Fregean Logics ⋆

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Abstract

According to Frege's principle the denotation of a sentence coincides with its truth-value. The principle is investigated within the context of abstract algebraic logic, and it is shown that taken together with the deduction theorem it characterizes intuitionistic logic in a certain strong sense.

A 2*nd-order matrix* is an algebra together with an algebraic closed set system on its universe. A *deductive system* is a 2nd-order matrix over the formula algebra of some fixed but arbitrary language. A 2nd-order matrix A is *Fregean* if, for any subset X of A, the set of all pairs $\langle a, b\rangle$ such that $X \cup \{a\}$ and $X \cup \{b\}$ have the same closure is a congruence relation on A. Hence a deductive system is Fregean if interderivability is compositional. The logics intermediate between the classical and intuitionistic propositional calculi are the paradigms for Fregean logics. Normal modal logics are non-Fregean while quasi-normal modal logics are generally Fregean.

The main results of the paper: Fregean deductive systems that either have the deduction theorem, or are protoalgebraic and have conjunction, are completely characterized. They are essentially the intermediate logics, possibly with additional connectives. All the full matrix models of a protoalgebraic Fregean deductive system are Fregean, and, conversely, the deductive system determined by any class of Fregean 2nd-order matrices is Fregean. The latter result is used to construct an example of a protoalgebraic Fregean deductive system that is not strongly algebraizable.

Key words: abstract algebraic logic, protoalgebraic logic, equivalential logic, algebraizable logic, self-extensional logic, Leibniz congruence, deduction theorem, quasivariety
1991 MSC: Primary: 03G99, Secondary: 03B22, 03C05, 08C15

⋆ Part of the research reported on here was done while both authors were in residence at the Centre de Recerca Matemàtica of the Institut d'Estudis Catalans.
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1 Supported by National Science Foundation grant CCR-9593168 and a grant from

Introduction

The origin of Fregean logic is Frege's principle of compositionality. Frege's seminal insight, as interpreted by Church [1], was to think of a (declarative) sentence in the same way as one thinks of a proper name. A sentence, like every proper name, must denote or name something (Church's rendering of Frege's *bedeuten*). Church calls the thing it denotes, i.e., its denotation (*Bedeutung*), its *truth-value*. According to Frege a sentence also has a *sense* (*Sinn*), which is also assumed to be compositional. But Frege viewed this concept as extra-linguistic and did not attempt to incorporate it in his formal system.

Frege's analysis of proper names when applied to the denotation of sentences leads to the *principle of compositionality* for truth-values: assume a constituent part φ of a sentence ϑ is replaced by another sentence φ' to give $\vartheta(\varphi'/\varphi)$. If φ and φ' both have the same truth-value, then so do ϑ and $\vartheta(\varphi'/\varphi)$. Logical systems that uphold the *Frege principle* are sometimes called *truth-functional* or *extensional*. Those that violate it are called *nontruth-functional* or *intensional*. Most modal logics are intensional in this sense.

The first one to formally analyze the Frege principle in a general setting was R. Suszko. In his view the denotation of a sentence is not its truth-value, but rather something more in keeping with Frege's notion of the sense of a sentence. [2] Moreover, he introduced a new binary connective Δ, called the *identity connective*, into the language with the idea that the sentence $\varphi \,\Delta\, \psi$ is to be interpreted as the proposition that φ and ψ have the same denotation in this new sense, which for the purposes of this introduction we will view as the proposition that φ and ψ have the same *meanings*. In Suszko's formal system, which he called *logic with identity*, the principal axioms governing the identity-of-meaning connective Δ express its compositionality. Suszko's system also includes all the classical connectives, in particular the biconditional $\leftrightarrow$. As in Frege's system, $\varphi \leftrightarrow \psi$ is to be interpreted as the proposition that φ and ψ have the same truth-value. It is easily shown that the two binary connectives $\leftrightarrow$ and Δ are both compositional only if the sentences $\varphi \leftrightarrow \psi$ and $\varphi \,\Delta\, \psi$ are themselves logically equivalent for all sentences φ and ψ. Thus Frege's principle that $\leftrightarrow$ is compositional can be formalized in Suszko's system as the proposition

$$(x \leftrightarrow y)\,\Delta\,(x\,\Delta\, y).$$

Suszko calls this the *Fregean axiom*. When adjoined to the other axioms of

the Ministerio de Educación y Ciencia of the government of Spain.

[2] Suszko looked to Wittgenstein for support for this view. For him the denotation of a sentence is what the sentence says about a certain "situation". This term was chosen by Suszko to interpret Wittgenstein's *Sachlage*—the state of affairs. See [2] and [3].

logic with identity it gives *Fregean logic*, and extensions of logic with identity in which it fails to hold are called *non-Fregean*.

In this paper we investigate the Fregean axiom within the framework of *abstract algebraic logic*. This is a developing area of algebraic logic in which the focus is on the process by which a class of algebras is associated with a given logical system and on the connection between its metalogical and algebraic properties. We consider a much wider class of deductive systems than those encompassed by Suszko's logic of identity. In particular, we consider deductive systems that are not assumed a priori to have special connectives dedicated to representing identity of meanings or of truth-values. This requires that we begin our investigation with a coherent explanation of how these two key notions are to be represented in an arbitrary deductive system. We now discuss, very briefly, some consequences of the theory of referential frames that serve this purpose; for a more detailed account of referential frames see [4].

A deductive system $\mathcal{S}$ is characterized by its language type Λ, the set Fm_Λ of formulas over Λ, and by the consequence relation $\vdash_{\mathcal{S}}$ that, for each set Γ of formulas, specifies which formulas φ are consequences of Γ. A set T of formulas is a *theory* of $\mathcal{S}$ if it is closed under consequence, i.e., $\varphi \in T$ whenever $T \vdash_{\mathcal{S}} \varphi$. The set of all theories of $\mathcal{S}$ is denoted by $\mathrm{Th}\,\mathcal{S}$. By the theory *axiomatized* by an arbitrary set Γ of formulas we mean the set of all formulas φ such that $\Gamma \vdash_{\mathcal{S}} \varphi$.

$\mathcal{S}$ is viewed as an "uninterpreted" logic. Its interpretations take the form of matrices. A (*logical*) *matrix* is a structure of the form $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra (of the same language type as $\mathcal{S}$), the *underlying algebra* of $\mathfrak{A}$, and $F \subseteq A$, the *designated set* of $\mathfrak{A}$. An *interpretation of* $\mathcal{S}$ is a matrix $\mathfrak{A}$ together with a mapping $h\colon \mathrm{Fm} \to A$ from the set of Fm of formulas into the universe A of the underlying algebra of $\mathfrak{A}$. $h(\varphi)$ is to be thought of as the "sense" or "meaning" of the formula φ under the interpretation, and φ is "true" or "false" depending on whether or not $h(\varphi) \in F$. Several natural assumptions are made about interpretations. First of all, the meaning function h is assumed to be a homomorphism from the algebra of formulas $\mathbf{Fm}$ into $\boldsymbol{A}$; this is the *principle of compositionality of meaning*. Secondly, truth and meaning are assumed to be connected by another well-known principle, due to Leibniz. According to the Leibniz principle identity can be characterized in second-order logic by the formula

$$x \approx y \quad \text{iff} \quad \forall P(P(x) \leftrightarrow P(y)),$$

where P ranges over all unary predicates. The principle is adapted to the interpretations of a deductive system $\mathcal{S}$ by restricting attention to predicates that are "definable" in $\mathcal{S}$ by some formula $\vartheta(x)$ with a designated variable x ($\vartheta(x)$ may have other variables that are treated as parameters). Thus we assume that, if the formulas φ and ψ have different meanings in an interpretation, then they can be distinguished by some predicate, i.e., for some formula

$\vartheta(x)$, $\vartheta(\varphi/x)$ and $\vartheta(\psi/x)$ have different truth-values. This is also known as the *principle of contextual differentiation*. Finally, we assume the class of interpretations is sound and complete for the consequence relation in the sense that $\Gamma \vdash_{\mathcal{S}} \varphi$ iff φ is true in every interpretation in which each $\psi \in \Gamma$ is true.

A consequence of these assumptions is that the global identity-of-truth-value and identity-of-meaning relations can be characterized entirely in terms of the consequence relation, without direct reference to the interpretations. In fact, the identity-of-truth-value relation of $\mathcal{S}$ is given by

$$\mathbf{\Lambda}\,\mathcal{S} = \{\, \langle \varphi, \psi \rangle : \forall \Gamma \subseteq \mathrm{Fm}_\Lambda \big(\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \vdash_{\mathcal{S}} \psi \big) \,\},$$

and the identity-of-meaning relation by

$$\mathbf{\Omega}\,\mathcal{S} = \{\, \langle \varphi, \psi \rangle : \forall \Gamma \subseteq \mathrm{Fm}_\Lambda \; \forall \vartheta(x) \in \mathrm{Fm}_\Lambda \big(\Gamma \vdash_{\mathcal{S}} \vartheta(\varphi/x) \Leftrightarrow \Gamma \vdash_{\mathcal{S}} \vartheta(\psi/x) \big) \,\}.$$

$\mathbf{\Lambda}\,\mathcal{S}$ and $\mathbf{\Omega}\,\mathcal{S}$ are called the *Frege relation* and *Leibniz congruence* of $\mathcal{S}$, respectively. The Fregean axiom for $\mathcal{S}$ takes the form $\mathbf{\Lambda}\,\mathcal{S} = \mathbf{\Omega}\,\mathcal{S}$. Arbitrary deductive systems with this property have been identified and investigated in the literature under the name *self-extensional* (see [3]).

The paradigms for self-extensional deductive systems are the classical and intuitionistic propositional calculi. But these systems have a stronger property: every interpreted classical and intuitionistic logic also satisfies the Fregean axiom, and it is this that is taken to be the defining property of a Fregean deductive system. For any theory T of a deductive system $\mathcal{S}$ define:

$$\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,T = \{\, \langle \varphi, \psi \rangle : \forall \Gamma \subseteq \mathrm{Fm}_\Lambda \big(T, \Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow T, \Gamma \vdash_{\mathcal{S}} \psi \big) \,\},$$

$$\widetilde{\mathbf{\Omega}}_{\mathcal{S}} T = \{\, \langle \varphi, \psi \rangle : \forall \Gamma \subseteq \mathrm{Fm}_\Lambda \; \forall \vartheta(x) \in \mathrm{Fm}_\Lambda \big(T, \Gamma \vdash_{\mathcal{S}} \vartheta(x/\varphi) \Leftrightarrow T, \Gamma \vdash_{\mathcal{S}} \vartheta(x/\psi) \big) \,\}.$$

$\widetilde{\mathbf{\Omega}}_{\mathcal{S}}\,T$ is called the *Suszko congruence* of T with respect to $\mathcal{S}$; it can be expressed in the following more perspicuous form by means of the consequence operator $\mathrm{Clo}_{\mathcal{S}}$ of $\mathcal{S}$. $\widetilde{\mathbf{\Omega}}_{\mathcal{S}}\,T = \{\, \langle \varphi, \psi \rangle : \forall \vartheta(x) \in \mathrm{Fm}_\Lambda \big(\mathrm{Clo}_{\mathcal{S}}(T, \vartheta(x/\varphi)) = \mathrm{Clo}_{\mathcal{S}}(T, \vartheta(x/\psi)) \big) \,\}$. Similarly, $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,T = \{\, \langle \varphi, \psi \rangle : \mathrm{Clo}_{\mathcal{S}}(T, \varphi) = \mathrm{Clo}_{\mathcal{S}}(T, \psi) \,\}$.

A deductive system $\mathcal{S}$ is *Fregean* if $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\,T = \widetilde{\mathbf{\Omega}}_{\mathcal{S}}\,T$ for every theory T of $\mathcal{S}$.

The main result of the paper is that the Fregean axiom together with the deduction theorem is the characteristic property of the intuitionistic calculus. In Theorems 2.20 and 2.22 it is shown that every Fregean deductive system with the uniterm deduction-detachment theorem (Def. 1.38) and every protoalgebraic (Def. 1.14) deductive system with conjunction is equivalent in a strong sense to an axiomatic extension of the appropriate fragments of the intuitionistic propositional calculus, possibly with arbitrarily many additional connectives λ that are compatible with intuitionistic logical equivalence in the sense that, if the rank of λ is n, then $\big((x_0 \leftrightarrow y_0) \wedge \cdots \wedge (x_{n-1} \leftrightarrow y_{n-1})\big) \rightarrow (\lambda x_0 \ldots x_{n-1} \leftrightarrow \lambda y_0 \ldots y_{n-1})$ is a theorem of the deductive system.

Another important theme of the paper has to do with an old problem in abstract algebraic logic: why is it that almost all the algebraizable logics in the literature have varieties of algebras as their algebraic counterparts, when the general theory of algebraizable logics (as recently developed in abstract algebraic logic) indicates that quasivarieties are the natural algebraic counterparts? (Algebraizable deductive systems whose algebraic counterpart is a variety are said to be *strongly algebraizable*.) The paper contains some new insights into the solution of this problem, both in the form of original results and of elaborations of recent important results on this problem due to J. M. Font and R. Jansana [5]. In particular we show that every Fregean deductive system with the uniterm deduction-detachment theorem is strongly algebraizable and that its algebraic counterpart is termwise definitionally equivalent to a variety of Hilbert algebras with compatible operations (Thm. 2.23 and Cor. 2.24). Similarly, it is shown that every protoalgebraic, Fregean deductive system with conjunction is strongly algebraizable, provided that it has at least one theorem, and that its algebraic counterpart is termwise definitionally equivalent to a variety of Brouwerian semilattices with compatible operations (Thm. 2.25 and Cor. 2.26.) Partial generalizations of these results to self-extensional systems are given in Thms. 2.29 and 2.32.

The other central topic of the paper is an investigation of the relationship between Fregean deductive systems and their matrix semantics. A semantic version of the Fregean property is defined (Def. 2.13) and it is proved that, if a protoalgebraic deductive system is Fregean, then every full 2nd-order model (Def. 3.1) of it is Fregean (Cor. 3.5). Conversely, the deductive system determined by any class of Fregean 2nd-order matrices is Fregean (Cor. 3.8). The latter result is used to verify that a particular algebraizable, Fregean deductive system is not strongly algebraizable; the example is due to P. Idziak. We also outline a proof that the $\{\leftrightarrow, \neg\}$-fragment of the intuitionistic propositional calculus is Fregean and algebraizable but not strongly algebraizable. We conclude that the behavior of protoalgebraic, Fregean deductive systems that fail to have the uniterm deduction-detachment theorem differs strikingly from those that do. A protoalgebraic, Fregean deductive system may be either strongly algebraizable or not, but the uniterm deduction-detachment theorem guarantees strong algebraizability in this context by Thm. 2.23. The fact that the deduction-detachment system is uniterm is essential. An example of a Fregean deductive system with the multiterm deduction-detachment theorem is given in [6].

