year 1985. In addition, Sarma [1977] has described an input–output model developed by IBM whose purpose is to forecast the industrial implications in the overall American economy.

Although input–output models were originally introduced by Leontief in order to model the structure of a national economy, at the present time they have been used, and are being used, in several other areas as well. For example, they have been used in connection with certain cooperate planning problems (see Stone [1970] or Sandberg [1974a]), they serve as a basis for solving some cost evaluation problems (see Hibbs [1972]), and they play a central role in studies of environmental pollution (see Gutmanis [1972]).

In the present chapter, Section 3 is devoted to an extensive treatment of the open input–output model while the closed model is discussed in Section 4. In each of these sections we shall primarily be concerned with nonnegative solutions to appropriate systems of linear equations that are derived from the models.

As usual, the last two sections are devoted to the exercises and the notes, respectively.

Next, in Section 2, we lay the framework for the material to be presented later by illustrating many of the important concepts by a simple example, that of analyzing the flow of goods in an economy with three aggregated industries—agriculture, manufacturing, and services.

2 A SIMPLE APPLICATION

There are four goals of this section: (1) to acquaint the reader with the way in which economists construct and analyze an input–output table, (2) to
illustrate, by this application, the way in which input–output models can be used in economic forecasting, (3) to acquaint the reader with certain notation and terminology conventions, and most importantly, (4) to provide material related to nonnegativity for reference in later sections of this chapter.

We begin by discussing an example of an input–output table showing the flow of goods and services among different branches of an economic system during a particular time period. For an alternate example, see Sarma [1977].

In order to engage in production, each sector or industry must obtain some inputs, which might include raw materials, semifinished goods, and capital equipment bought from other industries. In addition, business taxes must be paid and labor hired. Very often, some intermediate products, which are used as inputs to produce other goods and services as against final or outside products which do not reenter the production processes, are purchased from other industries. The output produced by each industry is sold either to outside users or to other industries or sectors which use the goods as inputs. A table summarizing the origin of all the various inputs and the destination of all the various outputs of all the industries in an economy is called an input–output table.

As an example, we consider a simple hypothetical economy consisting of three sectors: (1) agriculture, (2) manufactures, and (3) services. Each of the sectors produces precisely one kind of output: agricultural goods, manufactured goods, or services. These three sectors are to be interdependent, in that they purchase inputs from and sell outputs to each other. No government, foreign imports, or capital equipment will be involved here. All finished goods and services which do not reenter into production processes are used by the outside sector consisting of consumers, etc.

Following the previously stated assumptions, we give a hypothetical table summarizing the flow of goods and services, measured in dollars.

<table>
<thead>
<tr>
<th>Input from</th>
<th>Output to</th>
<th>(1) Agriculture</th>
<th>(2) Manufactures</th>
<th>(3) Services</th>
<th>Outside demand</th>
<th>Output (in dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Agriculture</td>
<td>15</td>
<td>20</td>
<td>30</td>
<td>35</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>(2) Manufactures</td>
<td>30</td>
<td>10</td>
<td>45</td>
<td>115</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>(3) Services</td>
<td>20</td>
<td>60</td>
<td>—</td>
<td>70</td>
<td>150</td>
<td></td>
</tr>
</tbody>
</table>

Here, the data in any row show the distribution of input to various sectors and users while the data in a column indicate the sources of inputs needed for production. For example, reading across the first row (agriculture), we find that, of the gross output of $100 of agricultural goods produced, $15 is to be
used in further agricultural production, $20 of goods is sold to manufactures, $30 of goods is sold to services, and finally $35 of agriculture goods go to satisfy the outside demand. Similarly, reading down the second column, we see that in order to produce $200 of gross output, manufactures has to input $20 of agriculture goods, $10 of its own goods, and $60 of services.

In order to analyze Table 2.1, we will need the following notation.

Subscripts:

1. agriculture, (2) manufactures, (3) services.

\( x_i \): Gross output of sector \( i \).

\( x_{ij} \): Sales of sector \( i \) to sector \( j \).

\( d_i \): Final demand on sector \( i \).

Then the basic row relation of Table 2.1 is

\[
\begin{align*}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},
\end{align*}
\]

(2.2)

This of course says that the gross output of a sector consists of the intermediate product sold to various production sectors and the final outside product which takes into account the consumers and the open sector.

As usual, we collect the gross output produced by all sectors in the vector

\[
x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

and the final demand in the vector

\[
d := \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
\]

Now it is much more convenient in the analysis if the input–output table in question is converted into one indicating the input requirements for the production of one unit of output for each sector. This table is often called a technical input–output table and its entries are called the input coefficients for the economy, which were mentioned in the introduction to this chapter. In order to construct the technical input–output table associated with Table 2.1, the various inputs of each sector are divided by the gross output of that sector. For example, each entry of the first column of Table 2.1 is divided by the gross output of agriculture, which is $100.

Letting \( t_{ij} \) denote the input coefficient indicating the amount of product \( i \) needed to produce one unit output of product \( j \), we have

\[
t_{ij} = x_{ij}/x_j, \quad 1 \leq i, j \leq 3.
\]

(2.3)

The technical table thus derived from Table 2.1 is then the following.
Next, rewriting (2.3) as

\[(2.5) \quad x_{ij} = t_{ij}x_j, \quad 1 \leq i,j \leq 3,\]

and substituting (2.5) into (2.2) we obtain

\[(2.6) \quad x_i = t_{i1}x_1 + t_{i2}x_2 + t_{i3}x_3 + d_i, \quad i = 1,2,3.\]

Then letting \( T = (t_{ij}) \), we have

\[(2.7) \quad T = \begin{bmatrix} 0.15 & 0.10 & 0.20 \\ 0.30 & 0.05 & 0.30 \\ 0.20 & 0.30 & 0.00 \end{bmatrix},\]

so that the matrix \( A = I - T \) is given by

\[(2.8) \quad A = \begin{bmatrix} 0.85 & -0.10 & -0.20 \\ -0.30 & 0.95 & -0.30 \\ -0.20 & -0.30 & 1.00 \end{bmatrix}.\]

