A Growth Cycle

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The Model

Presented here is a starkly schematized and hence quite unrealistic model of cycles in growth rates. This type of formulation now seems to me to have better prospects than the more usual treatment of growth theory or of cycle theory, separately or in combination. Many of the bits of reasoning are common to both, but in the present paper they are put together in a different way.

The following assumptions are made for convenience:

(1) steady technical progress (disembodied);

(2) steady growth in the labour force;

(3) only two factors of production, labour and 'capital' (plant and equipment), both homogeneous and non-specific;

(4) all quantities real and net;

(5) all wages consumed, all profits saved and invested.

These assumptions are of a more empirical, and disputable, sort:

(6) a constant capital-output ratio;

(7) a real wage rate which rises in the neighbourhood of full employment.

No. (5) could be altered to constant proportional savings, thus changing the numbers but not the logic of the system. No. (6) could be softened but it would mean a serious complicating of the structure of the model.

Symbols used are:

q is output;

k is capital;

w is wage rate;

$a = a_0 e^{\alpha t}$ is labour productivity; α constant;

σ is capital-output ratio (inverse of capital productivity);

w/a is workers' share of product, $(l - w/a)$ capitalists';

Surplus = profit = savings = investment= $(l - w/a)q = \dot{k}$.

Profit rate = $\dot{k}/k = \dot{q}/q = (l - w/a)/\sigma$.

$n = n_0 e^{\beta t}$ is labour supply, β constant;

$l = q/a$ is employment.

Writing $(q/l)^{\cdot}$ for $d/dt(q/l)$, we have

$$(\dot{q}/l)/q/l = \dot{q}/q - \dot{l}/l = \alpha,$$

so that

$$\dot{l}/l = (l - w/a)/\sigma - \alpha.$$

Call

$$u = w/a, \quad v = l/n,$$

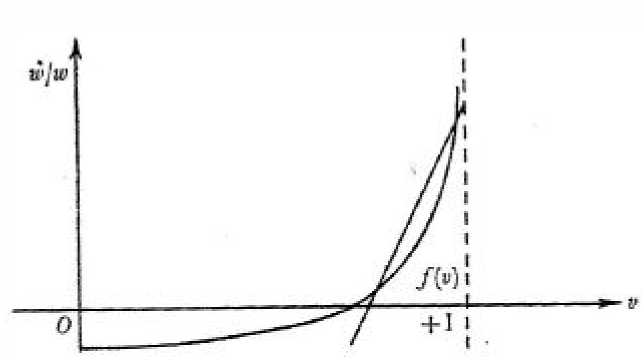

Figure 1:

so that

$$\dot{v}/v = (l - u)/\sigma - (\alpha + \beta).$$

Assumption (7) may be written as

$$\dot{w}/w = f(v)$$

as shown in fig.1.

The following analysis can be carried out using such an $f(v)$, with a change in degree but not in kind of results. Instead, in the interest of lucidity and ease of analysis, I shall take a linear approximation (as shown in fig.1),

$$\dot{w}/w = -\gamma + \rho v$$

and this does quite satisfactorily for moderate movements of v near the point $+1$. Both γ and ρ must be large. Since

$$\dot{u}/u = \dot{w}/w - \alpha, \quad \dot{u}/u = -(\alpha + \gamma) + \rho v.$$

From this and the equation above for v, we have a convenient statement of our model.

$$\dot{v} = [(1/\sigma - (\alpha + \beta)) - (1/\sigma)u]v, \tag{1}$$

$$\dot{u} = [-(\alpha + \gamma) + \rho v]u. \tag{2}$$

In this form we recognize the Volterra case of prey and predator (*Theorie Mathematique de la Lutte pour la Vie*, Paris, 1931). To some extent the similarity is purely formal, but not entirely so. It has long seemed to me that Volterra's problem of the symbiosis of two populations - partly complementary, partly hostile - is helpful in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form.

Eliminating time and performing a first integration we get

$$(1/\sigma)u + \rho v - [1/\sigma - (\alpha + \rho)] \log u - (\gamma + \alpha) \log v = \text{ constant.}$$

Letting

$$\theta_1 = 1/\sigma; \quad \eta_1 = 1/\sigma - (\alpha + \beta),$$

$$\theta_2 = \rho; \qquad \eta_2 = \gamma + \alpha,$$

we can transform this into

$$\phi(u) = u^{\eta_1} e^{-\theta_1 u} = H v^{-\eta_2} e^{\theta_2 v} = H\psi(v), \tag{3}$$

where H is arbitrary constant, depending on initial conditions. Since $1/\sigma > \alpha + \beta$, all coefficients are positive. By differentiating,

$$d\phi/du = (-\theta_1 + \eta_1/u)\phi, \quad d\psi/dv = (\theta_2 - \eta_2/v)\psi,$$

so that we can see that these functions have the sorts of shapes given in fig.2.