Outline of the paper

The first section contains a survey of the basic elements of abstract algebraic logic that are needed for a systematic study of Fregean logics. The notions of protoalgebraic, equivalential, and algebraizable deductive systems are reviewed. The main novelty of the section is a fairly detailed discussion of the notion of a regularly algebraizable deductive system (Def. 1.29). The highlight of this part is a proof of the fact that a deductive system is regularly algebraizable if and only if it is the assertional logic (Def. 1.32) of a relatively point-regular quasivariety (Def. 1.31). See Thm. 1.34. The deduction-detachment theorem in abstract algebraic logic is discussed in the last part of the section.

It turns out that the property of being Fregean is expressible as a Gentzen-style, or what we call a *2nd-order*, inference rule. The basic properties of Fregean deductive systems are most conveniently developed within the theory of 2nd-order rules and 2nd-order matrices. This is done in the first part of Section 2. The properties of the Frege, Leibniz, and Suszko relations and their relationships are also developed here. The latter part of the section contains the characterizations of Fregean deductive systems with the uniterm deduction-detachment theorem and, alternatively, conjunction; the results on strong algebraizability mentioned previously can also be found here.

The discussion of the matrix semantics of Fregean systems is contained in the last section.

Acknowledgments and connections with other work

As explained above, the study of Fregean logics was initiated by R. Suszko and his collaborators. His formal system of logic with identity, both with and without the Fregean axiom, has been investigated in a number of papers ([7–14].) The first published work on Fregean logic, within the context of abstract algebraic logic, is the monograph of Font and Jansana [5]; an extended abstract can be found in [15]. About the same time, and essentially independently of Font and Jansana, the present authors were obtaining many similar results. However we concentrated almost exclusively on protoalgebraic Fregean logics while the scope of Font and Jansana's work was much broader. The study of nonprotoalgebraic Fregean logic requires the "2nd-order" methods that are developed in Sections 2 and 3. These were pioneered by the Barcelona algebraic logic group, in particular by Font, Jansana, A. Torrens, and V. Verdú. The present authors wish to acknowledge the influence this work has had on their own. A detailed investigation of protoalgebraic Fregean deductive systems with the multiterm deduction-detachment system can be found in [6].

The paper [16] on essentially the purely algebraic aspects of Fregean logic appeared earlier and had some influence on the metalogical developments. The algebraic theory has been further developed in [17–19].

1 Elements of Abstract Algebraic Logic

1.1 Closed-set systems

Let A be a nonempty set. A family $\mathcal{C}$ of subsets of A that is closed under the intersection of arbitrary subfamilies is called a *closed-set system over* A. If $\mathcal{C}$ is also closed under the union of subfamilies that are (upward) directed (under inclusion), then it is called an *algebraic closed-set system.* All closed-set systems considered here are automatically assumed to be algebraic unless otherwise indicated. An algebraic closed set system $\mathcal{C}$ forms an algebraic lattice $\langle \mathcal{C}, \cap, \vee \rangle$ under set-theoretic inclusion. Since empty subfamilies are allowed, every closed-set system over A contains A. Closed-set systems will be represented by the calligraphic letters $\mathcal{C}, \mathcal{D}, \ldots$. The closed sets of a closed-set system $\mathcal{C}$ (i.e., the members of $\mathcal{C}$) will be called *filters* and represented by upper case Latin letters $F, G, \ldots$. The closure operator associated with a given closed-set system $\mathcal{C}$ is denoted by $\mathrm{Clo}_{\mathcal{C}}$. Thus $\mathrm{Clo}_{\mathcal{C}} : \mathcal{P}(A) \to \mathcal{P}(A)$, and, for each $X \subseteq A$, $\mathrm{Clo}_{\mathcal{C}} X = \bigcap \{ F : X \subseteq F \in \mathcal{C} \}$. The algebraicity of $\mathcal{C}$ is reflected in the fact that $\mathrm{Clo}_{\mathcal{C}} X = \bigcup \{ \mathrm{Clo}_{\mathcal{C}} X' : X' \subseteq_{\omega} X \}$. ($X' \subseteq_{\omega} X$ means that X' is a finite subset of X.)

Let $\mathcal{C}$ be an algebraic closed-set system over a nonempty set A. If B is a nonempty subset of A, then $\{ F \cap B : F \in \mathcal{C} \}$ is an algebraic closed-set system over B. If $h: B \to A$, then $h^{-1}(\mathcal{C}) := \{ h^{-1}(F) : F \in \mathcal{C} \}$ is an algebraic closed-set system over B.

Given any $F \in \mathcal{P}(A)$, $\{F, A\}$ is the smallest closed-set system containing F; it is obviously algebraic. If $\mathcal{C}$ is a closed-set system over A that contains F, then we define $[\, F \,)_{\mathcal{C}} := \{ G : F \subseteq G \in \mathcal{C} \}$. It is called the *principal* (*2nd-order*) *filter of* $\mathcal{C}$ *generated by* F. $[\, F \,)_{\mathcal{C}}$ is obviously also an algebraic closed-set system.

1.2 Deductive systems

By a *language type* we mean a set Λ of connectives or operation symbols, depending on whether we are viewing them from a logical or algebraic perspective. Each connective has associated with it a natural number, called its *rank* or *arity*. Λ is *pointed* if it contains a distinguished constant (i.e., nullary operation) symbol, which is usually denoted by $\top$. A fixed, denumerable set Va of *variable symbols* is assumed, and the set of *Λ-formulas* (*Λ-terms* in an algebraic context) is formed in the usual way. The set of Λ-formulas is denoted by Fm_Λ, and the corresponding *algebra of formulas* by $\mathbf{Fm}_\Lambda$. (The subscript may be omitted when only one language type is under consideration.) For any set X of variables, $\mathrm{Fm}_\Lambda(X)$ is the set of formulas in which only variables from X occur, and $\mathbf{Fm}_\Lambda(X)$ is the corresponding subalgebra of $\mathbf{Fm}_\Lambda$. The operation of simultaneously substituting fixed but arbitrary formulas for variables is identified with the unique endomorphism of $\mathbf{Fm}_\Lambda$ it determines. Formulas are represented by lower case Greek letters $\varphi, \psi, \ldots$, and sets of formulas by upper case Greek letters $\Gamma, \Delta, \ldots$.

Definition 1.1 *Let k be a nonzero natural number. By a k-*dimensional deductive system*, or more simply a k-*deductive system*, we mean an ordered pair*

$$\mathcal{S} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S} \rangle,$$

where Λ is an arbitrary language type and $\mathrm{Th}\,\mathcal{S}$ is an (algebraic) closed-set system over Fm^k_Λ, the k-th Cartesian power of Fm_Λ, that is substitution-invariant *in the following sense. $\sigma^{-1}(T) \in \mathrm{Th}\,\mathcal{S}$ for every $T \in \mathrm{Th}\,\mathcal{S}$, or more succinctly,*

$$\sigma^{-1}(\mathrm{Th}\,\mathcal{S}) \subseteq \mathrm{Th}\,\mathcal{S}, \quad \textit{for every substitution } \sigma. \qquad \square$$

The closure operator $\mathrm{Clo}_{\mathrm{Th}\,\mathcal{S}} : \mathcal{P}(\mathrm{Fm}^k_\Lambda) \to \mathrm{Fm}^k_\Lambda$ is called the *consequence operator* of $\mathcal{S}$ and will be denoted by the simpler expression $\mathrm{Cn}_\mathcal{S}$. The *consequence relation* $\vdash_\mathcal{S}$ of a k-deductive system $\mathcal{S}$ is the binary relation between $\mathcal{P}(\mathrm{Fm}^k_\Lambda)$ and Fm^k_Λ defined by $\Gamma \vdash_\mathcal{S} \varphi$ iff $\varphi \in \mathrm{Clo}_\mathcal{S}(\Gamma)$, for all $\Gamma \subseteq \mathrm{Fm}^k_\Lambda$ and $\varphi \in \mathrm{Fm}^k_\Lambda$. A deductive system $\mathcal{S}$ is often identified with one of the two ordered pairs $\langle \mathbf{Fm}_\Lambda, \mathrm{Clo}_{\mathrm{Th}\,\mathcal{S}} \rangle$ or $\langle \mathbf{Fm}_\Lambda, \vdash_\mathcal{S} \rangle$. The reason for our choosing to identify it with the pair $\langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S} \rangle$ will become clear below in Section 2.

The basic syntactic unit of a k-deductive system is a k-tuple of Λ-formulas; these are called *k-formulas*. If $\varphi = \langle \varphi_0, \ldots, \varphi_{k-1} \rangle$ is a k-formula and $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ is a substitution, then the *σ-substitution instance of* φ, $\sigma(\varphi)$, is defined to be $\langle \sigma(\varphi_0), \ldots, \sigma(\varphi_{n-1}) \rangle$.

The filters of $\mathcal{S}$, i.e., the members of $\mathrm{Th}\,\mathcal{S}$, are called *theories of $\mathcal{S}$*, or *$\mathcal{S}$-theories*. Theories are represented by the uppercase Latin letters $T, S, \ldots$.

The defining properties of a k-deductive $\mathcal{S}$ system in terms of its consequence relation are as follows. For all $\Gamma, \Delta \subseteq \mathrm{Fm}^k_\Lambda$ and $\varphi \in \mathrm{Fm}^k_\Lambda$,

$$\Gamma \vdash_\mathcal{S} \varphi \text{ for all } \varphi \in \Gamma \tag{1}$$
$$\Gamma \vdash_\mathcal{S} \varphi \text{ and } \Gamma \subseteq \Delta \text{ imply } \Delta \vdash_\mathcal{S} \varphi; \tag{2}$$
$$\Gamma \vdash_\mathcal{S} \varphi \text{ and } \Delta \vdash_\mathcal{S} \psi \text{ for every } \psi \in \Gamma \text{ imply } \Delta \vdash_\mathcal{S} \varphi; \tag{3}$$
$$\Gamma \vdash_\mathcal{S} \varphi \text{ implies } \Gamma' \vdash_\mathcal{S} \varphi \text{ for some } \Gamma' \subseteq_\omega \Gamma; \tag{4}$$
$$\Gamma \vdash_\mathcal{S} \varphi \text{ implies } \sigma(\Gamma) \vdash_\mathcal{S} \sigma(\varphi) \text{ for every substitution } \sigma. \tag{5}$$

Note that (4), which expresses the property that $\mathcal{S}$ is *finitary*, is equivalent to the algebraicity of $\mathrm{Th}\,\mathcal{S}$. Similarity, (5) expresses the substitution-invariance of $\mathcal{S}$. We also note that condition (2) is a consequence of the other conditions.

1-deductive systems, the ones we will be mostly concerned with in this paper, can be identified with deductive systems in the classical sense of Tarski. These include all the familiar sentential logics together with their various fragments and refinements—for example, the classical and intuitionistic propositional calculi, the intermediate logics, the various modal logics (including Lewis's S4 and S5), and the multiple-valued logics of Łukasiewicz and Post. The substructural logics such as BCK logic, relevance logic, and linear logic can also be formulated as 1-deductive systems, although they are often formulated as Gentzen-type systems.

The deductive systems of equational logic however can be most naturally formulated as 2-deductive systems. k-deductive systems were first considered in [20] and systematically used in [21] as a vehicle for studying algebraizability and the deduction theorem in the context of abstract algebraic logic.

By a *k-dimensional sequent*, or simply a *k-sequent*, or more simply a *sequent* when k is clear from context, we mean a pair $\langle \Gamma, \varphi \rangle$ where Γ is a set of k-formulas and φ is a single k-formula; the sequent is *finite* if Γ is finite and *proper* if Γ is nonempty. The k-sequent $\langle \Gamma, \varphi \rangle$ is usually written in the traditional form $\dfrac{\Gamma}{\varphi}$. A k-formula φ is a *theorem* of a k-deductive system $\mathcal{S}$ if $\vdash_\mathcal{S} \varphi$ (i.e., $\emptyset \vdash_\mathcal{S} \varphi$). The set of all theorems is denoted by $\mathrm{Thm}\,\mathcal{S}$. The k-sequent $\dfrac{\Gamma}{\varphi}$ is a *rule* of $\mathcal{S}$ if $\Gamma \vdash_\mathcal{S} \varphi$.

A k-formula ψ is *directly derivable* from a set Δ of k-formulas by the k-sequent $\dfrac{\Gamma}{\varphi}$ if there is a substitution $\sigma\colon \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ such that $\sigma(\Gamma) \subseteq \Delta$ and $\sigma(\varphi) = \psi$. Every pair of sets Ax, of k-formulas, and Ru, of finite k-sequents, determines a k-deductive system $\mathcal{S}$ in the usual way: for $\Delta \subseteq \mathrm{Fm}^k_\Lambda$ and $\varphi \in \mathrm{Fm}^k_\Lambda$, $\Delta \vdash_\mathcal{S} \varphi$ iff φ is contained in the smallest set of k-formulas that includes Δ, contains all substitution instances of each k-formula in Ax, and is closed under direct derivability with respect to each k-sequent in Ru. The pair Ax and Ru is called a system of *axioms* and *inference rules* for $\mathcal{S}$, and $\mathcal{S}$ is said to be *presented by* Ax *and* Ru. Every k-deductive system can be presented by some, and in fact, many different systems of axioms and inference rules.

Let $\mathcal{S}$ and $\mathcal{S}'$ be k-deductive systems over the language types Λ and Λ', respectively. $\mathcal{S}'$ is an *expansion of* $\mathcal{S}$ if $\Lambda \subseteq \Lambda'$ and $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$. An expansion is called an *extension* if $\Lambda = \Lambda'$; it is said to be *axiomatic* if a presentation of $\mathcal{S}'$ can be obtained from a presentation of $\mathcal{S}$ by adjoining new axioms but no new inference rules. An expansion is *conservative* if $\vdash_{\mathcal{S}'} \cap \left(\mathcal{P}(\mathrm{Fm}^k_\Lambda) \times \mathrm{Fm}^k_\Lambda\right) = \vdash_{\mathcal{S}}$. In this situation $\mathcal{S}$ is called a *fragment* of $\mathcal{S}'$.

The most important example of a 2-deductive system is equational (or quasi-equational) logic. In this context a 2-formula $\langle \varphi, \psi \rangle$ is to be interpreted as an equation $\varphi \approx \psi$.

Free equational logic. Let Λ be any language type. The axioms and inference rules of the system of free equational logic over Λ are the following.

(A1) $\langle x, x \rangle$;

(R1) $\dfrac{\langle x, y \rangle}{\langle y, x \rangle}$;

(R2) $\dfrac{\langle x, y \rangle, \langle y, z \rangle}{\langle x, z \rangle}$;

(R3$_\lambda$) $\dfrac{\langle x_0, y_0 \rangle, \ldots, \langle x_{n-1}, y_{n-1} \rangle}{\langle \lambda x_0 \ldots x_{n-1},\ \lambda y_0 \ldots y_{n-1} \rangle}$, for each $\lambda \in \Lambda$, n the rank of λ.