As we have seen, Table 2.1, or alternatively Table 2.4, summarizes the gross purchases or sales of products among the three sectors of the hypothetical economy. It also describes the technology of production. From Table 2.1, we see, for example, that the production of $150 of services requires the input of $30 of agricultural goods and $45 of manufactured goods. Here $70 of the services go to satisfy the outside demand. This leads us to ask what the gross output of services should be if the technical input coefficients are held fixed while the outside demand is allowed to change. This economic forecasting is based upon the assumption that when the level of output is changed, the amounts of all inputs required are also changed proportionately. This is called the assumption of fixed proportion of factor inputs. Thus we are assuming that Table 2.4 remains fixed even though the outside demand and, accordingly, the gross output columns of Table 2.1 may change. Thus in order to forecast what level of output \( x_i \) each of the three sectors should have in order to satisfy the input requirements and the outside demands \( d_i \), we need only solve the linear system

\[(2.9) \quad Ax = d_i,\]
where \( A \) is given by (2.8). If the economic system is to be feasible, then (2.9) must have a nonnegative solution for each outside demand vector \( d \). It will be seen in Section 3, that this is just the requirement that \( A \) be a nonsingular M-matrix. By condition (N$_{38}$) of Theorem 6.2.3 then, we require that \( A^{-1} \) exist and satisfy \( A^{-1} \succeq 0 \). In our case, the inverse of the matrix \( A \) given by (2.8) is

\[
(2.10) \quad A^{-1} = \begin{bmatrix}
1.3459 & 0.2504 & 0.3443 \\
0.5634 & 1.2676 & 0.4930 \\
0.4382 & 0.4304 & 1.2167
\end{bmatrix},
\]

where the entries are rounded to four decimal places. Then since \( A^{-1} \succeq 0 \), the output vector \( x = A^{-1}d \) is nonnegative for each outside demand vector \( d \succeq 0 \). Thus this particular economic system is feasible. For example, if the outside demand on agriculture is \$100, that on manufactures is \$200 and that on services is \$300, then \( d = (100, 200, 300)^T \) so that the output vector \( x \) is computed using (2.10) to be

\[
x = A^{-1}d = \begin{bmatrix}
287.96 \\
457.76 \\
494.91
\end{bmatrix}.
\]

Thus \$287.96 of agriculture, \$457.76 of manufactures, and \$494.91 of services are required to satisfy the input demands of each of the three sectors and the outside sector.

Now suppose the demand for commodity (1), agriculture, is increased to \$300, so that \( \tilde{d} = (300, 200, 300) \). Then a new output vector \( \tilde{x} \) is given by

\[
\tilde{x} = A^{-1}\tilde{d} = \begin{bmatrix}
557.14 \\
570.44 \\
582.55
\end{bmatrix}.
\]

Here we see that the output of each sector increases, but that the output of commodity (1), agriculture, increases by the greater amount. It will be shown in Section 3 that this will always be the case if the row sums of the production matrix \( T \) satisfy

\[
\sum_{j=1}^{n} t_{ij} < 1,
\]

and \( T \) is irreducible.

Under these conditions it will be shown that if only the demand of commodity \( j \) increases, then the output of commodity \( j \) increases by the greatest amount, although all outputs may increase. In our example, each row sum of the production matrix \( T \) given by (2.7) is, of course, strictly less than one, and \( T \) is irreducible.
3 THE OPEN MODEL

As we have mentioned earlier, interindustry analysis of the Leontief type is concerned primarily with systems in which the products of economic factors (machines, materials, labor, etc.) are themselves used as factors to produce further goods. Various Leontief-type models have been presented, the simplest of which is the open model which will be discussed in the present section.

As before, we assume that the economy is divided into $n$ sectors, each producing one commodity to be consumed by itself, by other industries and by the outside sector. Then identifying the $i$th sector with the $i$th commodity, we have the following notation:

- $x_i$: Gross output of sector $i$.
- $x_{ij}$: Sales of sector $i$ to sector $j$.
- $d_i$: Final demand on sector $i$.
- $t_{ij}$: Input coefficient, the number of units of commodity $i$ required to produce one unit of commodity $j$.

Then the overall input–output balance of the entire economy can be expressed in terms of the $n$ equations:

\[
(3.1) \quad x_i = \sum_{j=1}^{n} x_{ij} + d_i, \quad 1 \leq i \leq n.
\]

Now assuming that if the level of output is changed then the amounts of all inputs required are also changed proportionally; that is, assuming a fixed proportion of factor inputs, the input coefficients $t_{ij}$ are constant and satisfy

\[
(3.2) \quad t_{ij} = x_{ij}/x_j, \quad 1 \leq ij \leq n.
\]

The system of linear equations (3.1) then becomes

\[
(3.3) \quad x_i = \sum_{j=1}^{n} t_{ij}x_j + d_i, \quad 1 \leq i \leq n.
\]

Letting $T = (t_{ij})$ and

\[
(3.4) \quad A := I - T
\]

as before, the overall input–output balance of the entire economy is expressed in terms of the system of $n$ linear equations in $n$ unknowns:

\[
(3.5) \quad AX = d,
\]

where $x = (x_i)$ is the output vector and $d = (d_i)$ is the final demand vector.

The model just described is called the open Leontief model and the matrix $T = (t_{ij})$ is called the input matrix for the model. Then $A$, given by (3.4),
is in $Z^{n \times n}$; that is, $a_{ij} \leq 0$ for all $i \neq j$. Matrices in $Z^{n \times n}$ are often called matrices of Leontief type, or sometimes essentially nonpositive matrices by economists. Clearly the features of the open Leontief model are completely determined by the properties of $A$.

The economic model just described also has an associated price–valuation system, which gives the pricing or value side of the input–output relationship. Our notation will be

\[ p_j: \text{ Price of the } j\text{th commodity.} \]
\[ v_j: \text{ Value added per unit output of the } j\text{th commodity.} \]

Then

\[ \sum_{i=1}^{n} t_{ij}p_i, \quad 1 \leq j \leq n, \]

is the unit cost of the $j$th commodity, so that

\[ p_j = \sum_{i=1}^{n} t_{ij}p_i \]

is the net revenue per unit output of the $j$th commodity; that is, the value added per unit output, $v_j$. Then letting $p$ denote the vector with entries $p_j$ and $v$ denote the vector with entries $v_j$, the relationship just stated is described by the system of linear equations

\[(3.6) \quad p^t - p^tT = v^t \]

or

\[(3.7) \quad p^tA = v^t, \]

where $A = I - T$, as before. The vector $p$ is called the price vector and the vector $v$ is called the value added vector of the associated open Leontief model. Obviously $v \geq 0$ and the economist is interested in solving the system (3.6), or (3.7), for price vectors $p \geq 0$.

A link between the two systems (3.5) and (3.6) is given by the relation

\[ \sum_{i=1}^{n} v_i x_i = \sum_{j=1}^{n} p_j d_j, \]

which may be interpreted by the following economic statement: The "national income" and the "national product" are equal.