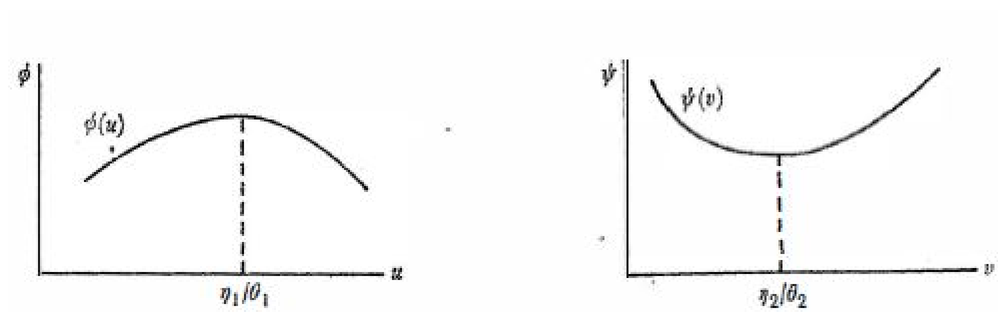

Figure 2:

Our problem as stated in (3) is to equate $\phi(u)$ to $\psi(v)$ multiplied by a constant H. This can be done neatly in the four quadrant positive diagram in fig.3. We draw through the origin a straight line, A, with the slope $\phi/\psi = H$ (arbitrary since dependent on the given initial condition). Then in symmetrical quadrants we place the two curves and ϕ and ψ equating these two through the constant of proportionality gives a possible pair of values for u and v. All possible pairs of u and v constitute a solution, which may be plotted in the remaining quadrant. It can be shown, and indeed is quite obvious, that these solution points lie on a closed, positive curve, B, in u, v space. By going back to equations (1) and (2) we can find in what order the points succeed each other and hence in what direction we traverse curve B, as indicated by arrows in fig.3. A second integration will yield u and v as functions of time, thus allowing us to determine the second arbitrary factor, the point on B at which we start. By varying the slope of A we can generate a family of closed curves broadly similar to B, thus yielding all the possible solutions. One initial condition selects the curve, a second fixes the starting point, and then we traverse some particular curve B in the direction of the