In free equational logic the 2-formula $\langle \varphi, \psi \rangle$ is identified with the equation $\varphi \approx \psi$ and the 2-sequent

$$\frac{\gamma_0 \approx \delta_0, \ldots, \gamma_{n-1} \approx \delta_{n-1}}{\varphi \approx \psi}$$

with the quasi-equation $\gamma_0 \approx \delta_0 \wedge \cdots \wedge \gamma_{n-1} \approx \delta_{n-1} \to \varphi \approx \psi$. The theories of free equational logic are exactly the congruences on the formula algebras $\mathbf{Fm}_\Lambda$.

Applied equational logic. Each quasivariety defines an extension of free equational logic in the following way.

Definition 1.2 *Let* Q *be a quasivariety over the language type* Λ *that is defined by the identities* Id *and quasi-identities* Qd. *By the* equational logic [3]

[3] Equational logic in this sense differs from the equational logic of identities as it is commonly understood in universal algebra. The latter applies only to varieties so that the set Qd of proper quasi-identities is empty. Moreover, the free equational logic in this sense includes the rule of substitution, i.e., $\langle \varphi, \psi \rangle \,/\, \langle \sigma(\varphi), \sigma(\psi) \rangle$ for

of Q*, in symbols* $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$*, we mean the* 2*-deductive system presented by* (A1)*,* (R1)*,* (R2)*, and* (R3$_\lambda$)*, for* $\lambda \in \Lambda$*, together with the following axioms:*

$$\langle \varphi, \psi \rangle, \quad \textit{for every identity } \varphi \approx \psi \in \mathrm{Id},$$

and inference rules:

$$\frac{\langle \gamma_0, \delta_0 \rangle, \dots, \langle \gamma_{n-1}, \delta_{n-1} \rangle}{\langle \varphi, \psi \rangle}, \quad \begin{array}{l} \textit{for every quasi-identity} \\ \left(\bigwedge_{i<n} \gamma_i \approx \delta_i \right) \to \varphi \approx \psi \in \mathrm{Qd}. \end{array} \qquad \square$$

The theories of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ are the Q-*congruences* on $\mathbf{Fm}_\Lambda$, i.e., those congruences Θ on $\mathbf{Fm}_\Lambda$ such that $\mathbf{Fm}_\Lambda/\Theta \in \mathsf{Q}$. The set of Q-congruences on $\mathbf{Fm}_\Lambda$ is denoted by $\mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$; thus

$$\mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda = \mathrm{Th}\,\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q} \quad \text{and} \quad \boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q} = \langle \mathbf{Fm}_\Lambda, \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda \rangle.$$

The $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$-theory generated by $E \subseteq \mathrm{Fm}^2_\Lambda$ is the K-congruence generated by E, i.e., $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}}(E) = \mathrm{Cg}^{\mathbf{Fm}_\Lambda}_{\mathsf{Q}}(E)$.

In the following theorem we show that, conversely, if $\mathcal{C}$ is any algebraic closed-set system of congruences of $\mathbf{Fm}_\Lambda$ that is substitution-invariant, then $\mathcal{C} = \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$ for a unique quasivariety Q. A consequence of this way of looking at equational logic is a useful intrinsic characterization of those sets of congruences on the formula algebra that are the relative congruences of some quasivariety.

Theorem 1.3 *Let* Λ *be an arbitrary language type. Let* $\mathcal{C} \subseteq \mathrm{Co}\,\mathbf{Fm}_\Lambda$*. Then* $\mathcal{C} = \mathrm{Co}_{\mathsf{Q}}\,\mathrm{Fm}_\Lambda$ *for some quasivariety* Q *iff the following conditions hold.*

(i) $\mathcal{C}$ *is closed under intersection;*
(ii) $\mathcal{C}$ *is closed under the union of upper-directed sets;*
(iii) $\mathcal{C}$ *is substitution-invariant.*

PROOF. If $\mathcal{C} = \mathrm{Co}_{\mathsf{Q}}\,\mathrm{Fm}_\Lambda$, then it is obvious that conditions (i)–(iii) hold. For the reverse implication, assume (i)–(iii) hold, i.e. that $\mathcal{C}$ is an algebraic closed-set system of congruences on $\mathbf{Fm}_\Lambda$ that is substitution-invariant. Then $\mathcal{S} = \langle \mathbf{Fm}_\Lambda, \mathcal{C} \rangle$ is a 2-deductive system, and since theories of $\mathcal{S}$ are congruences, (A1) is a theorem of $\mathcal{S}$ and (R1)–(R3$_\lambda$), for $\lambda \in \Lambda$, are rules of $\mathcal{S}$. Thus $\mathcal{S}$ is an extension of the free equational logic, i.e., an applied equational logic. Let Q be the quasivariety whose identities coincide with the theorems of $\mathcal{S}$ and whose quasi-identities coincide with the rules of $\mathcal{S}$. Then $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$. $\square$

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every substitution σ. Consequently only identities are derivable.

This characterization can also be expressed in purely lattice theoretic terms. A complete lattice $\boldsymbol{L}$ is a *complete meet-subsemilattice* of a complete lattice $\boldsymbol{M}$ if $\bigwedge^{\boldsymbol{L}} S = \bigwedge^{\boldsymbol{M}} S$ for every subset S of L; it is said to be *continuous* if $\bigvee^{\boldsymbol{L}} S = \bigvee^{\boldsymbol{M}} S$ for every upward directed subset S of L.

Corollary 1.4 *Let $\boldsymbol{\mathcal{C}}$ be a sublattice of the complete lattice $\mathbf{Co}\,\mathbf{Fm}_\Lambda$ of all congruences on $\mathbf{Fm}_\Lambda$. Then $\boldsymbol{\mathcal{C}} = \mathbf{Co}_{\mathsf{Q}}\,\mathrm{Fm}_\Lambda$ for some quasivariety Q iff $\boldsymbol{\mathcal{C}}$ is substitution-invariant and a complete, continuous meet-subsemilattice of $\mathbf{Co}\,\mathbf{Fm}_\Lambda$.* $\square$

The mapping $\mathsf{Q} \mapsto \boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ is a one-one correspondence between quasivarieties over Λ and extensions of the free equational logic over Λ by additional axioms and inference rules.

It will be useful to define the equational logic of an arbitrary class K of Λ-algebras. For such a class $\mathbf{S}\,\mathsf{K}$ and $\mathbf{I}\,\mathsf{K}$ respectively denote the class of all subalgebras and isomorphic images of members of K. By a K-*congruence* on a Λ-algebra $\boldsymbol{A}$ (not necessarily a member of K) we mean a congruence Θ on $\boldsymbol{A}$ such the $\boldsymbol{A}/\Theta \in \mathbf{I}\,\mathbf{S}\,\mathsf{K}$. The set of all K-congruences on $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$. We define $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{K}$ to be $\langle \mathbf{Fm}_\Lambda, \mathrm{Co}_{\mathsf{K}}\,\mathbf{Fm}_\Lambda \rangle$. $\mathrm{Co}_{\mathsf{K}}\,\mathbf{Fm}_\Lambda$ is not in general closed under intersection or directed union, but it is closed under inverse substitution, and it does have an associated consequence relation: for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{Fm}^2_\Lambda$,

$$E \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \varphi \approx \psi \in \bigcap \Big\{ \Theta \in \mathrm{Co}_{\mathsf{K}}\,\mathbf{Fm}_\Lambda : E \subseteq \Theta \Big\}.$$

Alternatively, $E \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{K}} \varphi \approx \psi$ iff for every homomorphism $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$, $h(\vartheta) = h(\xi)$ for all $\vartheta \approx \xi \in E$ implies $h(\vartheta) = h(\xi)$. Note that in general $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{K}$ is not finitary, but it is substitution invariant.

1.3 Algebraizable 1-deductive systems

A general theory of algebraizability of logic was presented in [22] and subsequently refined and extended in a number of papers by several different authors [23,24,5,21]. We restrict ourselves here to 1-deductive systems and to the notion of algebraizability as originally presented in [22]. In current terminology this is called *finite algebraizability.* From now on, when we speak of a "deductive system" (without reference to its dimension) we mean a 1-deductive system.

In the following development it is convenient to use the expression $K \approx L$ as an abbreviation for a set of equations $\{\, \kappa_i \approx \lambda_i : i \in I \,\}$.

Definition 1.5 *Let $\mathcal{S}$ be a 1-deductive system over Λ. $\mathcal{S}$ is* finitely algebraizable *if there is a quasivariety Q over Λ, a finite nonempty system*

$$E(x,y) = \Big\{\varepsilon_0(x,y), \ldots, \varepsilon_{n-1}(x,y)\Big\}$$

of binary formulas (i.e., formulas in two variables), and a finite nonempty system

$$K(x) \approx L(x) = \Big\{\kappa_0(x) \approx \lambda_0(x), \ldots, \kappa_{m-1}(x) \approx \lambda_{m-1}(x)\Big\}$$

of equations (2-formulas) in one variable such that the following equivalences hold between the rules of $\mathcal{S}$ and of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$.

$$\text{For all } \Gamma \cup \{\varphi\} \subseteq_{\boldsymbol{\omega}} \mathrm{Fm}_\Lambda, \quad \Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \tag{6}$$
$$\Big\{\kappa_j(\psi) \approx \lambda_j(\psi) : \psi \in \Gamma, j < m\Big\} \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} \kappa_j(\varphi) \approx \lambda_j(\varphi), \quad \text{for all } j < m.$$

$$\text{For all } \Gamma \approx \Delta \cup \{\varphi \approx \psi\} \subseteq_{\boldsymbol{\omega}} \mathrm{Fm}^2_\Lambda, \quad \Gamma \approx \Delta \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} \varphi \approx \psi \quad \text{iff} \tag{7}$$
$$\Big\{\varepsilon_i(\gamma,\delta) : \gamma \approx \delta \in \Gamma \approx \Delta, i < n\Big\} \vdash_{\mathcal{S}} \varepsilon_i(\varphi,\psi), \quad \text{for all } i < n.$$

$$\Big\{\varepsilon_i\Big(\kappa_j(x),\, \lambda_j(x)\Big) : i < n, j < m\Big\} \vdash_{\mathcal{S}} x, \quad \text{and} \quad x \vdash_{\mathcal{S}} \varepsilon_i\Big(\kappa_j(x),\, \lambda_j(x)\Big), \quad \text{for all } j < m, i < n. \tag{8}$$

$$\Big\{\kappa_j\Big(\varepsilon_i(x,y)\Big) \approx \lambda_j\Big(\varepsilon_i(x,y)\Big) : i < n, j < m\Big\} \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} x \approx y, \quad \text{and} \tag{9}$$
$$x \approx y \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} \kappa_j\Big(\varepsilon_i(x,y)\Big) \approx \lambda_j\Big(\varepsilon_i(x,y)\Big), \text{ for all } i < n, j < m. \quad \square$$

We note that, since Q is a quasivariety, $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ as well as $\mathcal{S}$ is a finitary deductive system. So, if the equivalences (6) and (7) hold for finite sets of equations and formulas, they also hold for infinite sets.

In the sequel when we say simply "algebraizable" we will mean finitely algebraizable in the above sense.

These four conditions are abbreviated respectively as follows.

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad K(\Gamma) \approx L(\Gamma) \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} K(\varphi) \approx L(\varphi). \tag{10}$$
$$\Gamma \approx \Delta \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} \varphi \approx \psi \quad \text{iff} \quad E(\Gamma,\Delta) \vdash_{\mathcal{S}} E(\varphi,\psi). \tag{11}$$
$$x \dashv\vdash_{\mathcal{S}} E\Big(K(x), L(x)\Big). \tag{12}$$
$$x \approx y \dashv\vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}} K\Big(E(x,y)\Big) \approx L\Big(E(x,y)\Big). \tag{13}$$

Similar abbreviations, which should be self-explanatory, will be used in the sequel without further elaboration.

It is not difficult to show ([22, Corollary 2.9]) that each of the conditions (11) and (12) is derivable from the two conditions (10) and (13), and vice versa.

Thus (10) and (13) together are sufficient for algebraizability. Similarly, (11) and (12) are sufficient.

Definition 1.6 *Let $\mathcal{S}$ be a* 1*-deductive system and* Q *a quasivariety over the language type* Λ*.*

(i) *A finite system $K(x) \approx L(x)$ of equations in one variable is said to be a* faithful interpretation *of (the consequence relation of) $\mathcal{S}$ in (the consequence relation of) $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ if the equivalence (10) holds for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\Lambda$.*

(ii) *A finite system $E(x,y)$ of binary formulas is said to be a* faithful interpretation *of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ in $\mathcal{S}$ if the equivalence (11) holds for all $\Gamma \approx \Delta \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}^2_\Lambda$.*

(iii) *The interpretations $K(x) \approx L(x)$ and $E(x,y)$ are* inverses *of one another if the entailments* (12) *and* (13) *both hold.* □

Corollary 1.7 *A 1-deductive system $\mathcal{S}$ is* (*finitely*) *algebraizable iff there is an invertible faithful interpretation $E(x,y)$ of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ in $\mathcal{S}$ for some quasivariety Q.* □

So a 1-deductive system $\mathcal{S}$ is algebraizable if it is definitionally equivalent in a natural way to the equational logic of some quasivariety Q. We will see below in Cor. 1.26 that Q is uniquely determined by $\mathcal{S}$. It is called the *equivalent quasivariety of* $\mathcal{S}$.

Definition 1.8 *An algebraizable 1-deductive system whose equivalent quasivariety is a variety is said to be* strongly algebraizable. □

Let $\boldsymbol{A}$ be a Λ-algebra. A subset F of A^k is called a *filter* of $\boldsymbol{A}$ *over* a k-deductive system $\mathcal{S}$, an $\mathcal{S}$*-filter of* $\boldsymbol{A}$, if F contains all interpretations of each theorem of $\mathcal{S}$ and is closed under all interpretations of each rule of $\mathcal{S}$. More precisely, for every homomorphism $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$, we have that $h(\varphi) \in F$ for each theorem φ of $\mathcal{S}$, and $h(\psi_0), \dots, h(\psi_{n-1}) \in F$ imply $h(\varphi) \in F$ for each rule $\dfrac{\psi_0, \dots, \psi_{n-1}}{\varphi}$ of $\mathcal{S}$. The set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ is denoted by $\mathrm{Fi}_\mathcal{S}\,\boldsymbol{A}$; it is an (algebraic) closed-set system over A. Thus it forms an algebraic lattice under inclusion, which is denoted by $\mathbf{Fi}_\mathcal{S}\,\boldsymbol{A}$.