While the solvability of the original output system (3.5) for the nonnegative outputs $x_i$, $1 \leq i \leq n$, means the feasibility of the model, that of solving the price system (3.7) in the nonnegative prices $p_j$, $1 \leq j \leq n$, means profitability. This leads to the following definitions.
(3.8) **Definitions**  An open Leontief model with input matrix $T$ is said to be *feasible* if the system (3.5) has a nonnegative solution for the output vector $x$, for each open demand vector $d$. It is said to be *profitable* if the system (3.7) has a nonnegative solution for the price vector $p$, for each value added vector $v$.

The duality of these concepts is made apparent in the following elementary theorem, which is based upon the theory of M-matrices developed in Chapter 6. Recall that $A$ is a nonsingular M-matrix if and only if $A = sI - B$, $s > 0$, $B \geq 0$, with $s > \rho(B)$, the spectral radius of $B$.

(3.9) **Theorem**  Consider an open Leontief model with input matrix $T$ and let $A = I - T$. Then the following statement are equivalent:

1. The model is feasible.
2. The model is profitable.
3. $A$ is a nonsingular M-matrix.

**Proof**  We show first that (1) and (3) are equivalent. If (1) holds, then by choosing the demand vector $d$ to be positive, it follows that $A$ satisfies:

\[(3.10) \quad \text{There exists } x > 0 \text{ with } Ax = d > 0.\]

But this is just condition (I.28) of Theorem 6.2.3, which characterizes nonsingular M-matrices. Thus (3) holds since $A \in \mathbb{Z}^{n \times n}$. Conversely, if (3) holds, then by condition (N.38) of Theorem 6.2.3, $A^{-1} \geq 0$. Thus (3.5) has the nonnegative solution $x = A^{-1}d$ for each $d \geq 0$, so that (1) holds. The equivalence of (2) and (3) is established in a similar manner by using the fact that $A$ is a nonsingular M-matrix if and only if the same is true for $A'$.

Recall that 50 characterizations for $A \in \mathbb{Z}^{n \times n}$ to be a nonsingular M-matrix were given in Theorem 6.2.3. Thus we can state the following corollary.

(3.11) **Corollary**  Consider an open Leontief model with input matrix $T$ and let $A = I - T$. Then the following statements are equivalent:

1. The model is feasible.
2. The model is profitable.
3. $A$ satisfies one (and thus all) of the conditions $(A_1)$–$(Q_{50})$ of Theorem 6.2.3.

Now condition $(A_1)$ of Theorem 6.2.3, which states that $A \in \mathbb{Z}^{n \times n}$ is an M-matrix if and only if all the principal minors of $A$ are positive, was established in the economics text by Hawkins and Simon,[1949]. Their result has consequently been known in the economics literature as the *Hawkins–Simon condition* for the feasibility (or profitability) of the open Leontief
model. Actually, condition \((A_1)\) was established much earlier for nonsingular \(M\)-matrices by Ostrowski [1937].

We remark also that if the input matrix \(T\) of an open Leontief model is irreducible, then by Theorem 6.2.7, the model is feasible (or profitable) if and only if
\[
A^{-1} > 0
\]
or equivalently
\[
Ax > 0 \quad \text{for some} \quad x > 0.
\]

In this case, (3.12) is equivalent to the statement that the model has all positive inputs for any nonzero demand vector and/or the model has all positive prices for any nonzero value added vector. In this regard, we note that the input matrix \(T\), given by (2.7) for the example discussed in Section 2, is irreducible. Thus the economy discussed there is feasible, if and only if \(A^{-1} > 0\), where \(A = I - T\); but this in turn is verified by (2.10).

We also mention in passing that if \(T\) is the input matrix for a feasible (profitable) open Leontief model, then since \(\rho(T) < 1\), the output and price vectors may be computed from the series
\[
\sum_{k=0}^{\infty} T^k d \quad \text{and} \quad \sum_{k=0}^{\infty} v^T T^k,
\]
respectively, by Lemma 6.2.1. However, these methods are usually not practical.

This section is concluded with a discussion of the effects that changes in the open demands of a feasible model may have on the final outputs, and the effects that changes in the value added requirements on a feasible model may have on the prices.

First we shall need the following technical lemma.

(3.14) **Lemma** Let \(A\) be a nonsingular \(M\)-matrix of order \(n\) whose row sums are all nonnegative; that is, \(Ae \geq 0\) where \(e := (1, \ldots, 1)^t\). Then the entries of \(A^{-1}\) satisfy
\[
(A^{-1})_{ii} \geq (A^{-1})_{ik}, \quad 1 \leq i, k \leq n.
\]

**Proof** Since
\[
A^{-1} = \frac{1}{\det A} \text{Adj} A,
\]

where \(\text{Adj} A\) is the transposed matrix of the cofactors of \(A\), it suffices to show that if \(C_{ij}\) is the cofactor of the \(i\)th row and \(j\)th column of \(A\), then
\[
C_{ii} \geq C_{ik}, \quad 1 \leq i, k \leq n.
\]
For that purpose we write $A = sI - B$, $s > 0$, $B \geq 0$. Then $s > \rho(B)$ and, by assumption, $s \geq \max_i \sum_{j=1}^n b_{ij}$. We consider two cases. Assume that $s > \max_i \sum_{j=1}^n b_{ij}$. Now we can replace any zero entries of $B$ by a small positive number $\delta$ to form a new positive matrix $\tilde{B} = (\tilde{b}_{ij})$, but that still

$$s > \max_i \sum_{j=1}^n \tilde{b}_{ij}.$$ 

Thus if we can prove $C_{ii} \geq \tilde{C}_{ik}$ for all $i$ and $k$ in this case, then $C_{ii} \geq C_{ik}$ for all $i$ and $k$ in the original case, by continuity in $\delta$, letting $\delta$ approach zero. As a result, it suffices to assume that $B \gg 0$, which we shall do.

Consider $s \neq k$ with $k$ fixed (but arbitrary). Then replace all the elements of the $i$th row of $B$ by zeros, except the $(i,i)$th and $(i,k)$th which we replace by $s/2$. Denote the new matrix by $W$; it is clearly irreducible, and moreover has all row sums not exceeding $s$, and all but the $i$th less than $s$. Thus $\rho(W) < s$ by the Perron-Frobenius theorem and Theorem 2.1.11. Then

$$-\frac{s}{2} C_{ik} + \frac{s}{2} C_{ii} = \det(sI - W) > 0.$$ 

since $sI - W$ is a nonsingular M-matrix and recalling that the cofactors remain the same as for $sI - B$. Therefore $C_{ii} \geq C_{ik}$.

Second, if $s = \max_i \sum_{j=1}^n b_{ij}$, then we take any $\varepsilon > 0$ and consider $s + \varepsilon$ in place of $s$ for $A$. Case one then applies and (3.15) follows from continuity by letting $\varepsilon \to 0$. Thus the lemma is proved. 