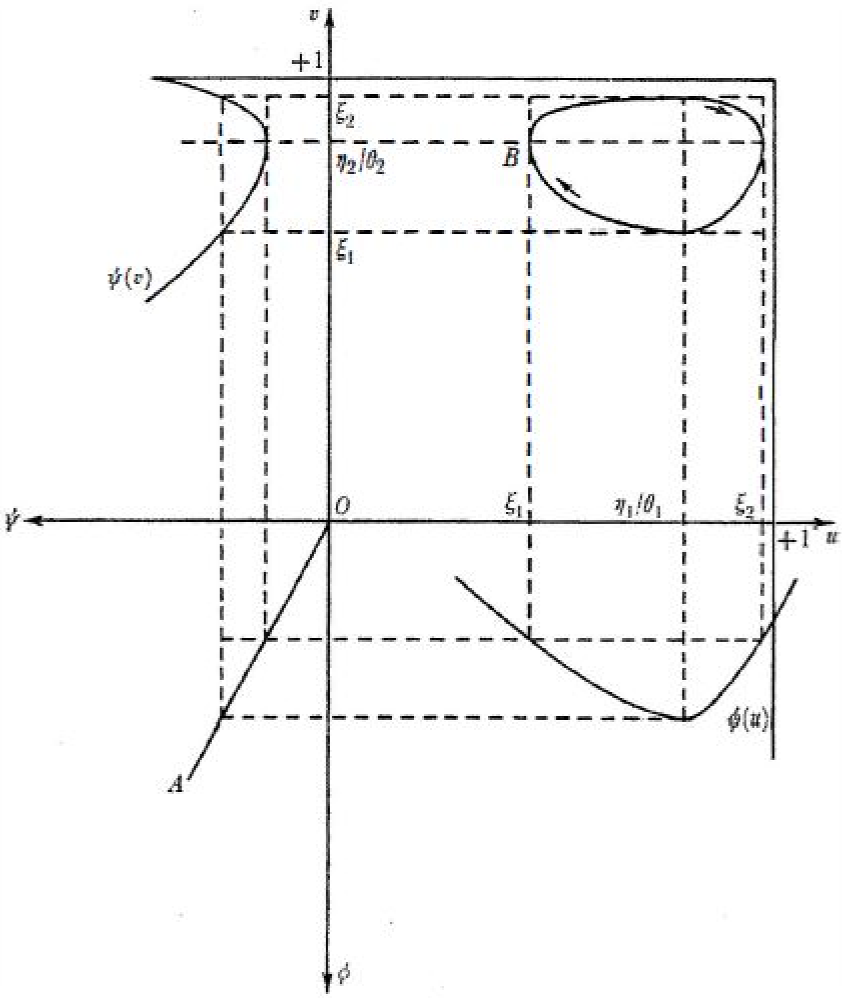

Figure 3:

arrows forever, in the absence of given outside changes. There remains only to spell out the meaning of the motion.

Hence we may classify our model as a non-linear conservative oscillator of, fortunately, a soluble type. As the representative point travels around the closed curve B, u vibrates between ξ_1 and ξ_2, and v between ξ_1 and ξ_2. Both u and v must be positive and v must, by definition, be less than unity; u normally will be also but may, exceptionally, be greater than unity (wages and consumption greater than total product by virtue of losses and disinvestment). Over the stretch 0 to + 1 on the u axis, the point u indicates the distribution of income, workers' share to the left, capitalists' to the right. The capitalists' share, multiplied by a constant, $1/\sigma$, gives us the profit rate and the rate of growth in output, $\dot{q}/q$. When profit is greatest, $u = \xi_1$, employment is average, $v = \eta\theta_2$,, and the high growth rate pushes employment to its maximum ξ_2 which squeezes the profit rate to its average value η_1/θ_1. The deceleration in growth lowers employment (relative) to its average value again, where profit and growth are again at their nadir ξ_2. This low growth rate leads to a fall in output and employment to well below full employment, thus restoring profitability to its average value because productivity is now rising faster than wage rates. This is, I believe, essentially what Marx meant by the contradiction of capitalism and its transitory resolution in booms and slumps. It is, however, un-Marxian in asserting that profitability is restored not (necessarily) by a fall in real wages but rather by their failing to rise with productivity. Real wages must fall in relation to productivity; they may fall absolutely as well, depending on the severity of the cycle. The improved profitability carries the seed of its own destruction by engendering a too vigorous expansion of output and employment, thus destroying the reserve army of labour and strengthening labour's bargaining power. This inherent conflict and complementarity of workers and capitalists is typical of symbiosis.

An undisturbed system has constant average values η_1/θ_1 for u and for v, hence a constant long-run average distribution of income and degree of unemployment. Much more remarkable is the fact that a *disturbed* system still has the same constant long-run values. The time averages of u and of v are independent of initial conditions. We can see this from the fact that a rotation of A (an outside change) will only make the curve B larger or smaller but will not alter its central point. Therefore continual shocks will alter the shape of the cycle but not the long-run average values. Output and employment both will show alternating rates of growth. Whether they actually decrease or merely rise less rapidly will depend on the severity of the cycle. For a mild cycle the growth rate may decrease but never become negative, in other cases there may be a sharp fall. However, the increases must predominate over the decreases, since the time average of $1 - u$ is positive and hence so also is that of $\dot{q}/q$. Likewise employment grows in the long-run at the same rate as labour supply, since the time average of v is constant. Similarily the equality of the growth rate in wages to that in productivity follows from the constancy of u. By contrast the profit rate is equal to $1 - u$ and therefore tends to constancy. We may look at this as standing Ricardo (and Marx) on his head. Progress first accrues as profits but profits lead to expansion and expansion forces wages up and profits down. Therefore we have a Malthusian Iron Law of Profits. This is because of the tendency of capital, though not capitalists, to breed excessively. By contrast labour is something of a rent good since the supply, though variable, does not seem to be a function of wages. Hence it is the sole ultimate beneficiary from technical progress. By now there would, I suppose, be considerable agreement that what happened in history is: wage rates went up; profit rates stayed down. It is to the explanation of this that the present paper is addressed.