We note that $\mathrm{Th}\,\mathcal{S} = \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_\Lambda$. We also note that, for every quasivariety Q, the ($\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$)-filters of $\boldsymbol{A}$ are exactly the Q-congruences on $\boldsymbol{A}$, i.e., the congruences Θ such that $\boldsymbol{A}/\Theta \in \mathsf{Q}$. Thus $\mathrm{Co}_\mathsf{Q}\,\boldsymbol{A} = \mathrm{Fi}_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}}\,\boldsymbol{A}$. Since Q is closed under isomorphism and subdirect products, generated Q-congruences exist, and, for every $R \subseteq A^2$, $\mathrm{Cg}^{\boldsymbol{A}}_\mathsf{Q}(X) = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}}\,\boldsymbol{A}}(X)$. The algebraic lattice of Q-congruences is denoted by $\mathbf{Co}_\mathsf{Q}\,\boldsymbol{A}$.

Definition 1.9 *Let $\boldsymbol{A}$ be a Λ-algebra and $F \subseteq A^k$. Let*

$$\boldsymbol{\Omega}_{\boldsymbol{A}} F := \Big\{ \langle a, b\rangle \in A^2 : \varphi^{\boldsymbol{A}}(a,\bar{c}) \in F \text{ iff } \varphi^{\boldsymbol{A}}(b,\bar{c}) \text{ for all } \varphi(x,\bar{z}) \in \mathrm{Fm}^k_\Lambda \text{ and } \bar{c} \in A^{|k|} \Big\}.$$

$\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is clearly a congruence relation on $\boldsymbol{A}$. It is called the Leibniz congruence of F on $\boldsymbol{A}$. □

Lemma 1.10 ([25, Lemma 1.5]) *Let $\boldsymbol{A}$ be a Λ-algebra and $F \subseteq A^k$. $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is the largest congruence Θ on $\boldsymbol{A}$ that is* compatible with F *in the following sense. If $\langle a_0, \dots, a_{k-1}\rangle \in F$ and $a_i \equiv b_i \pmod{\Theta}$ for all $i < k$, then $\langle b_0, \dots, b_{k-1}\rangle \in F$.* □

It is easy to check that there is always a largest congruence with this property. As a mapping from $\mathcal{P}(A^k)$ to $\mathrm{Co}\,\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is called the *Leibniz operator on $\boldsymbol{A}$*. When $\boldsymbol{A}$ is the formula algebra $\mathbf{Fm}_\Lambda$ the superscript on $\boldsymbol{\Omega}$ is normally omitted; thus $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathbf{Fm}_\Lambda}$. In the following lemma we verify a fundamental property of the Leibniz congruence, its invariance under surjective homomorphisms.

Lemma 1.11 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Λ-algebras and $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ a surjective homomorphism. Then for any $F \subseteq B^k$,*

$$\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big) = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F).$$

PROOF. $h^{-1}(F) := \{ \langle a_0, \dots, a_{k-1}\rangle : \langle h(a_0), \dots, h(a_{k-1})\rangle \in F \}$. It is easy to check that $h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$ is a congruence on $\boldsymbol{A}$ compatible with $h^{-1}(F)$; so $\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big) \supseteq h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$.

Let $\Theta = h^{-1}(\mathrm{Id}_A)$, the relation kernel of h. Θ is compatible with $h^{-1}(F)$, so that $\Theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big)$. Thus $h\Big(\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big)\Big)$ is a congruence of $\boldsymbol{B}$ that is compatible with $h\,h^{-1}(F)$, which equals F because h is surjective. Consequently $h\Big(\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big)\Big) \subseteq \boldsymbol{\Omega}_{\boldsymbol{B}} F$, and hence

$$\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big) = h^{-1}h\Big(\boldsymbol{\Omega}_{\boldsymbol{A}}\Big(h^{-1}(F)\Big)\Big) \subseteq h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F). \qquad □$$

It is known that a 1-deductive system $\mathcal{S}$ is algebraizable with equivalent quasivariety Q iff $\boldsymbol{\Omega}$ induces an isomorphism between the lattices $\mathbf{Th}\,\mathcal{S}$ and $\mathbf{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$; see [22, Theorems 3.7(ii) and 4.1]. It then follows from Cor. 1.4 that a 1-deductive system $\mathcal{S}$ is algebraizable iff $\boldsymbol{\Omega}$ is an isomorphism between $\mathbf{Th}\,\mathcal{S}$ and a substitution-invariant, complete, continuous meet-subsemilattice of $\mathbf{Co}\,\mathbf{Fm}_\Lambda$. (The condition that the meet-subsemilattice be substitution-invariant turns out to be redundant.) This is one of the main results of [22]; see Theorem 4.2. It is further proved in [22] that, if a 1-deductive system $\mathcal{S}$

is algebraizable, then, for every Λ-algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is an isomorphism between $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and a continuous sublattice of $\mathbf{Co}\,\boldsymbol{A}$ ([22, Theorem 5.1]).

This characterization gives rise in turn to a useful representation of the equivalent quasivariety of any algebraizable deductive system. The mode of representation is quite general and can be used to associate a class of Λ-algebras with every deductive system of any given dimension.

By a *k-matrix* (*over* Λ) we mean an ordered pair $\mathfrak{A} = \langle \boldsymbol{A}, F_{\mathfrak{A}} \rangle$, where $\boldsymbol{A}$ is a Λ-algebra and $F_{\mathfrak{A}}$ is a subset of A^k. $F_{\mathfrak{A}}$ is called the *designated filter* of $\mathfrak{A}$ and $\boldsymbol{A}$ the *underlying algebra of* $\mathfrak{A}$. The subscript $_{\mathfrak{A}}$ on the designated filter is usually omitted. 1-matrices are referred to simply as *matrices.*

$\mathfrak{A}$ is *reduced* if $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F_{\mathfrak{A}} = \mathrm{Id}_A$, the identity congruence. If $\mathfrak{A}$ is an arbitrary k-matrix, $\mathfrak{A}^*$ is defined to be the quotient matrix $\langle \boldsymbol{A}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\, F_{\mathfrak{A}},\ F_{\mathfrak{A}}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\, F_{\mathfrak{A}} \rangle$. As a consequence of the commutativity of the Leibniz operator with inverse surjective homomorphisms (Lem. 1.11) we have that $\mathfrak{A}^*$ is always reduced; it is called the *reduction of* $\mathfrak{A}$. For any given class K of matrices, $\mathsf{Alg}\,\mathsf{K}$ is defined to be the class of underlying algebras of members of K and K^* is the class of reduced members of K.

Let $\mathcal{S}$ be an arbitrary k-deductive system. A k-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$ is a *model,* more precisely a *matrix model, of* $\mathcal{S}$ if $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$. The class of models of $\mathcal{S}$ is denoted by $\mathsf{Mod}\,\mathcal{S}$, and the class of all reduced models (i.e., models that are reduced as matrices) by $\mathsf{Mod}^*\,\mathcal{S}$. With each k-deductive system we associate the class of algebras $\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$, that is the class of underlying algebras of reduced models of $\mathcal{S}$. We will see below in Cor. 1.26 that, if $\mathcal{S}$ is an algebraizable 1-deductive system, then $\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$ is its equivalent quasivariety.

The following easy lemma exactly characterizes the ($\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$)-congruences on the formula algebra as the Leibniz congruences.

Lemma 1.12 *Let $\mathcal{S}$ be a 1-deductive system.*

$$\mathrm{Co}_{\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}}(\mathbf{Fm}_\Lambda) = \{\, \boldsymbol{\Omega}\, T : T \in \mathrm{Th}\,\mathcal{S} \,\}.$$

PROOF. The inclusion from right to left is obvious. Let $\Theta \in \mathrm{Co}_{\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}}\,\mathbf{Fm}_\Lambda$, i.e., $\mathbf{Fm}_\Theta/\Theta \in \mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$. There exists a $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}\,\mathcal{S}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F = \mathrm{Id}_A$ and $\boldsymbol{A} \cong \mathbf{Fm}_\Lambda/\Theta$. Let $h\colon \mathbf{Fm}_\Lambda \twoheadrightarrow \boldsymbol{A}$ be a surjective homomorphism such that Θ is its relation kernel. Then $\Theta = h^{-1}(\mathrm{Id}_A) = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{A}}\, F) = \boldsymbol{\Omega}\, h^{-1}(F)$, by Lem. 1.11, and $h^{-1}(F) \in \mathrm{Th}\,\mathcal{S}$. □

Let $\mathcal{S}$ be an algebraizable 1-deductive system. In [22, Theorem 2.17] an algorithm is given for constructing a presentation of the equational logic of

$\mathsf{Alg\,Mod}^* \mathcal{S}$ by identities and quasi-identities from a given presentation of $\mathcal{S}$ by axioms and inference rules. The following is an improvement of this result.[4]

Theorem 1.13 *Let $\mathcal{S}$ be a 1-deductive system presented by a set of axioms* Ax *and a set of proper inference rules* Ru. *Assume $\mathcal{S}$ is algebraizable with equivalent quasivariety* Q. *Let*

$$E(x,y) = \Big\{ \varepsilon_0(x,y), \dots, \varepsilon_{n-1}(x,y) \Big\}$$

be a faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ in $\mathcal{S}$, and let $K(x) \approx L(x) = \Big\{ \kappa_0(x) \approx \lambda_0(x), \dots, \kappa_{m-1}(x) \approx \lambda_{m-1}(x) \Big\}$ be a faithful interpretation of $\mathcal{S}$ in $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ that is the inverse of $E(x,y)$. Then the equational logic $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ of Q is presented by adjoining to

(i) *the axioms and rules of free equational logic* ((A1),(R1),(R2), *and* (R3$_\lambda$) *for each $\lambda \in \Lambda$),*

the identities

(ii) $K(\varphi) \approx L(\varphi)$, *for each $\varphi \in$* Ax,

together with the following quasi-identities.

(iii) $\dfrac{K(\psi_0) \approx L(\psi_0), \dots, K(\psi_{p-1}) \approx L(\psi_{p-1})}{K(\varphi) \approx L(\varphi)}$, *for each sequent* $\dfrac{\psi_0, \dots, \psi_{p-1}}{\varphi}$ *in* Ru*;*

(iv) $\dfrac{K\big(E(x,y)\big) \approx L\big(E(x,y)\big)}{x \approx y}$

PROOF. It is easy to see that each equation and quasi-equation of (i)–(iv) is an identity and quasi-identity of Q, so we only have to prove that, conversely, every identity and quasi-identity of Q is a consequence of (i)–(iv). We first show that the equations $K\big(E(x,x)\big) \approx L\big(E(x,x)\big)$, i.e., $\kappa_j\big(\varepsilon_i(x,x)\big) \approx \lambda_j\big(\varepsilon_i(x,x)\big)$, for $i < n, j < m$, are all provable (i.e., derivable from the empty set) with respect to (w.r.t.) (i)–(iv). $\varepsilon_i(x,x)$ is a theorem of $\mathcal{S}$ for each $i < n$. Let $\eta_0, \dots, \eta_k = \varepsilon_i(x,x)$ be a proof w.r.t. Ax, Ru. Then the sequence of sets of equations $K(\eta_0) \approx L(\eta_0), \dots, K(\eta_k) \approx L(\eta_k)$ includes, in the obvious sense, a proof of all the equations $\kappa_j\big(\varepsilon_i(x,x)\big) \approx \lambda_j\big(\varepsilon_i(x,x)\big)$ w.r.t. (ii) and (iii).

Suppose now that $\dfrac{\gamma_0 \approx \delta_0, \dots, \gamma_{k-1} \approx \delta_{k-1}}{\vartheta \approx \psi}$ is a quasi-identity of Q. We showed above that $K\big(E(\gamma_i,\gamma_i)\big) \approx L\big(E(\gamma_i,\gamma_i)\big)$ is provable w.r.t. (ii) and (iii). But

[4] This improvement was pointed out to the authors by J. M. Font and R. Jansana.

$K\Big(E(\gamma_i,\delta_i)\Big) \approx L\Big(E(\gamma_i,\delta_i)\Big)$ is derivable from $\gamma_i \approx \delta_i$ and $K\Big(E(\gamma_i,\gamma_i)\Big) \approx L\Big(E(\gamma_i,\gamma_i)\Big)$ w.r.t. (A1) and (R3$_\lambda$) for $\lambda \in \Lambda$. So, for each $i < k$, $K\Big(E(\gamma_i,\delta_i)\Big) \approx L\Big(E(\gamma_i,\delta_i)\Big)$ is derivable from $\gamma_i \approx \delta_i$ alone w.r.t. (i)–(iii). We next observe that, from the assumption $\gamma_0 \approx \delta_0, \dots, \gamma_{k-1} \approx \delta_{k-1} \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathbf{Q}} \vartheta \approx \psi$, we have that $E(\gamma_0,\delta_0), \dots, E(\gamma_{k-1},\delta_{k-1}) \vdash_{\mathcal{S}} E(\vartheta,\psi)$. So each equation of $K\Big(E(\vartheta,\psi)\Big) \approx L\Big(E(\vartheta,\psi)\Big)$ is derivable from $K\Big(E(\gamma_i,\delta_i)\Big) \approx L\Big(E(\gamma_i,\delta_i)\Big)$, for $i < k$, w.r.t. (iii), and hence from the $\gamma_i \approx \delta_i$, for $i < k$, w.r.t. (i)—(iii). An application of (iv) gives the desired derivation of $\vartheta \approx \psi$ from $\gamma_0 \approx \delta_0, \dots, \gamma_{k-1} \approx \delta_{k-1}$. □

1.4 Protoalgebraic and equivalential deductive systems

As previously noted, algebraizability can be characterized by conditions on the Leibniz operator $\boldsymbol{\Omega}$ on $\mathcal{S}$-theories. By progressively relaxing these restrictions we obtain a hierarchy of increasingly wider classes of deductive systems that still retain much of the algebraic character of algebraizable systems [25,20,4,26,24,23,27]. We consider two such classes here.

Definition 1.14 *A k-deductive system $\mathcal{S}$ is* protoalgebraic *if $\boldsymbol{\Omega}\colon \mathrm{Th}\,\mathcal{S} \to \mathrm{Co}\,\mathbf{Fm}_\Lambda$ is monotonic, i.e,*

$$T \subseteq S \text{ implies } \boldsymbol{\Omega}\,T \subseteq \boldsymbol{\Omega}\,S, \quad \text{for all } \mathcal{S}\text{-theories } T \text{ and } S. \qquad \square$$

It is an easy consequence of the commutativity of the Leibniz operator with inverse surjective homomorphisms that, if $\mathcal{S}$ is protoalgebraic, then $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotonic on $\mathcal{S}$-filters of $\boldsymbol{A}$ for every countably generated Λ-algebra $\boldsymbol{A}$, and it is not difficult to see that this is in fact the case for all Λ-algebras.