Now consider an open Leontief input–output model with input matrix $T$. Then even if the model is feasible, so that $A = I - T$ is a nonsingular M-matrix, it does not always follow that the row sums of $T$ are all at most one. But if this is the case, we can prove the following theorem.

**Theorem (3.16)** Let $T$ be the input matrix for a feasible open Leontief model and suppose that

$$Te \leq e, \quad e = (1, \ldots, 1)^\prime.$$ 

Then if the demand for commodity $i$ alone increases, none of the outputs decrease and the output of commodity $i$ increases and increases by the greatest amount, although other outputs may increase by the same amount.

**Proof** As before we let $A = I - T$, and let $x$ and $d$ denote the output and demand vectors, respectively. Then $A$ is a nonsingular M-matrix by Theorem 3.9. Moreover

$$Ae = (I - T)e = e - Te \geq 0$$ 

by assumption, so that Lemma 3.14 applies.
Now suppose that the $i$th term of the demand vector $d$ is increased by the amount $\delta$. Then the resulting demand vector becomes

$$\bar{d} = d + \delta e_i,$$

where $e_i$ denotes the $i$th unit vector. From (3.5), $x = A^{-1}d$ and the new output vector $\bar{x}$ becomes

$$\bar{x} = A^{-1}\bar{d} = A^{-1}(d + \delta e_i) = A^{-1}d + \delta A^{-1}e_i = x + \delta(A^{-1})_i,$$

where $(A^{-1})_i$ denotes the $i$th column of $A^{-1}$. Then since $A^{-1} \geq 0$ by condition (N$_{38}$) of Theorem 6.2.3, it follows from

$$\bar{x}_k = x_k + \delta(A^{-1})_{ki}, \quad 1 \leq k \leq n,$$

that $\bar{x}_i > x_i$ and, also, none of the outputs decrease. Moreover by Lemma 3.14,

$$\delta(A^{-1})_{ki} \leq \delta(A^{-1})_{ii}, \quad 1 \leq k \leq n,$$

so that $x_i$ increases by the greatest amount, although other outputs may increase by the same amount. ■

(3.17) **Corollary** If the input matrix $T$ for an open model is irreducible and $Te \ll e$, then if the demand of commodity $i$ alone increases all the outputs increase and the output of commodity $i$ increases by the greater amount.

*Proof* The result follows since $A^{-1} \gg 0$ and since strict inequality holds in (3.15) for each $i$ and $k, k \neq i$. ■

Now since the linear systems (3.5) and (3.7) have dual analyses, one would expect that a similar relationship between the values added and the prices will hold, except in this case column sums rather than row sums are considered. But if an open Leontief model is profitable, then no sector of the economy operates at a loss and moreover at least one sector operates at a profit. In terms of the input matrix $T$, this means that

(3.18)

$$\sum_{i=1}^{n} t_{ij} \leq 1, \quad 1 \leq j \leq n,$$

with strict inequality in at least one $j$. This leads to the following.

(3.19) **Theorem** If the value added in commodity $i$ alone of a feasible open Leontief model is increased, then none of the prices decrease and the price of commodity $i$ increases by the greatest amount, although other prices may increase by the same amount.
Proof By Theorem 3.9 the model is profitable so that by (3.18) the column sums of the input matrix \( T \) for the model are at most one. Now letting \( A = I - T \), and letting \( p \) and \( v \) denote the price and value added vectors, respectively, the linear system (3.7) can be written in the form

\[ A^t p = v. \]

The proof is then completed in a manner similar to the proof of Theorem 3.16, by replacing \( A \) by \( A^t \).

For the irreducible case, we give a result which is dual to Corollary 3.17. The proof is similar to that of Corollary 3.17 and is omitted.

(3.20) Corollary If the input matrix \( T \) for an open model is irreducible and \( T^e \ll e \), then if the value added in commodity \( i \) alone is increased, all of the prices increase and the price of commodity \( i \) increases by the greater amount.

Two remarks concerning Theorems 3.16 and 3.19 are now in order: (i) It is not necessarily true that an increase in demand for a single commodity forces a greatest increase in the supply of that commodity compared to others; and (ii) it is always true that an increase in the value added in a single commodity forces a greatest increase in the price in that commodity compared to others.

To illustrate (i), consider an open Leontief model with input matrix \( T \) given by

\[ T = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}. \]

Then since the maximum column sum of \( T \) is less than one, the model is profitable and thus feasible, and the matrix

\[ A = I - T = \begin{bmatrix} 4 & -5 \\ -2 & 6 \end{bmatrix} \]

is a nonsingular M-matrix. Then

\[ A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}, \]

which agrees with our assertions in this section. Note also that the sum of the entries in the first row of \( T \) is greater than one. Thus the model does not satisfy Theorem 3.16. In fact, if the demand for commodity 2 increases, by a single unit, then the supply vector increases by

\[ \begin{bmatrix} \frac{5}{2} \\ 2 \end{bmatrix}, \]

which is a greater increase in the supply of commodity 1 than of commodity 2.
On the other hand, Theorem 3.19 is obviously satisfied here since the column sums of $T$ are less than one. Also, since $T$ is irreducible, Corollary 3.20 holds, so that an increase in a single value added component results in an increase in each price component, with a greater increase in the given component. This is clearly illustrated by the fact that $A^{-1}$ is strictly row diagonally dominant here.

Now consider the example developed in Section 2. The matrix $T$, given by (2.7), has all row sums as well as all column sums less than one. Then Corollary 3.17, as well as Corollary 3.20, is satisfied by this model, since $T$ is irreducible.

4 THE CLOSED MODEL

If the outside sector of the open input–output model is absorbed into the system as just another industry, then the system will become a closed model. In such a model, final demand and primary input do not appear; in their place will be the input requirements and the output of the newly conceived industry. Here all commodities will now be intermediate in nature, since everything that is produced is produced only for the input requirements of the sectors or industries within the model itself.

Thus in the closed Leontief input–output system, the consumer or open sector will be regarded as a production sector. In the particular formulation of the closed model discussed here, the open sector's “inputs” are various consumption goods and services, and its “output” is labor. In this system, final demand and items such as employment and the wage rate are all treated as unknown and their equilibrium values are solved for simultaneously with the rest of the variables.