The following lemma is a reformulation of a minor generalization of the correspondence theorem for the filters of a protoalgebraic 1-deductive system ([25, Theorem 2.4]).

Lemma 1.15 *Let $\mathcal{S}$ be a protoalgebraic k-deductive system. Let $\boldsymbol{A}, \boldsymbol{B}$ be Λ-algebras and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a surjective homomorphism. Then for every $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}$,*

$$h^{-1}\Big([\,F\,)_{\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}}\Big) = [\,h^{-1}(F)\,)_{\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}}.$$

PROOF. Obviously $h^{-1}\Big([\,F\,)_{\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}}\Big) \subseteq [\,h^{-1}(F)\,)_{\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}}$, whether or not $\mathcal{S}$ is protoalgebraic. For the reverse inclusion, suppose $h^{-1}(F) \subseteq G \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$. Note that $F \subseteq h(G)$ since h is surjective. Let Θ be the relation kernel of h, i.e., $\Theta = h^{-1}(\mathrm{Id}_A)$. Then obviously Θ is compatible with $h^{-1}(F)$. So $\Theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}\,h^{-1}(F)$ by definition of $\boldsymbol{\Omega}_{\boldsymbol{A}}$, and $\boldsymbol{\Omega}_{\boldsymbol{A}}\,h^{-1}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}\,G$ by protoalgebraicity. So Θ is compatible with G, which implies $h(G) \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}$ and $h^{-1}h(G) = G$. So $G \in h^{-1}\Big([\,F\,)_{\mathrm{Fi}_{\mathcal{S}}\boldsymbol{B}}\Big)$. $\square$

If $\mathcal{S}$ is an arbitrary k-deductive system, then $\mathsf{Alg\,Mod}^*\,\mathcal{S}$ is closed under isomorphism, but not in general under subalgebras, products, or subdirect products.

Theorem 1.16 ([20, Theorem 9.3]) *Let $\mathcal{S}$ be a k-deductive system. $\mathcal{S}$ is protoalgebraic iff $\mathsf{Mod}^*\,\mathcal{S}$ is closed under subdirect products. Thus, if $\mathcal{S}$ is protoalgebraic, then $\mathsf{Alg\,Mod}^*\,\mathcal{S}$ is closed under subdirect products.* $\square$

Let $\Delta(x,y) = \{\,\delta_i(x,y) : i \in I\,\}$ be a (possibly infinite) set of 1-formulas in two variables. $\Delta(x,y)$ is called a *protoequivalence system* [5] for a 1-deductive system $\mathcal{S}$ if the following are theorems and a rule of $\mathcal{S}$.

$$\Delta(x,x) \tag{14}$$

$$\frac{x,\ \Delta(x,y)}{y}. \tag{MP$_\Delta$}$$

The latter sequent is called *Δ-modus ponens* or *Δ-detachment*, and Δ is said to have the *modus ponens* or *detachment property* (with respect to $\mathcal{S}$) if (MP$_\Delta$) is a rule of $\mathcal{S}$.

Every protoequivalence system for $\mathcal{S}$ includes a finite subset that is also a protoequivalence system for $\mathcal{S}$. This easily follows from the assumption that $\mathcal{S}$ is finitary. Consequently, in the sequel we normally assume that protoequivalence systems are finite.

The following syntactical characterization is due to [20], but see [4, Theorem 1.1.3] for a simpler proof.

Theorem 1.17 *A 1-deductive system is protoalgebraic iff it has a protoequivalence system.* $\square$

For each language type Λ and each nonzero k there is a unique protoalgebraic k-deductive system with no theorems; it is presented by the single inference rule $\dfrac{x}{y}$ (and no axioms). The system has exactly two theories, $\emptyset$ and Fm^k_Λ. The protoequivalence system for this deductive system is empty. The protoequivalence systems for every other protoalgebraic k-deductive system are all nonempty. We will be interested only in such systems in the sequel. Thus in the sequel we assume that all protoalgebraic deductive systems have at least one theorem.

[5] Elsewhere in the literature a system of this kind is referred to either as an *equivalence system* ([20]) or as an *implication system.*

In the following, $\bar{u}$ denotes a generally infinite sequence $u_0, u_1, u_2, \ldots$, without repetitions, of variables different from x and y. Let

$$E(x, y, \bar{u}) = \{\, \varepsilon_i(x, y, \bar{u}) : i \in I \,\}$$

be a system of 1-formulas over Λ in two variables, x and y, and an arbitrary number of variables from the list $\bar{u}$; the latter variables are called *parameters.* Of course, each individual formula $\varepsilon_i(x, y, \bar{u})$ actually contains only a finite number of parameters, but the set of parameters that occurs in at least one of the members of $E(x, y, \bar{u})$ maybe infinite and generally is. For all $\varphi, \psi \in \mathrm{Fm}_\Lambda$, let $\forall\bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta})$ stand for the set of substitution instances $\sigma\big(\varepsilon_i(x, y, \bar{u})\big)$, where i ranges over all of I and σ ranges over all substitutions such that $\sigma(x) = \varphi$ and $\sigma(y) = \psi$; i.e.,

$$\forall\bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}) := \Big\{\, \varepsilon_i(\varphi, \psi, \bar{\vartheta}) : i \in I, \bar{\vartheta} \in \mathrm{Fm}_\Lambda^{|\bar{u}|} \,\Big\}.$$

$E(x, y, \bar{u})$ is called an *equivalence system with parameters* of a 1-deductive system $\mathcal{S}$ if it is nonempty and each $\varepsilon_i(x, x, \bar{u})$ is theorem of $\mathcal{S}$, and hence, because of the substitution-invariance of $\mathcal{S}$, each member of

$$\forall\bar{\vartheta}\, E(x, x, \bar{\vartheta}) \tag{15}$$

is a theorem of $\mathcal{S}$. In addition, all of the following are rules of $\mathcal{S}$.

$$\frac{\forall\bar{\vartheta}\, E(x, y, \bar{\vartheta})}{\forall\bar{\vartheta}\, E(y, x, \bar{\vartheta})}, \quad \text{i.e.,} \frac{\forall\bar{\vartheta}\, E(x, y, \bar{\vartheta})}{\varepsilon_i(y, x, \bar{\xi})} \text{ for each } i \in I \text{ and } \bar{\xi} \in \mathrm{Fm}_\Lambda^{|\bar{u}|}, \tag{16}$$

$$\frac{\forall\bar{\vartheta}\, E(x, y, \bar{\vartheta}),\ \forall\bar{\vartheta}\, E(y, z, \bar{\vartheta})}{\forall\bar{\vartheta}\, E(x, z, \bar{\vartheta})}, \tag{17}$$

$$\frac{\forall\bar{\vartheta}\, E(x_0, y_0, \bar{\vartheta}), \ldots, \forall\bar{\vartheta}\, E(x_{n-1}, y_{n-1}, \bar{\vartheta})}{\forall\bar{\vartheta}\, E(\lambda x_0, \ldots, x_{n-1},\ \lambda y_0, \ldots, y_{n-1}, \bar{\vartheta})} \quad \text{for all } \lambda \in \Lambda \ (n \text{ is the rank of } \lambda), \tag{18}$$

$$\frac{x,\ \forall\bar{\vartheta}\, E(x, y, \bar{\vartheta})}{y}. \tag{19}$$

The infinite system of rules (19) is called $E(x, y, \bar{u})$-*modus ponens* or $E(x, y, \bar{u})$-*detachment.*

It is easy to see that if $E(x, y, \bar{u})$ is an equivalence system with parameters for $\mathcal{S}$, then $E(x, y, x, x, x, \ldots)$, the set of formulas obtained by replacing every parameter by x, is a protoequivalence system for $\mathcal{S}$ and hence $\mathcal{S}$ is protoalgebraic. The next theorem shows that the converse holds, every protoalgebraic system with at least one theorem has an equivalence system with parameters. This property gives an alternative characterization of protoalgebraic 1-deductive systems.

Theorem 1.18 ([20, Theorem 3.10]) *Assume that $\mathcal{S}$ is a protoalgebraic 1-deductive system. Let $\Delta(x,y) = \{\delta_0(x,y), \dots, \delta_{n-1}(x,y)\}$ be a protoequivalence system for $\mathcal{S}$. Let $\bar{u}$ be a sequence without repetitions of all variables in* $\mathrm{Va} \setminus \{x,y\}$*. Then*

$$\begin{aligned} E(x,y,\bar{u}) := \ & \Big\{ \delta_i\Big(\xi(x,\bar{u}),\, \xi(y,\bar{u})\Big) : i < n, \xi(x,\bar{u}) \in \mathrm{Fm}_\Lambda(\mathrm{Va} \setminus \{y\} \Big\} \\ & \cup \Big\{ \delta_i\Big(\xi(y,\bar{u}),\, \xi(x,\bar{u})\Big) : i < n, \xi(x,\bar{u}) \in \mathrm{Fm}_\Lambda(\mathrm{Va} \setminus \{y\} \Big\} \end{aligned}$$

is an equivalence system with parameters for $\mathcal{S}$. □

Let $E(x,y,\bar{u}) = \{\, \varepsilon_i(x,y,\bar{u}) : i \in I \,\}$ be a system of binary formulas with parameters. For any algebra $\boldsymbol{A}$ and all $a, b \in A$ we denote by $\forall \bar{d}\, E^{\boldsymbol{A}}(a,b,\bar{d})$ the set of all elements of A of the form $h\Big(\varepsilon_i(x,y,\bar{u})\Big)$ where i ranges over I and h over all homomorphisms $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ such that $h(x) = a$ and $h(y) = b$, i.e.,

$$E^{\boldsymbol{A}}(a,b,\bar{u}) = \{\, \varepsilon^{\boldsymbol{A}}(a,b,\bar{d}) : \bar{d} \in A^{|\bar{u}|} \,\}.$$

A nonempty system $E(x,y,\bar{u}) = \{\, \varepsilon_i(x,y,\bar{u}) : i \in I \,\}$ of binary formulas with parameters is said to *define Leibniz congruences* in a 1-deductive system $\mathcal{S}$ if, for every algebra $\boldsymbol{A}$ and $F \subseteq A$,

$$\boldsymbol{\Omega}_{\boldsymbol{A}} F = \Big\{ \langle a, b \rangle \in A^2 : \ \forall \bar{d}\, E^{\boldsymbol{A}}(a,b,\bar{d}) \subseteq F \Big\}.$$

Theorem 1.19 *$E(x,y,\bar{u})$ is an equivalence system with parameters for a* 1*-deductive system $\mathcal{S}$ iff $E(x,y,\bar{u})$ defines Leibniz congruences in $\mathcal{S}$.*

PROOF. Suppose $E(x,y,\bar{u})$ is an equivalence system with parameters for a 1-deductive system $\mathcal{S}$. Fix an arbitrary algebra $\boldsymbol{A}$ and $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ and define

$$\Phi := \Big\{ \langle a, b \rangle \in A^2 : \forall \bar{d}\, E^{\boldsymbol{A}}(a,b,\bar{d}) \subseteq F \Big\}.$$

The theorems and rules (15)–(18) of $\mathcal{S}$ guarantee that Φ is a congruence relation, and the $E(x,y,\bar{u})$-detachment rule, (19), guarantees that Φ is compatible with F. So $\Phi \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F$. To show the inclusion in the opposite direction assume $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$. Then, for each $i \in I$ and $\bar{d} \in A^{|\bar{u}|}$, $\varepsilon_i^{\boldsymbol{A}}(a,a,\bar{d}) \equiv \varepsilon_i^{\boldsymbol{A}}(a,b,\bar{d}) \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$. But $\varepsilon_i^{\boldsymbol{A}}(a,a,\bar{d}) \in F$ by (15). Thus $\varepsilon_i^{\boldsymbol{A}}(a,b,\bar{d}) \in F$ since $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is compatible with F. This shows $\Phi = \boldsymbol{\Omega}_{\boldsymbol{A}} F$. Hence $E(x,y,\bar{u})$ defines Leibniz congruences in $\mathcal{S}$.

Suppose now that $E(x,y,\bar{u})$ defines Leibniz congruences in $\mathcal{S}$. Since $\varphi \equiv \varphi \pmod{\boldsymbol{\Omega}\, T}$ for every $T \in \mathrm{Th}\,\mathcal{S}$ and $\varphi \in \mathrm{Fm}_\Lambda$, we have that $\forall \bar{\vartheta}\, E(\varphi,\varphi,\bar{\vartheta}) \subseteq T$ for every $T \in \mathrm{Th}\,\mathcal{S}$, i.e., (15) is a set of theorems of $\mathcal{S}$. Suppose $\forall \bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta}) \subseteq T$. Then $\varphi \equiv \psi \pmod{\boldsymbol{\Omega}\, T}$. Hence $\psi \equiv \varphi \pmod{\boldsymbol{\Omega}\, T}$, and consequently $\forall \bar{\vartheta}\, E(\psi,\varphi,\bar{\vartheta}) \subseteq T$ by assumption. Since this holds for every $T \in \mathrm{Th}\,\mathcal{S}$ and all $\varphi, \psi \in \mathrm{Fm}_\Lambda$, (16) is an infinite rule of $\mathcal{S}$. In a similar way we get that (17) and (18) are infinite rules of $\mathcal{S}$. Finally, assume $\{\varphi\} \cup \forall\bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}) \subseteq T$. Then $\varphi \equiv \psi \pmod{\boldsymbol{\Omega}\, T}$ and hence $\psi \in T$ since $\boldsymbol{\Omega}\, T$ is compatible with T. So $E(x, y, \bar{u})$-detachment is a rule of $\mathcal{S}$, and hence $E(x, y, \bar{u})$ is an equivalence system with parameters for $\mathcal{S}$. □

One useful consequence of this theorem is that, under special circumstances, an equivalence system with parameters gives a faithful interpretation of the equational logic of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\, \mathsf{Alg\,Mod}^*\, \mathcal{S}$ in $\mathcal{S}$ in a restricted sense; compare Def. 1.6(ii). In order to prove this we need the following technical lemma.

The class of all countable models of a deductive system $\mathcal{S}$ is denoted $\mathsf{Mod}_{\boldsymbol{\omega}_1}\, \mathcal{S}$, and, for any class K of Λ-algebras, the class of all countable subalgebras of members of K is denoted by $\mathbf{S}_{\boldsymbol{\omega}_1}\, \mathsf{K}$.