Because there is no variable determined from outside, we ask a different question in the closed system. Given the technology of production, what are the equilibrium output and price levels such that there is no unsatisfied demand? To analyze this problem we begin with an equation that looks similar to Eq. (3.5), $Ax = d$, where $x_i$ is the output of the $i$th sector and $d_i$ is the outside demand on the $i$th sector, and where $A = I - T$, where $T$ is the input matrix for the model. However, final demands are now considered to be inputs of the consumer sector. Each component $d_i$ of $d$ is now related to a level of employment $\theta$, so that consumers purchase an amount $d_i$ which varies with the level of employment $\theta$. To achieve this we define fixed technical coefficients $c_i$ by

$$c_i = d_i/\theta,$$

so that

$$d_i = c_i \theta.$$
Then assuming that the original open model contains \( n - 1 \) sectors, the input–output relations in the closed model are described by the \( n - 1 \) simultaneous linear equations

\[
(4.1) \quad x_i = \sum_{j=1}^{n-1} t_{ij} x_j + c_i \theta, \quad i = 1, \ldots, n - 1.
\]

Now the total labor supply; that is, the level of employment \( \theta \), is the sum of the labor used by each sector,

\[
\theta = \sum_{i=1}^{n} L_i
\]

where \( L_i \) is the amount of labor used by sector \( i \), and where \( i = n \) is the consumer sector. Now defining fixed labor input coefficients \( l_i \) by

\[
l_i := \frac{L_i}{x_i}, \quad i = 1, \ldots, n - 1 \quad \text{and} \quad l_n := \frac{L_n}{\theta},
\]

the total labor supply can be expressed in terms of a linear equation

\[
(4.2) \quad \theta = \sum_{j=1}^{n} l_j x_j + l_n \theta.
\]

Then attaching (4.2) to the linear system (4.1) as the last equation, we have

\[
x_1 = t_{1,1} x_1 + \cdots + t_{1,n-1} x_{n-1} + c_1 \theta,
\]

\[
x_2 = t_{2,1} x_1 + \cdots + t_{2,n-1} x_{n-1} + c_2 \theta,
\]

\[
\vdots
\]

\[
x_{n-1} = t_{n-1,1} x_1 + \cdots + t_{n-1,n-1} x_{n-1} + c_{n-1} \theta,
\]

\[
\theta = l_1 x_1 + \cdots + l_{n-1} x_{n-1} + l_n \theta.
\]

To write this linear system in matrix notation, we let

\[
T := \begin{bmatrix}
    t_{11} & t_{12} & \cdots & t_{1,n-1} & c_1 \\
    t_{21} & t_{22} & \cdots & t_{2,n-1} & c_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    t_{n-1,1} & t_{n-1,2} & \cdots & t_{n-1,n-1} & c_{n-1} \\
    l_1 & l_2 & \cdots & l_{n-1} & l_n
\end{bmatrix}
\]

Now for notational convenience we define

\[
t_{i\sigma} := c_i, \quad i = 1, \ldots, n - 1, \quad \text{and} \quad t_{n\beta} := l_j, \quad j = 1, \ldots, n.
\]

We also define

\[
x_i' = (x_1, \ldots, x_{n-1}, \theta)'
\]
and let \( x_n = \epsilon \). Then the input–output relationships for the closed model just described are given by the system of linear equations

\[
(4.3) \quad x = Tx,
\]

and \( T \) is called the input matrix for the model. Or letting \( A := I - T \), the linear system (4.3) can be written as

\[
(4.4) \quad Ax = 0,
\]

where \( 0 \) denotes the zero vector of dimension \( n \). Since the linear system (4.4) is homogeneous, it either has only the trivial solution \( x = 0 \) or it has infinitely many solutions. As before, we shall be interested in solutions where the output coefficients \( x_i \) are nonnegative and, in this case, positive.

This closed Leontief input–output system takes into account the demand factors as well as supply factors. In a certain sense, the level of employment, \( \epsilon \), represents the final demand on the system. This final demand is not given but is determined with the other supply variables, the total outputs \( x_i \) of each of the \( n - 1 \) original industries. Thus by solving for the final demand, \( \epsilon \), and output requirements \( x_i \) simultaneously, the closed model takes into account both the impact of demand on supply and that of supply on demand. Thus the equilibrium output levels calculated from a closed model incorporate not only the outputs required to meet a given final demand but also the outputs required to meet the change in the final demand which is induced by changes in production.

Now let \( x \) be a vector of outputs produced by this system. Then since

\[
\sum_{j=1}^{n} t_{ij} x_j
\]

is the quantity of the \( i \)th input necessary to produce this output bundle, \( y := Tx \) is the vector of inputs. But since outputs \( x_i \) produce the only source of inputs, the system cannot operate unless \( y \leq x \). Moreover, production processes will be assumed to be irreversible so that \( x \) is necessarily non-negative and, in fact, positive in our case. This leads to the following definition.

\begin{definition}
A closed Leontief model with input matrix \( T \) is said to be feasible if there exists some \( x \) such that
\[
(4.6) \quad Tx \leq x, \quad x \gg 0.
\]
Such an \( x \), if it exists, is called a feasible output solution to the model.
\end{definition}

Now if the model is feasible, we search for some solution to the model, which we are willing to consider an output equilibrium. With this in mind we give the following.
(4.7) Definition A vector \( x \) is called an **output equilibrium vector** for a closed Leontief model with input matrix \( T \) if

\[
Tx = x, \quad x \gg 0.
\]

It follows from Definition 4.7 then that a closed input–output model has an output equilibrium vector if and only if the homogeneous system (4.4) has a positive solution; that is,

\[
Ax = 0, \quad x \gg 0,
\]

is consistent. In this case the model is then feasible. However, we will see that a feasible model does not necessarily possess an output equilibrium vector.

It turns out that the theory of singular M-matrices, developed in Chapter 6, is quite useful in analyzing the feasibility of a closed Leontief model. Recall that \( A \) is an M-matrix with **property c** provided that \( A \) has a representation \( A = sI - B, s > 0, B \geq 0 \), where the powers of \( B/s \) converge to some matrix (see Definition 6.4.10).

(4.10) Theorem A closed Leontief model with input matrix \( T \) is feasible only if \( A = I - T \) is an M-matrix with **property c**.

**Proof** By Definition 4.5, the model is feasible only if (4.6) holds, so that \( Ax \geq 0 \) for some \( x \gg 0 \). But since \( A \in \mathbb{Z}^{\text{sym}} \), it follows that \( A \) is an M-matrix with **property c**, by Exercise (6.4.14).

Now by applying Theorem 6.4.12 and Exercise 6.4.13, we have the following.

(4.11) Corollary A closed Leontief model with input matrix \( T \) is feasible only if \( A = I - T \) satisfies one, and consequently all, of conditions \((A_1)\)–\((F_{13})\) of Theorem 6.4.12, and there exists a symmetric positive definite matrix \( W \) such that

\[
AW + WA^T
\]

is positive semidefinite.