Lemma 1.20 *Assume $\mathcal{S}$ is a protoalgebraic 1-deductive system over a countable language type Λ. Then $\mathbf{S}_{\boldsymbol{\omega}_1}\, \mathsf{Alg\,Mod}^*\, \mathcal{S} = \mathbf{S}\, \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1}\, \mathcal{S}$.*

PROOF. The inclusion from right to left is obvious. Let $\boldsymbol{A} \in \mathbf{S}_{\boldsymbol{\omega}_1}\, \mathsf{Alg\,Mod}^*\, \mathcal{S}$. Let $\langle \boldsymbol{B}, F\rangle \in \mathsf{Mod}^*\, \mathcal{S}$ such that $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{\Omega}_{\boldsymbol{B}}\, F = \mathrm{Id}_B$. If $\boldsymbol{\Omega}_{\boldsymbol{A}}(F \cap A) = \mathrm{Id}_A$, then $\langle \boldsymbol{A}, F \cap A\rangle \in \mathsf{Mod}^*\, \mathcal{S}$, and hence $\boldsymbol{A} \in \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1}\, \mathcal{S}$, and we are done. So we assume $\boldsymbol{\Omega}_{\boldsymbol{A}}(F \cap A) \supsetneq \mathrm{Id}_A$. Let $E(x, y, \bar{u})$ be an equivalence system for $\mathcal{S}$ with parameters; $E(x, y, \bar{u})$ exists by Thm. 1.18 since $\mathcal{S}$ is protoalgebraic. We define by recursion an increasing sequence $\boldsymbol{A} = \boldsymbol{A}_0 \subseteq \boldsymbol{A}_1 \subseteq \boldsymbol{A}_2 \subseteq \cdots \subseteq \boldsymbol{A}_n \subseteq \cdots \boldsymbol{B}$, $n < \boldsymbol{\omega}$, of countable subalgebras of $\boldsymbol{B}$ as follows. $\boldsymbol{A}_0 = \boldsymbol{A}$, as indicated. Suppose $\boldsymbol{A}_n$ is defined. For each pair a, a' of distinct elements of A_n, choose a term $\varepsilon(x, y, u_0, \ldots, u_{n-1}) \in E(x, y, \bar{u})$ and elements $b_0, \ldots, b_{m-1} \in B$ such that $\varepsilon^{\boldsymbol{B}}(a, b, b_0, \ldots, b_{m-1}) \notin F$; such a choice is possible by Thm. 1.19 since by assumption $\boldsymbol{\Omega}_{\boldsymbol{B}}\, F = \mathrm{Id}_B$. Let A'_n be the extension of A_n obtained by adjoining the $b_0, \ldots, b_{m-1}$ obtained in this way for every $\langle a, a'\rangle \in A_n^2 \setminus \mathrm{Id}_{A_n}$, and let $\boldsymbol{A}_{n+1}$ be the subalgebra of $\boldsymbol{B}$ generated by A'_n. Let $\boldsymbol{C} = \bigcup_{n<\boldsymbol{\omega}} \boldsymbol{A}_n$. Then $\boldsymbol{A} \subseteq \boldsymbol{C} \subseteq \boldsymbol{B}$ and $\boldsymbol{C}$ is countable.

We now show that $\boldsymbol{\Omega}_{\boldsymbol{C}}(F \cap C) = \mathrm{Id}_C$. Suppose $c, c' \in C$ with $c \neq c'$. Consider any $n \leq \boldsymbol{\omega}$ such that $c, c' \in A_n$. By definition of $\boldsymbol{A}_{n+1}$ there is an $\varepsilon(x, y, u_0, \ldots, u_{m-1}) \in E(x, y, \bar{u})$ and $b_0, \ldots, b_{m-1} \in A_{n-1} \subseteq C$ such that $\varepsilon^{\boldsymbol{C}}(a, b, b_0, \ldots, b_{m-1}) = \varepsilon^{\boldsymbol{B}}(a, b, b_0, \ldots, b_{m-1}) \notin F \cap C$. So $c \not\equiv c' \pmod{\boldsymbol{\Omega}_{\boldsymbol{C}}(F \cap C)}$ by Thm. 1.19, and hence $\boldsymbol{\Omega}_{\boldsymbol{C}}(F \cap C) = \mathrm{Id}_{\boldsymbol{C}}$. Thus $\langle \boldsymbol{C},\, F \cap C\rangle \in \mathsf{Mod}^*_{\boldsymbol{\omega}_1}\, \mathcal{S}$, and $\boldsymbol{A} \subseteq \boldsymbol{C} \in \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1}\, \mathcal{S}$. hence $\boldsymbol{A} \in \mathbf{S}\, \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1}\, \mathcal{S}$. □

We remark that this lemma also holds without the assumption that $\mathcal{S}$ is protoalgebraic, but we apply it only to protoalgebraic systems here.

Theorem 1.21 *Let $\mathcal{S}$ be a protoalgebraic 1-deductive system over a countable language type, and let $E(x, y, \bar{u})$ be a system of formulas in two variables and parameters. $E(x, y, \bar{u})$ is an equivalence system with parameters for $\mathcal{S}$ iff $\mathcal{S}$ has $E(x, y, \bar{u})$-detachment, and, for all $\varGamma \approx \varDelta \cup \{\varphi \approx \psi\} \subseteq_{\boldsymbol{\omega}} \mathrm{Fm}^2_{\varLambda}$,*

$$\varGamma \approx \varDelta \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}} \mathsf{Alg\,Mod}^* \mathcal{S}} \varphi \approx \psi \quad \textit{iff} \quad \forall \bar{\vartheta}\, E(\varGamma, \varDelta, \bar{\vartheta}) \vdash_{\mathcal{S}} \forall \bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}), \tag{20}$$

where $\forall \bar{\vartheta}\, E(\varGamma, \varDelta, \bar{\vartheta}) = \bigcup \Big\{ \forall \bar{\vartheta}\, E(\gamma, \delta, \bar{\vartheta}) : \gamma \approx \delta \in \varGamma \approx \varDelta \Big\}$.

PROOF. Assume first of all the $E(x, y, \bar{u})$ is an equivalence system with parameters for $\mathcal{S}$. Then by definition $\mathcal{S}$ has $E(x, y, \bar{u})$-detachment. We verify the equivalence (20).

$\Longleftarrow$: We prove the contrapositive. Suppose $\varGamma \approx \varDelta \nvdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}} \mathsf{Alg\,Mod}^* \mathcal{S}} \varphi \approx \psi$. Let X be the finite set of variables that appear in $\varGamma \approx \varDelta \cup \{\varphi \approx \psi\}$. Then there is a $\varTheta \in \mathrm{Co}\,\mathbf{Fm}_{\varLambda}(X)$ such that $\mathbf{Fm}_{\varLambda}(X)/\varTheta \in \mathsf{S}_{\boldsymbol{\omega}_1} \mathsf{Alg\,Mod}^* \mathcal{S}$ such that $\varGamma \approx \varDelta \subseteq \varTheta$ but $\vartheta \approx \psi \notin \varTheta$. By Lem. 1.20, $\mathsf{S}_{\boldsymbol{\omega}_1} \mathsf{Alg\,Mod}^* \mathcal{S} = \mathsf{S}\,\mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1} \mathcal{S}$. So there is a $\varTheta' \in \mathrm{Co}\,\mathbf{Fm}_{\varLambda}$ such that $\mathbf{Fm}_{\varLambda}/\varTheta' \in \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1} \mathsf{S}$ and $\varTheta' \cap \mathrm{Fm}^2(X) = \varTheta$. Since $\mathbf{Fm}_{\varLambda}/\varTheta' \in \mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1} \mathsf{S}$, by Lem. 1.12 $\varTheta' = \boldsymbol{\Omega}\, T$ for some $\mathcal{S}$-theory T. So $\varTheta' = \{ \langle \xi, \eta \rangle \in \mathrm{Fm}^2_{\varLambda} : \forall \bar{\vartheta}\, (\xi, \eta, \bar{\vartheta}) \subseteq T \}$ by Thm. 1.19. Thus $\forall \bar{\vartheta}\, E(\varGamma, \varDelta, \bar{\vartheta}) \subseteq T$, but $\forall \bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}) \nsubseteq T$, i.e., $\forall \bar{\vartheta}\, E(\varGamma, \varDelta, \bar{\vartheta}) \nvdash_{\mathcal{S}} E(\varphi, \psi, \bar{\vartheta})$.

$\Longrightarrow$: Again we prove the contrapositive. Suppose $\forall \bar{\vartheta}\, (\varGamma, \varDelta, \bar{\vartheta}) \nvdash_{\mathcal{S}} \forall \bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta})$. Let T be the $\mathcal{S}$-theory generated by $\forall \bar{\vartheta}\, E(\varGamma, \varDelta, \bar{\vartheta})$, and let $\varTheta = \boldsymbol{\Omega}\, T$. By Lem. 1.12, $\mathbf{Fm}_{\varLambda}/\varTheta \in \mathsf{Alg\,Mod}^* \mathcal{S}$, and by Thm. 1.19 $\varGamma \approx \varDelta \subseteq \varTheta$ but $\varphi \approx \psi \notin \varTheta$. So $\varGamma \approx \varDelta \nvdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}} \mathsf{Alg\,Mod}^* \mathcal{S}} \varphi \approx \psi$.

Assume now that $\mathcal{S}$ has $E(x, y, \bar{u})$-detachment and the equivalence (20) holds. The remaining defining conditions for an equivalence system with parameters, (15)–(18), are easily verified. Consider, for example, the transitivity rule (17). For all $\varphi, \xi, \psi \in \mathrm{Fm}_{\varLambda}$,

$$\forall \bar{\vartheta}\, E(\varphi, \xi, \bar{\vartheta}) \cup \forall \vartheta\, E(\xi, \psi, \bar{\vartheta}) \vdash_{\mathcal{S}} \forall \vartheta\, E(\varphi, \psi, \bar{\vartheta})$$
$$\text{iff} \quad \varphi \approx \xi,\ \xi \approx \psi \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}} \mathsf{Alg\,Mod}^* \mathcal{S}} \varphi \approx \psi.$$

The right-hand entailment obviously holds. Thus the left-hand entailment, the transitivity rule, holds. The other defining rules of an equivalence system with parameters are verified similarly. □

We note some features of this proof for future reference. We only used the premiss that the language type is countable to apply Lem. 1.12. So the conclusion of the theorem holds, without any condition on the cardinality of the language type, whenever $\mathsf{S}_{\boldsymbol{\omega}_1} \mathsf{Alg\,Mod}^* \mathcal{S} = \mathsf{S}\,\mathsf{Alg\,Mod}^*_{\boldsymbol{\omega}_1} \mathcal{S}$, in particular whenever $\mathsf{Alg\,Mod}^* \mathcal{S}$ is closed under subalgebras. More generally, it holds without qualification if the consequence relation $\vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg\,Mod}^*\mathcal{S}}$ in the equivalence (20) is replaced by $\vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg\,Mod}^*_{\omega_1}\mathcal{S}}$.

Recall that $\mathrm{Thm}\,\mathcal{S}$ is the smallest theory of $\mathcal{S}$, the set of theorems of $\mathcal{S}$.

Corollary 1.22 *Let $\mathcal{S}$ be a protoalgebraic 1-deductive system over a countable language type Λ. Then the set of all identities of $\mathsf{Alg\,Mod}^*\,\mathcal{S}$ coincides with $\boldsymbol{\Omega}\,\mathrm{Thm}\,\mathcal{S}$.*

PROOF. By the theorem, $\varphi \approx \psi$ is an identity of $\mathsf{Alg\,Mod}^*\,\mathcal{S}$ iff the entailment $\vdash_{\mathcal{S}} \forall\bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta})$ holds, i.e., iff $\forall\bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta}) \subseteq \mathrm{Thm}\,\mathcal{C}$. The conclusion of the corollary now follows immediately from Thm. 1.19. □

The quotient algebra $\mathbf{Fm}_\Lambda/\,\boldsymbol{\Omega}(\mathrm{Thm}\,\mathcal{S})$ is called the *Lindenbaum-Tarski algebra* of $\mathcal{S}$.

An equivalence system $E(x,y) = \{\,\varepsilon_i(x,y) : i \in I\,\}$ for $\mathcal{S}$ with an empty sequence of parameters is called simply an *equivalence system.*[6] Thus $E(x,y)$ is an equivalence system for a 1-deductive system $\mathcal{S}$ if it is a nonempty protoequivalence system (i.e., (14) and (MP_Δ), with $E(x,y)$ in place of $\Delta(x,y)$, are respectively theorems and a rule of $\mathcal{S}$), and in addition the following are rules of $\mathcal{S}$.

$$\frac{E(x,y)}{E(y,x)}, \tag{21}$$

$$\frac{E(x,y),\, E(y,z)}{E(x,z)}, \tag{22}$$

$$\frac{E(x_0,y_0),\ldots,E(x_{n-1},y_{n-1})}{E(\lambda x_0,\ldots,x_{n-1},\,\lambda y_0,\ldots,y_{n-1})} \quad \text{for all } \lambda \in \Lambda \ (n \text{ is the rank of } \lambda). \tag{23}$$

In contrast to protoequivalence systems, an equivalence system may not include a finite subset that is also an equivalence system. We remark that (21) and (22) are provable from the remaining inference rules defining an equivalence system, although this is not completely obvious.

A 1-deductive system is said to be (*finitely*) *equivalential* if has a (finite) equivalence system ([28,26]). Obviously, every equivalential system is protoalgebraic. The significance of equivalential systems should be apparent after the next two theorems and their corollaries.

[6] also called a *weak congruence system* ([20]).

Theorem 1.23 ([20, Theorem 13.12]) *Let $\mathcal{S}$ be a 1-deductive system.*

(i) *$\mathcal{S}$ is equivalential iff $\mathsf{Mod}^* \mathcal{S}$ is closed under the formation of submatrices and direct products.*
(ii) *$\mathcal{S}$ is finitely equivalential iff $\mathsf{Mod}^* \mathcal{S}$ is closed under the formation of submatrices and reduced products.* □

Corollary 1.24 ([26, Corollary I.13]) *$\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ is a quasivariety for every finitely equivalential 1-deductive system $\mathcal{S}$.* □

Theorem 1.25 *Let $\mathcal{S}$ be a 1-deductive system. Then a nonempty set $E(x, y)$ of binary formulas is an equivalence system for $\mathcal{S}$ iff $\mathcal{S}$ has $E(x, y)$-detachment and $E(x, y)$ is a faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ in $\mathcal{S}$, i.e., for all $\Gamma \approx \Delta \cup \{\varphi \approx \psi\} \subseteq \mathrm{Fm}^2_\Lambda$,*

$$\Gamma \approx \Delta \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}} \varphi \approx \psi \quad \textit{iff} \quad E(\Gamma, \Delta) \vdash_{\mathcal{S}} E(\varphi, \psi). \tag{24}$$

PROOF. By Thm. 1.21 the equivalence (24) holds under the hypothesis that the language type is countable. But since $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ is closed under subalgebras, $\mathbf{S}_{\boldsymbol{\omega}_1}\,\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S} = \mathbf{S}\,\mathsf{Alg}\,\mathsf{Mod}^*_{\boldsymbol{\omega}_1} \mathcal{S}$ without any restriction on the cardinality of the language type. See the remarks following the proof of Thm. 1.21 □

Corollary 1.26 *If a 1-deductive system $\mathcal{S}$ is algebraizable, then its equivalent quasivariety is unique and must be $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$.*

PROOF. Assume $\mathcal{S}$ is algebraizable with equivalent quasivariety Q and let $E(x, y)$ be a finite, faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$ in $\mathcal{S}$. It is easy to verify that that (14), (())MP$_\Delta$, (21)–(23) are all theorems or rules of $\mathcal{S}$ and hence that $E(x, y)$ is a finite equivalence system for $\mathcal{S}$. Thus by the theorem $E(x, y)$ is also a faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ in $\mathcal{S}$. Thus $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q} = \boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$, and hence $\mathsf{Q} = \mathsf{Mod}^* \mathcal{S}$ since they are both quasivarieties. □

1.5 Regularly algebraizable deductive systems

The deductive systems of traditional algebraic logic are all algebraizable in the special sense that "truth" is represented by a single element in the Lindenbaum-Tarski algebra, in opposition to the case for an arbitrary algebraizable deductive system where it may have a multitude of representations.