However, the converse of Theorem 4.10 does not hold; that is, \( A = I - T \) may be an M-matrix with **property c** while there is no \( x \gg 0 \) with \( Ax \geq 0 \).

Such is the case for a closed Leontief model with input matrix

\[
T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},
\]

for then setting

\[
A = I - T = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}
\]

it follows that there is no \( x \gg 0 \) with \( Ax \geq 0 \).
In order to further study feasible Leontief models and to investigate the existence of an output equilibrium vector, it will be convenient to permute \( T \) into a special block triangular form. If \( T \) is any square, reducible matrix, then there is a permutation matrix \( P \) for which \( P T P^t \) is in reduced triangular block form:

\[
P T P^t = \begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22} \\
\vdots & \vdots & \ddots \\
T_{k1} & T_{k2} & \cdots & T_{kk}
\end{bmatrix},
\]

where each block \( T_{ii} \) is either square and irreducible, or a \( 1 \times 1 \) null matrix. For our purposes here, it is convenient to define a \( 1 \times 1 \) null matrix to be irreducible, and if \( T \) is irreducible, we write simply \( T = (T_{11}) \). We first establish the following.

(4.13) **Lemma** Suppose the input matrix \( T \) of a closed Leontief model is irreducible and let \( A = I - T \). Then the following statements are equivalent:

1. The model is feasible.
2. The model has an output equilibrium vector \( x \) which is unique up to positive scalar multiples.
3. \( A \) is an M-matrix with “property c.”

**Proof** That (1) implies (3) was established in Theorem 4.10. Now assuming (3), it follows that there exists \( x \gg 0 \) with \( A x = 0 \) by Theorem 6.4.16 (2). But since \( T \) is irreducible and \( T x = x \), it follows that \( x \) is unique up to positive scalar multiples by the Perron–Frobenius theorem, Theorem 2.1.4. Thus (2) holds. Finally, the implication (2) implies (1) is clear from Definitions 4.5 and 4.7.

Next, we investigate feasible Leontief models in terms of the reduced normal form (4.12) of the input matrix.

(4.14) **Theorem** Let \( T \) be the input matrix for a closed Leontief model and suppose that \( P T P^t \) has the reduced triangular block form (4.12). Let \( A = I - P T P^t \) be partitioned conformally with \( P T P^t \). Then the model is feasible if and only if \( A \) is an M-matrix and for each \( i \), \( 1 \leq i \leq k \),

\[
A_{ii} \text{ singular implies } T_{ij} = 0 \quad \text{for all } i \neq j.
\]

**Proof** Assume first that the model is feasible. Then \( A \) is an M-matrix by Theorem 4.10, and there exists a partitioned vector \( x = (y^1, \ldots, y^k)^t \gg 0 \) such that \( P T P^t x \leq x \), with \( A x \geq 0 \), so that for \( A_{ii} = I - T_{ii} \),

\[
\sum_{j \neq i} A_{ij} y^j \geq 0 \quad \text{for all } 1 \leq i \leq k.
\]
Because \( A_{ij} \leq 0 \) for all \( j < i \), it follows that \( A_{ii}y_i \geq 0 \) for all \( 1 \leq i \leq k \). But then if \( A_{ii} \) is singular, it follows that \( A_{ii}y_i = 0 \) from Theorem 6.4.16 (5), since \( A_{ii} \) is irreducible. It follows further that \( A_{ij}y_j = 0 \) for all \( j < i \), since \( A_{ij} \leq 0 \). Thus \( A_{ij} = 0 \) for all \( j \neq i \) and consequently \( T_{ij} = -A_{ij} = 0 \) for all \( j \neq i \), so that (4.15) holds.

Conversely assume that \( A = I - PTP^t \) is an M-matrix and that (4.15) holds. If \( A_{ii} \) is singular then by construction, \( A_{ii} \) is a singular irreducible M-matrix, and as such, there exists \( y_i \gg 0 \) for which \( A_{ii}y_i = 0 \) by Theorem 6.4.16(2). Similarly, if \( A_{ii} \) is nonsingular, then \( A_{ii} \) is a nonsingular M-matrix since \( A \) is an M-matrix, so that there exists \( y_i \gg 0 \) for which \( A_{ii}y_i \gg 0 \) by Theorem 6.2.3, part (I_2i). Recalling that (4.15) holds, it follows that the \( y_i \)'s can be scaled so that \( x = (y_1', \ldots, y_k') \) satisfies \( x \gg 0 \), and (4.16) holds. Thus \( Ax \geq 0 \) so that \( PTP^tx \leq x \) and consequently, \( T(P^tx) \leq P^tx, P^tx \gg 0 \) and so the model is feasible. □

As mentioned earlier, a feasible closed Leontief model does not necessarily have an output equilibrium vector. For example, if \( T \) is the input matrix and \( A = I - T \) is a nonsingular M-matrix, then the model is feasible since there exists \( x \gg 0 \) with \( Ax \gg 0 \), by Theorem 6.2.3, part (I_2i); but of course there is no \( x \) with \( Ax = 0 \).

Our final result characterizes those closed Leontief models possessing an output equilibrium vector, in terms of the reduced triangular block form (4.12) for the input matrix \( T \).

**Theorem** Let \( T \) and \( A \) be as in Theorem 4.14. Then the model has an output equilibrium vector if and only if \( A \) is an M-matrix and for each \( i, 1 \leq i \leq k \),

\[
A_{ii} \text{ is singular } \quad \text{if and only if} \quad T_{ij} = 0 \quad \text{for all } \ j \neq i.
\]

**Proof** If the model has an output equilibrium vector then the model is feasible; thus by Theorem 4.14, \( A \) is an M-matrix and \( A \) satisfies (4.15). Now suppose that \( i \) is an index such that \( T_{ij} = 0 \) for all \( i \neq j \). Then by assumption there is a partitioned vector \( x = (y_1, \ldots, y_k) \gg 0 \) such that \( Ax = 0 \). Then \( A_{ii}y_i = 0 \) and thus \( A_{ii} \) is singular, establishing (4.18).

For the converse, assume that \( A \) is an M-matrix and that (4.18) holds. Suppose first that \( i \) is an index such that \( A_{ii} \) is singular. Then we can choose \( y_i \gg 0 \) so that \( A_{ii}y_i = 0 \), since \( A_{ii} \) is a singular irreducible M-matrix. In addition, \( A_{ii} = 0 \) for all \( i \neq j \) by (4.18). We now assume, without loss of generality, that the blocks \( A_{ii} \) have been ordered so that all the singular blocks come first; that is, \( A_{11}, \ldots, A_{gg} \) are all the singular diagonal blocks of \( A \). Let \( y_i \gg 0 \) be such that \( A_{ii}y_i = 0, i = 1, \ldots, g \). Now if \( g = k \), the result holds.
If $k > g$ define

$$y_h = A^{-1} \sum_{j=1}^{h-1} A_{kj} y_j, \quad h = g + 1, \ldots, k.$$ 

Now $A_{hh}^{-1} \gg 0$ since $A_{hh}$ is a nonsingular irreducible M-matrix. Moreover $A_{kj} \neq 0$ for some $1 \leq j \leq h - 1$ by (4.18). Thus by induction, all the vectors $y_j$ are positive, and moreover,

$$\sum_{j=1}^{h} A_{kj} y_j = 0, \quad h = g + 1, \ldots, k.$$ 

Consequently, letting $x = (y_1, \ldots, y_h)$, we see that $x \gg 0$ and $Ax = 0$. Thus $TP^x = P^x$ so that $P^x$ is an output equilibrium vector for the model, completing the proof. \qed

We remark that an alternate proof of Theorem 4.17 can be given by using the concepts of basic and final classes of a nonnegative matrix (see Theorem 2.3.10).

This section is concluded by noting that if the input matrix $T$ for a feasible closed Leontief model is irreducible, then the output equilibrium vector $x$ can be computed by the methods discussed in Chapter 8 for computing the stationary distribution vector associated with an ergodic Markov chain. In particular, for any $\alpha > 0$, the powers of $T_z = (1 - \alpha)I + \alpha T$ converge, by Theorem 8.4.9, to a matrix $L$. Moreover in this case

$$L = xe, \quad e = (1, \ldots, 1),$$

where

$$Tx = x, \quad x \gg 0.$$ 

Then the iterative procedure

$$x^{k+1} = Tx^k, \quad k = 0, 1, \ldots,$$

converges to $x$ for any $x^0 = e_i$, the $i$th unit vector. Here Theorem 8.4.32 can often be used to choose $x$ in such a way as to optimize the asymptotic convergence rate of (4.19).

Moreover, we note that the direct computational methods discussed in Exercises 8.5.18 and 8.5.19 can also be modified in this case in order to compute an output equilibrium vector for a closed Leontief model, where the input matrix $T$ is once again assumed to be irreducible.

Finally, suppose that the input matrix $T$ for a closed Leontief model is irreducible and has the property that

$$\sum_{i=1}^{n} t_{ij} = 1, \quad 1 \leq j \leq n.$$ (4.20)
Then since $T^i$ is then a stochastic matrix, the entire analysis developed in Chapter 8 for the stationary distribution vector associated with an ergodic Markov chain carries over here to the study of the output equilibrium vector, with $T$ replaced by $T^i$. In economic terms, condition (4.20) means simply that each sector of the economy is in equilibrium in terms of internal factors.

5 EXERCISES

(5.1) Consider an open Leontief model with input matrix $T$ given by

$$T = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}.$$ 

Show that if $A = I - T$ then $A^{-1}$ is approximated by

$$A^{-1} \approx \frac{1}{0.38} \begin{bmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.64 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{bmatrix},$$

where the entries are rounded to two decimal places.

(5.2) Show that the model with input matrix $T$ described in Exercise 5.1 is feasible and compute the outputs $x_1, x_2,$ and $x_3$ associated with the open demands

$$d_1 = 10, \quad d_2 = 5, \quad d_3 = 6.$$ 

(5.3) Show that Corollaries 3.17 and 3.20 apply to the model given in Exercise 5.1.

(5.4) Determine if the open Leontief model with input matrix $T$ given by

$$T = \begin{bmatrix} 0.5 & 0.6 \\ 0.3 & 0.7 \end{bmatrix}$$

is feasible.

(5.5) Show that if an open Leontief model with input matrix $T$ is feasible, then the sum of the entries in at least one column of $T$ is less than one.

(5.6) Consider an open Leontief model in which each sector supplies exactly one other sector and in which each sector has exactly one supplier. Show by example that the input matrix for such a model is not necessarily irreducible.
(5.7) Consider a closed Leontief model whose input-output relationships are described by the simultaneous equations
\[ \begin{align*}
    x_1 &= 0.2x_1 + 0.2x_2, \\
    x_2 &= 0.3x_2 + 0.3\varepsilon, \\
    \varepsilon &= 0.8x_1 + 0.5x_2 + 0.7\varepsilon,
\end{align*} \]
where \( x_1 \) and \( x_2 \) are the outputs of the two internal sectors and where \( \varepsilon \) is the level of employment.
Show that the model described here is feasible.

(5.8) Show that the closed model described in Exercise 5.7 has an output equilibrium vector, \((x_1, x_2, \varepsilon)^T\), which is unique up to positive scalar multiples. Compute the output equilibrium vector where \( x_1 + x_2 + \varepsilon = 1 \).

(5.9) Give an example of a closed Leontief model having at least two linearly dependent output equilibrium vectors.

(5.10) Show that a closed Leontief model having a symmetric input matrix \( T \) is feasible if and only if \( A = I - T \) is an M-matrix. Show by example, however, that such a model need not have an output equilibrium vector and give necessary and sufficient conditions for such a vector to exist in terms of the reduced normal form (4.12) for \( T \).

(5.11) Let \( A \) be a nonsingular M-matrix and consider the linear systems \( Ax_i = d_i \), for vectors \( d_i > 0 \), \( i = 1, 2 \), and let
\[ \begin{align*}
    \Delta d &= d_2 - d_1, \\
    \Delta x &= x_2 - x_1.
\end{align*} \]
Then \( x_i > 0 \), \( i = 1, 2 \), and the following assertions hold

(a) \( \Delta d \neq 0 \iff \Delta x \neq 0 \);
(b) \( \Delta d > 0 \rightarrow \Delta x > 0 \);
(c) \( \Delta d < 0 \rightarrow \Delta x < 0 \);

and moreover \( \Delta x \gg 0 \) and \( \Delta x \ll 0 \), respectively, in (b) and (c) if \( A \) is irreducible.

In view of the open Leontief model,

(a) means that the production of at least one commodity changes if and only if the demand for at least one commodity changes,
(b) means that if the demand for at least one commodity increases then the production of at least one commodity increases (all increase if \( A \) is irreducible), and
(c) means the same as (b) for a decrease in demand (Sierksma [1979]).
(5.12) In the notation of Exercise 5.11, assume in addition that $A$ is irreducible and let $\Delta x_j, \Delta d_j$ denote the $j$th components of $\Delta x$ and $\Delta d$, respectively.