By the E-G-*rule* relative to a given equivalence system $E(x, y)$ we mean the sequent

$$\frac{x, y}{E(x, y)}. \tag{25}$$

It follows immediately from Thm. 1.19 that, if $\mathcal{S}$ is equivalential, then

$$x \equiv y \pmod{\boldsymbol{\Omega}(\mathrm{Clo}_{\mathcal{S}}\{x, y\})} \quad \text{iff} \quad E(x, y) \subseteq \mathrm{Clo}_{\mathcal{S}}\{x, y\} \quad \text{iff} \quad x, y \vdash_{\mathcal{S}} E(x, y).$$

So the E-G-rule is equivalent to the so-called *meta G-rule*, that is, $x \equiv y$ $(\mathrm{mod}\ \boldsymbol{\Omega}(\mathrm{Clo}_{\mathcal{S}}\{x, y\}))$. Thus, if the E-G-rule is a rule of $\mathcal{S}$ for some equivalence system $E(x, y)$, then the E'-G-rule is a rule of $\mathcal{S}$ for every equivalence system $E'(x, y)$ of $\mathcal{S}$. We refer to all these rules collectively simply as the "G-rule".

Assume the G-rule is a rule of $\mathcal{S}$. Then $\varphi, \psi \vdash_{\mathcal{S}} E(\varphi, \psi)$. Hence, for every $\mathcal{S}$-theory T, $\varphi \equiv \psi$ $(\mathrm{mod}\ \boldsymbol{\Omega} T)$ for all $\varphi, \psi \in T$. Conversely, if $\varphi \equiv \psi$ $(\mathrm{mod}\ \boldsymbol{\Omega} T)$ for every theory T and all $\varphi, \psi \in T$, then in particular $x \equiv y$ $(\mathrm{mod}\ \boldsymbol{\Omega}(\mathrm{Clo}_{\mathcal{S}}\{x, y\}))$. So the G-rule holds. This shows that the G-rule is a rule of an equivalential deductive system iff T is an equivalence class of $\boldsymbol{\Omega} T$ for every theory T.

The following lemma gives a closely related characterization of the G-rule that proves useful in the sequel.

Lemma 1.27 *Let $\mathcal{S}$ be any equivalential 1-deductive system with an equivalence system $E(x, y)$. The G-rule is a rule of $\mathcal{S}$ iff $x \dashv\vdash_{\mathcal{S}} E(x, \top)$, where $\top$ is any theorem of $\mathcal{S}$.*

PROOF. Assume the G-rule is rule of $\mathcal{S}$ and let $\top$ be any theorem. We have $E(x, \top) \vdash_{\mathcal{S}} x$ by E-detachment. Conversely, $x, \top \vdash_{\mathcal{S}} E(x, \top)$ by the G-rule. Hence $x \vdash_{\mathcal{S}} E(x, \top)$ since $\top$ is a theorem.

Assume now that $x \dashv\vdash_{\mathcal{S}} E(x, \top)$. Then $x, y \vdash_{\mathcal{S}} E(x, \top), E(y, \top)$. But by the symmetry and transitivity of equivalence ((21) and (22)), $E(x, \top), E(y, \top) \vdash_{\mathcal{S}} E(x, y)$. So the G-rule is a rule of $\mathcal{S}$. □

Theorem 1.28 ([22], Corollary 4.8) *Every finitely equivalential deductive system $\mathcal{S}$ with the G-rule is algebraizable. Furthermore, the singleton $\{x \approx \top\}$ is a faithful interpretation of $\mathcal{S}$ in $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg\,Mod}^*\,\mathcal{S}$, where $\top$ can be taken to be any theorem of $\mathcal{S}$.*

PROOF. Let $E(x, y)$ be a finite equivalence system for $\mathcal{S}$. $E(x, y)$ is a faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Alg\,Mod}^*\,\mathcal{S}$ in $\mathcal{S}$ by Thm. 1.25. Thus it suffices to verify the invertibility condition (12), i.e., $x \dashv\vdash_{\mathcal{S}} E\big(K(x), L(x)\big)$, with

$K(x) \approx L(x) = \{x \approx \top\}$ This takes the form $x \dashv\vdash_{\mathcal{S}} E(x, \top)$, which holds by Lem. 1.27. □

For earlier, closely related results that anticipated this theorem see [26, Theorem II.1.2 and Proposition II.1.5].

Definition 1.29 *A finitely equivalential deductive system with the G-rule is said to be* regularly algebraizable. □

Theorem 1.30 *Let $\mathcal{S}$ be a 1-deductive system presented by a set* Ax *of axioms and a set* Ru *of proper inference rules. Assume $\mathcal{S}$ is regularly algebraizable with finite equivalence system $E(x, y) = \Big\{\varepsilon_0(x, y), \dots, \varepsilon_{n-1}(x, y)\Big\}$. Let $\top$ be a fixed but arbitrary theorem of $\mathcal{S}$. Then the unique equivalent quasivariety of $\mathcal{S}$ is defined by the identities*

(i) $\varphi \approx \top$*, for each* $\varphi \in \mathrm{Ax}$*;*

together with the following quasi-identities

(ii) $\dfrac{\psi_0 \approx \top, \dots, \psi_{p-1} \approx \top}{\varphi \approx \top}$*, for each inference rule* $\dfrac{\psi_0, \dots, \psi_{p-1}}{\varphi}$ *in* Ru*;*

(iv) $\dfrac{E(x, y) \approx \top}{x \approx y}$.

PROOF. By Thm. 1.28, $x \approx \top$ is a single defining equation for $\mathcal{S}$. The theorem is now an immediate corollary of Thm. 1.13. □

The equivalent quasivarieties of regularly algebraizable deductive systems turn out to be an important class of quasivarieties from the universal algebraic point of view. Loosely speaking there are the quasivarieties, and varieties, with a regular ideal structure (or filter structure in the present terminology). A quasivariety will be said to be *pointed* if it has a distinguished *constant term*, i.e., a term $\varphi(x_0, \dots, x_{n-1})$ with the property that $\varphi(x_0, \dots, x_{n-1}) \approx \varphi(y_0, \dots, y_{n-1})$ is an identity, where the $y_0, \dots, y_{n-1}$ are new variables distinct from $x_0, \dots, x_{n-1}$. Every pointed quasivariety is termwise definitionally equivalent to a quasivariety over a pointed language type in which $\top$ is the designated constant term. In the sequel we assume every pointed quasivariety is over a pointed language type and that $\top$ is the distinguished constant term.

Definition 1.31 *A pointed quasivariety* Q *is said to be* relatively point-regular *if each* Q*-congruence* Θ *on* $\mathbf{Fm}_\Lambda$ *is uniquely determined by its* $\top$*-equivalence class* $\top/\Theta$. □

Let $\boldsymbol{A}$ be a Λ-algebra. If Q is relatively point-regular, then every Q-congruence on $\boldsymbol{A}$ is completely determined by its $\top^{\boldsymbol{A}}$-congruence class (this follows for countably generated $\boldsymbol{A}$ directly from the one-one correspondence between Q-congruences on $\mathbf{Fm}_\Lambda/\Theta$ and Q-congruences on $\mathbf{Fm}_\Lambda$ that include Θ).

If Q is a pointed quasivariety, then $\{\,\top/\Theta : \Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda\,\}$ is an algebraic closed-set system over $\mathbf{Fm}_\Lambda$. Moreover, given any substitution $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$, and any $\Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$, $\sigma^{-1}(\top/\Theta) = \top/\,\sigma^{-1}(\Theta)$, as is easily verified since $\sigma(\top) = \top$. So $\{\,\top/\Theta : \Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda\,\}$ is substitution-invariant. These observations justify the following:

Definition 1.32 *Let* Q *be pointed quasivariety. The* assertional logic *of* Q *is the 1-deductive system*

$$\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q} = \Big\langle \mathbf{Fm}_\Lambda,\ \{\,\top/\Theta : \Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda\,\}\Big\rangle. \qquad \square$$

Corollary 1.33 *Let* Q *be a pointed quasivariety. Then, for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\Lambda$,

$$\Gamma \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}} \varphi \quad \textit{iff} \quad \Gamma \approx \top \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}} \varphi \approx \top,$$

where $\Gamma \approx \top = \{\,\psi \approx \top : \psi \in \Gamma\,\}$.

PROOF. Assume $\Gamma \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}} \varphi$. Then, for every $\Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$, $\Gamma \subseteq \top/\Theta$ implies $\varphi \in \top/\Theta$, i.e.,

$$\varphi \equiv \top \pmod{\mathrm{Cg}_{\mathsf{Q}}(\Gamma \approx \top)}. \tag{26}$$

Thus $\Gamma \approx \top \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}} \varphi \approx \top$.

Assume conversely that $\Gamma \approx \top \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}} \varphi \approx \top$. Then (26) holds, and hence $\Gamma \vdash_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}} \varphi$. $\square$

Thus $\{\,x \approx \top\,\}$ is faithful interpretation of $\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$ in $\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\,\mathsf{Q}$, and this condition clearly characterizes $\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$.

Theorem 1.34 *A 1-deductive system is regularly algebraizable iff it is the assertional logic of a relatively point-regular quasivariety. More precisely, a 1-deductive system* $\mathcal{S}$ *is regularly algebraizable iff* $\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$ *is a relatively point-regular quasivariety and* $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$.

PROOF. $\Rightarrow$: Assume $\mathcal{S}$ is regularly algebraizable with finite equivalence system $E(x,y)$. Let $\mathsf{Q} = \mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$. Then the G-rule guarantees that $\varphi \approx \psi$ is an identity of Q for any pair of theorems φ and ψ of $\mathcal{S}$. Thus Q is a pointed quasivariety with $\top$ representing any theorem. Then by Thm. 1.28 $\{x \approx \top\}$ is a faithful interpretation of $\mathcal{S}$ in $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$. But by by Cor. 1.33 $\{x \approx \top\}$ is also a faithful interpretation of $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ in $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$. So $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. It remains only to prove that Q is relatively point-regular. Consider any $\Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$. Clearly $\top/\Phi \subseteq \top/\Theta$ for every $\Phi \in \mathrm{Co}\,\mathbf{Fm}_\Lambda$ that is compatible with $\top/\Theta$; in particular, $\top/\,\boldsymbol{\Omega}(\top/\Theta) \subseteq \top/\Theta$. On the other hand, Θ is compatible with $\top/\Theta$, and thus $\Theta \subseteq \boldsymbol{\Omega}\,/(\top/\Theta)$, and hence $\top/\Theta \subseteq \top/\,\boldsymbol{\Omega}(\top/\Theta)$. So $\top/\,\boldsymbol{\Omega}(\top/\Theta) = \top/\Theta$. But $\boldsymbol{\Omega}$ is a bijection between $\mathrm{Th}\,\mathcal{S} = \{\,\top/\Theta : \Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda\,\}$ and $\mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$. So $\boldsymbol{\Omega}(\top/\Theta) = \Theta$ for every $\Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$. Hence each $\Theta \in \mathrm{Co}_{\mathsf{Q}}\,\mathbf{Fm}_\Lambda$ is uniquely determined by its $\top$-equivalence class.

$\Leftarrow$: Assume now that $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ for a relatively point-regular quasivariety Q. Let $\Theta = \mathrm{Cg}_{\mathsf{Q}}(\{x \approx y\})$, the Q-congruence on $\mathbf{Fm}_\Lambda$ generated by the 2-formula $\langle x, y\rangle$ where x, y are distinct variables. Since Q is relatively point-regular, $\Theta = \mathrm{Cg}_{\mathsf{Q}}(\top/\Theta \approx \top)$. By the substitution-invariance of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ and the compactness of Θ there is a finite subset $E(x,y)$ of $\top/\Theta$, in two-variables, such that $\Theta = \mathrm{Cg}_{\mathsf{Q}}(E(x,y) \approx \top)$. When reformulated in the form of sequents this equality says that

$$\frac{E(x,y) \approx \top}{x \approx y} \quad \text{and} \quad \frac{x \approx y}{E(x,y) \approx \top} \tag{27}$$

are rules of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$, i.e., quasi-identities of Q. It follows easily from this that $E(x,x) \approx \top$ (more precisely, each member of $E(x,x) \approx \top$) is a theorem of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ and

$$\frac{E(x,y) \approx \top}{E(y,x) \approx \top}, \qquad \frac{E(x,y) \approx \top,\ E(y,z) \approx \top}{E(x,z) \approx \top}$$

$$\frac{E(x_0,y_0) \approx \top, \ldots, E(x_{n-1},y_{n-1}) \approx \top}{E(\lambda x_0,\ldots,x_{n-1},\ \lambda y_0,\ldots,y_{n-1}) \approx \top}, \quad \lambda \in \Lambda,\ (n \text{ the rank of } \lambda$$

are all rules of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$. It follows from Cor. 1.33 that each formula of $E(x,x)$ is a theorem of $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ and and (21)–(23) are rules of $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. Moreover, since $\dfrac{x \approx \top,\ E(x,y) \approx \top}{y \approx \top}$ is obviously a consequence of (27) (together with the properties of equality), it is a rule of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$; hence E-detachment is a rule of $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. Finally, that $\dfrac{x \approx \top,\ y \approx \top}{E(x,y) \approx \top}$ is a rule of $\boldsymbol{\mathcal{S}}^{\text{EQL}}\,\mathsf{Q}$ is another obvious consequence of (27). Thus the G-rule is a rule of $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. So $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ is regularly algebraizable. □

Corollary 1.35 *Let Λ be an arbitrary pointed language type. There is a one-one correspondence between relatively point-regular quasivarieties and regularly algebraizable deductive systems. Every relatively point-regular quasivariety determines a unique regularly algebraizable 1-deductive system, its assertional logic. Conversely, every regularly algebraizable deductive system $\mathcal{S}$ is the assertional logic of a unique relatively point-regular quasivariety, its equivalent* *quasivariety* $\mathsf{Alg\,Mod}^* \mathcal{S}$. *Thus, for each regularly algebraizable deductive system* $\mathcal{S}$ *we have* $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\, \mathsf{Alg\,Mod}^* \mathcal{S}$ *and, conversely, for every relatively point-regular quasivariety* Q *we have* $\mathsf{Q} = \mathsf{Alg\,Mod}^* \boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\, \mathsf{Q}$. □

That the assertional logic of a relatively point-regular quasivariety Q is algebraizable with Q as its equivalent quasivariety was observed in [29] and independently in [30].