Show that

(a) $\Delta x \gg 0 \iff \Delta x_j > 0 \text{ for each } j \text{ with } \Delta d_j < 0$,
(b) $\Delta x \gg 0 \iff \Delta x_j < 0 \text{ for each } j \text{ with } \Delta d_j > 0$.

In terms of the open Leontief model

(a) means that if the production increases of all commodities for which the demand decreases, then all the productions increase and
(b) means that if the production decreases of all commodities for which the demand increases, then all the productions decrease.

Note also that in view of Exercise 5.11, if the production increases of an industry in which the demand decreases, these must be an industry for which the demand increases (Sierksma [1979]).

(5.13) Let $\alpha \in R$. Then a matrix $A \in R^{n \times n}$ will be called a matrix of class $M(\alpha)$ if

$$a_{ij} - a_{ij} = \alpha \quad \text{for all } i \neq j.$$ 

Thus a $3 \times 3$ matrix $A \in M(\alpha)$ has the form

$$A = \begin{bmatrix}
a_{11} & a_{22} - \alpha & a_{33} - \alpha \\
a_{11} - \alpha & a_{22} & a_{33} - \alpha \\
a_{11} - \alpha & a_{22} - \alpha & a_{33}
\end{bmatrix}.$$ 

Show that if $A \in M(\alpha)$ for some $\alpha \in R$, then

(a) $\det A = \alpha^{n-1} \left( \sum_{i=1}^{n} a_{ii} - (n - 1)\alpha \right)$;
(b) $\alpha^{n-1} A e = (\det A) e$ for $e = (1, \ldots, 1)'$.

Show in addition that if $\alpha \neq 0$, then

$$A \in M(\alpha) \iff \text{Adj}(A) \in M((\det A)/\alpha)$$

(Sierksma [1979]).

(5.14) Let $A$ be an irreducible, nonsingular $M$-matrix satisfying $A e \gg 0$. Show that in the notation of Exercises 5.11–5.13, the following statements
are equivalent:

(a) \( A \in M(x) \) for some \( x \in R \);
(b) \( \Delta x_i < \Delta x_j \iff \Delta d_i < \Delta d_j \) for each \( \Delta d \) with \( \Delta d_i \neq 0 \) and \( \Delta d_j \neq 0 \) and each \( i \) and \( j \);
(c) \( \Delta x_m = \max_i \Delta x_i \iff \Delta d_m = \max_i \Delta d_i \), for each \( \Delta d \neq 0 \).
(d) \( x_m = \min_i \Delta x_i \iff \Delta d_m = \min_i \Delta d_i \), for each \( \Delta d \neq 0 \).

In terms of the open Leontief model, (c) and (d) assert that if the input matrix \( T \) is irreducible and satisfies \( T e \ll e \), then the maximal and minimal changes in the demand and production will always occur in the same industry if and only if \( A \in M(x) \) for some \( x \in R \); where \( A = I - T \). Note also that this exercise relates Theorems 3.16 and 3.19 (Sierksma [1979]).

(5.15) Verify Exercise (5.14) for the input matrix

\[
T = \begin{bmatrix}
0 & \frac{1}{7} & \frac{2}{7} \\
\frac{1}{3} & \frac{1}{7} & \frac{3}{7} \\
\frac{1}{2} & \frac{1}{7} & \frac{2}{7}
\end{bmatrix}
\]

by showing that \( A = I - T \) is in \( M(x) \) for some \( x \in R \) and by computing \( \Delta x \) for \( \Delta d = (12, -12, 24)' \) and noting that (b)(c), and (d) then hold for this pair.

6 NOTES

(6.1) Leontief presented the first version of his input–output model in 1936, but developed the topic more fully in The Structure of the American Economy [1941]. A dynamic version was later presented by Leontief [1953], and a brief summarizing presentation of it was given by him [1966, Chapter VII]. The input–output literature is voluminous and an early bibliography has been compiled by Taskier [1961]. For elementary descriptions of input–output models, see Miernyk [1965] and Yan [1969]. A sophisticated analytical treatment of the subject has been given by Schumann [1968].

(6.2) The notation and terminology adopted in this chapter are along the lines of that used by Sarma [1977] and Yan [1969]. Many of the results in Sections 3 and 4 are based upon properties of nonsingular and singular M-matrices, respectively, developed in Chapter 6. Lemma 3.14 is essentially given in Seneta [1973, Theorem 2.3, p. 29], for the irreducible case. Also Theorem 3.16 is his Exercise 2.2, p. 34. Other results in Section 2 are fairly obvious in the terminology of nonsingular M-matrices.
The first input–output model presented by Leontief [1936] was a closed model. The particular "form" of the closed model developed in Section 4 is essentially along the lines of that given by Yan [1969, Chapter III]. However, Theorems 4.10, 4.14, and 4.17, concerning positive solutions to singular M-matrix equations, are believed to be new. For excellent papers on related topics, see Schneider [1956] and Carlson [1963].

(6.3) Other approaches to the analysis of input–output models have been given in the literature. For example, Dorfman et al. [1958] have taken a linear programming approach to the analysis of open Leontief models. Here the optimizing solution is the one and only efficient solution possible for the output vector. In addition, Kemeny and Snell [1960, Section 7.7] have studied closed Leontief models by associating with them a finite homogeneous Markov chain and then studying the model by investigating the properties of this chain. (See the concluding remarks to Section 4.) On the other hand, a nonlinear version of the linear input–output open model of Leontief has been developed and studied by Sandberg [1974a,b]. In the nonlinear version, each product \( t_{ij}x_j \) of the linear system (3.3) is replaced by a possibly nonlinear function \( t_{ij}(x_j) \) of \( x_j \) which is continuously differentiable. This approach enables one to give a comparative analysis of certain classes of input–output models.

(6.4) Input–output analysis is perhaps the most significant, but certainly not the only, area in economics in which nonnegative matrices and M-matrices play an important role. Another important application is to the study of the von Neumann model. The model was developed by von Neumann [1945/46] to introduce the concept of equilibrium growth. It provided the first proof that a solution to a general equilibrium model existed, and it was the first programming model. The main difference between von Neumann's model and the closed Leontief model is that in Leontief's model, the methods of production are given a priori, and in von Neumann's model, a larger number of possible production methods are available; one of the problems, then, is to choose the best technique. These production possibilities can be described by two nonnegative matrices \( A \) and \( B \), and one of the mathematical problems is to find the maximum scalar \( \alpha > 0 \) such that

\[
\alpha Az \leq Bz, \quad z \gg 0.
\]

As expected, the Perron–Frobenius theory of nonnegative matrices given in Chapter 2 plays an important role in the study of such problems. (See Hansen [1970, Chapter 16].)

An interesting application of the theory of M-matrices to the study of world goods prices has been developed. In a special reference to the protection