In the next theorem we get, as a consequence of the correspondence of Cor. 1.35, a Mal'cev-type characterization of relatively point-regular quasivarieties that generalizes the well-known characterization of point-regular varieties obtained in [31,32]; see also [33]. It was announced in [34]; proofs were given in [29] and independently in [30].

Theorem 1.36 *A pointed quasivariety* Q *is relatively point-regular iff there is a finite system* $E(x, y) = \{\varepsilon_0(x, y), \dots, \varepsilon_{n-1}(x, y)\}$ *of binary terms such that*

$$E(x, x) \approx \top \tag{28}$$

is a set of identities of Q*, and*

$$\frac{E(x, y) \approx \top}{x \approx y} \tag{29}$$

is a quasi-identity of Q.

PROOF. We verified in the proof of Thm. 1.34 that, if Q is relatively point-regular, then any finite set $E(x, y)$ of generators of $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}^{\mathrm{EQL}}\mathsf{Q}}(x \approx y)$ in the two variables x and y satisfies these conditions. Alternatively, assume Q is relatively point-regular, so that $\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$ is regularly algebraizable. Let $E(x, y)$ be a finite equivalence system for $\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$. Then (28) are identities and (29) is a quasi-identity of Q by Thm. 1.30. [7]

Assume now that (28) is a set of identities and (29) a quasi-identity of Q. Let $\varTheta$ be a Q-congruence of $\mathbf{Fm}_\varLambda$. Consider any $\varphi, \psi \in \mathrm{Fm}_\varLambda$. If $\varphi \equiv \psi \pmod{\varTheta}$, then $\varepsilon_i(\varphi, \psi) \equiv \top \pmod{\varTheta}$ by the identities (28). Thus $E(\varphi, \psi) \subseteq \top/\varTheta$. Conversely, if $E(\varphi, \psi) \subseteq \top/\varTheta$, then, by (29), $\varphi \equiv \psi \pmod{\varTheta}$. So, for every Q-congruence $\varTheta$, $\varTheta$ is completely determined by $\top/\varTheta$, i.e., Q is relatively point-regular. □

Another consequence of the correspondence is the following useful characterization of a relatively point-regular quasivariety in terms of its assertional

[7] There is an algebraic proof of this result in which the assertional logic enters only subliminally. It closely parallels the standard proof of the Mal'cev condition for point-regular varieties.

logic.

Theorem 1.37 *Let* Q *be a relatively point-regular quasivariety. For every* Λ*-algebra* $\boldsymbol{A}$ *we have* $\boldsymbol{A} \in \mathsf{Q}$ *iff* $\{\top^{\boldsymbol{A}}\} \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\mathsf{Q}}\,\boldsymbol{A}$ *and* $\boldsymbol{\Omega}_{\boldsymbol{A}}\{\top^{\boldsymbol{A}}\} = \mathrm{Id}_A$.

PROOF. Suppose $\boldsymbol{A} \in \mathsf{Q}$. $\mathsf{Q} = \mathsf{Alg\,Mod}^* \boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$. So there is an $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\mathsf{Q}}\,\boldsymbol{A}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}}\,F = \mathrm{Id}_A$. By the G-rule, $F = \{\top^{\boldsymbol{A}}\}$. Conversely, if $\{\top^{\boldsymbol{A}}\} \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\mathsf{Q}}\,\boldsymbol{A}$ and $\boldsymbol{\Omega}\{\top^{\boldsymbol{A}}\} = \mathrm{Id}_A$, then $\langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle \in \mathsf{Mod}^* \boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q}$. □

1.6 Deduction-detachment theorem

Generalizations of the deduction theorem of the classical and intuitionistic propositional calculi have played an important role in abstract algebraic logic; see [21,35–38].

The deduction theorem can be formulated for k-deductive systems; see [22, Definition 4.1]. We here we consider the notion only for the 1-deductive systems.

Definition 1.38 *Let* $\mathcal{S}$ *be a 1-deductive system. A nonempty set* $\Delta(x, y) = \{\delta_i(x, y) : i \in I\}$ *of binary formulas is called a* deduction-detachment system *for* $\mathcal{S}$ *if the following equivalence holds for all* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_\Lambda$.

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathcal{S}} \delta_i(\varphi, \psi), \textit{ for all } i \in I. \qquad \Box$$

The implication from right to left in above equivalence is obviously equivalent to $\mathcal{S}$ having the Δ-detachment property, i.e., to (MP_Δ) being a rule of $\mathcal{S}$. The implication in the opposite direction is called the *Δ-deduction theorem* or the *deduction property of Δ.*

Since $\mathcal{S}$ is finitary, there is a finite subset J of I such that $\dfrac{x,\ \{\,\delta_i(x, y) : i \in J\,\}}{y}$ is a rule of $\mathcal{S}$. Thus $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ iff $\Gamma \vdash_{\mathcal{S}} \delta_i(\varphi, \psi)$ for all $i \in J$. So every deduction-detachment system for $\mathcal{S}$ includes a finite subsystem that is also a deduction-detachment system. In the sequel we assume all deduction-detachment systems are finite.

A deductive system $\mathcal{S}$ is said to have the *multiterm* (or *generalized*) *deduction-detachment theorem* if it has a finite deduction-detachment system, and the *uniterm deduction-detachment theorem* if it has a deduction-detachment system with a single formula.

In the context of the Fregean deductive systems there is an important difference between multiterm and uniterm deduction-detachment systems; we shall see this in the next section. In the remaining part of this section however we concentrate on the multiterm case.

Corollary 1.39 *Every deductive system $\mathcal{S}$ with a deduction-detachment system is protoalgebraic. Moreover, every deduction-detachment system $\Delta(x, y)$ for $\mathcal{S}$ is also a protoequivalence system for $\mathcal{S}$.*

PROOF. Assume $\Delta(x, y)$ is a deduction-detachment system for $\mathcal{S}$. Since $x \vdash_{\mathcal{S}} x$ holds trivially, we have $\vdash_{\mathcal{S}} \Delta(x, x)$ by the deduction property of $\Delta(x, y)$. The other part of the definition of a protoequivalence system, namely the detachment property, is also part of the definition of a deduction-detachment system. $\square$

For any set X, $X^{(\boldsymbol{\omega})}$ denotes the set of all finite sequences of elements of X.

Let $\Delta(x, y) = \{\delta_0(x, y), \ldots, \delta_{n-1}(x, y)\}$. Let $\bar{\varphi} = \langle \varphi_0, \ldots, \varphi_{m-1} \rangle \in \mathrm{Fm}_{\Lambda}^{(\boldsymbol{\omega})}$ and $\psi \in \mathrm{Fm}_{\Lambda}$. Then $\Delta^*(\bar{\varphi}, \psi) \subseteq \mathrm{Fm}_{\Lambda}$ is defined by recursion on m as follows. If $m = 1$, then

$$\Delta^*(\bar{\varphi}, \psi) = \Delta(\varphi_0, \psi) \ (= \{\, \delta_i(\varphi_0, \psi) : i < n \,\}).$$

Assume $m > 1$. Let $\bar{\varphi} = \langle \varphi_0 \rangle \frown \bar{\varphi}'$, where $\bar{\varphi}' = \langle \varphi_1, \ldots, \varphi_{m-1} \rangle$, and define

$$\begin{aligned} \Delta^*(\bar{\varphi}, \psi) &= \Big\{\, \delta_i(\varphi_0, \xi) : i = 0, \ldots, n-1 \text{ and } \xi \in \Delta^*(\bar{\varphi}', \psi) \,\Big\} \\ &= \bigcup \Big\{ \Delta(\varphi_0, \xi) : \xi \in \Delta^*(\bar{\varphi}', \psi) \,\Big\}. \end{aligned}$$

The results presented in the following two lemmas can be found in [21], and in a more general context in [35].

Lemma 1.40 *Let $\Delta(x, y) = \{\, \delta_i(x, y) : i < n \,\}$ be a finite system of binary 1-formulas. For all $\varphi_0, \ldots, \varphi_{m-1}, \psi \in \mathrm{Fm}_{\Lambda}$, the sequent $\dfrac{\varphi_0, \ldots, \varphi_{m-1}}{\psi}$ is a consequence of the set of formulas $\Delta^*\big(\langle \varphi_0, \ldots, \varphi_{m-1} \rangle,\ \psi\big) \cup \{\varphi_0, \ldots, \varphi_{m-1}\}$ using only* (MP_{Δ}).

PROOF. By induction on m. Let $\bar{\varphi} = \langle \varphi_0, \ldots, \varphi_{m-1} \rangle$.

$m = 1$: $\Delta^*(\bar{\varphi}, \psi) = \Delta(\varphi_0, \psi)$. ψ is directly derivable from φ_0 and $\Delta(\varphi_0, \psi)$ by (MP_{Δ}).

$m > 1$: $\Delta^*(\bar{\varphi}, \psi) = \bigcup\Big\{ \Delta(\varphi_0, \xi) : \xi \in \Delta^*(\bar{\varphi}', \psi) \Big\}$, where $\bar{\varphi}' = \langle \varphi_1, \dots, \varphi_{m-1} \rangle$. From φ_0 and $\Delta^*(\bar{\varphi}, \psi)$ we can derive ξ for each $\xi \in \Delta^*(\bar{\varphi}', \psi)$ by (MP_Δ). Then by the induction hypothesis we can derive ψ from $\varphi_1, \dots, \varphi_{m-1}$ and $\Delta^*(\bar{\varphi}', \psi)$ by (MP_Δ). □

Lemma 1.41 *Let $\mathcal{S}$ be a deductive system with a deduction-detachment system $\Delta(x, y) = \{\, \delta_i(x, y) : i < n \,\}$. Then for each rule $\dfrac{\Gamma,\ \varphi_0, \dots, \varphi_{m-1}}{\psi}$ of $\mathcal{S}$ we have*

$$\Gamma \vdash_{\mathcal{S}} \Delta^*\Big(\langle \varphi_0, \dots, \varphi_{m-1} \rangle,\ \psi \Big).$$

PROOF. By induction on m. Let $\bar{\varphi} = \langle \varphi_0, \dots, \varphi_{m-1} \rangle$.

If $m = 1$, then $\Delta^*(\bar{\varphi}, \psi) = \Delta(\varphi_0, \psi)$ and the entailment $\Gamma \vdash_{\mathcal{S}} \Delta^*(\bar{\varphi}, \psi)$ follows from Γ and the entailment $\varphi_0 \vdash_{\mathcal{S}} \psi$ by the deduction-detachment theorem.

Assume $m > 1$. Let $\bar{\varphi}' = \langle \varphi_1, \dots, \varphi_{m-1} \rangle$ so that $\bar{\varphi} = \langle \varphi_0 \rangle \frown \bar{\varphi}'$.

$$\Delta^*(\bar{\varphi}, \psi) = \bigcup\Big\{ \Delta(\varphi_0, \xi) : \xi \in \Delta^*(\varphi', \psi) \Big\}.$$

By the induction hypothesis, it follows from $\Gamma,\ \varphi_0, \dots, \varphi_{m-1} \vdash_{\mathcal{S}} \psi$ that $\Gamma, \varphi_0 \vdash_{\mathcal{S}} \xi$ for each $\xi \in \Delta^*(\bar{\varphi}', \psi)$. Thus $\Gamma \vdash_{\mathcal{S}} \Delta(\varphi_0, \xi)$, for each $\xi \in \Delta^*(\bar{\varphi}', \psi)$, by the deduction-detachment theorem, i.e., $\Gamma \vdash_{\mathcal{S}} \Delta^*\Big(\langle \varphi_0, \dots, \varphi_{m-1} \rangle,\ \psi \Big)$. □

Theorem 1.42 *Let $\mathcal{S}$ be a deductive system with a deduction-detachment system $\Delta(x, y) = \{\delta_0(x, y), \dots, \delta_{n-1}(x, y)\}$. Then $\mathcal{S}$ has a presentation by axioms and inference rules in which the only proper inference rule is* (MP_Δ).

PROOF. Consider a fixed by arbitrary presentation Ax, Ru of $\mathcal{S}$, where Ax is a set of axioms and Ru is a set of proper inference rules. We construct a new presentation Ax′, Ru′ with the desired properties. Let Ax′ be the union of Ax and $\Delta^*\big(\langle \varphi_0, \dots, \varphi_{m-1} \rangle,\ \psi)$ for every inference rule $\dfrac{\varphi_0, \dots, \varphi_{m-1}}{\psi}$ in Ru. By Lem. 1.41, each member of Ax′ is a theorem of $\mathcal{S}$. Let Ru′ consist only of the inference rule (MP_Δ). Let $\mathcal{S}'$ be the deductive system axiomatized by Ax′, Ru′. Then, since Ax′ is a set of theorems of $\mathcal{S}$ and (MP_Δ) is a rule of $\mathcal{S}$, we have $\vdash_{\mathcal{S}'} \subseteq \vdash_{\mathcal{S}}$. On the other hand, Ax $\subseteq$ Ax′ and each inference rule of Ru is derivable in $\mathcal{S}'$ by Lem. 1.40. So $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$. Hence $\mathcal{S}' = \mathcal{S}$. □

Theorem 1.43 *Let $\mathcal{S}$ be a 1-deductive system that is regularly algebraizable and has the multiterm deduction-detachment theorem. Let*

$$E(x, y) = \Big\{ \varepsilon_0(x, y), \dots, \varepsilon_{n-1}(x, y) \Big\} \quad \textit{and} \quad \Delta(x, y) = \Big\{ \delta_0(x, y), \dots, \delta_{m-1}(x, y) \Big\}$$

be respectively a finite equivalence system and a finite deduction-detachment system for $\mathcal{S}$. Then the unique equivalent quasivariety of $\mathcal{S}$ is defined by the identities

$$E(x,x) \approx \top,$$

the two quasi-identities

$$\frac{x \approx \top,\ \Delta(x,y) \approx \top}{y \approx \top} \quad \textit{and} \quad \frac{E(x,y) \approx \top}{x \approx y},$$

and additional identities of the form $\varphi \approx \top$, where φ ranges over any fixed set of the axioms of a presentation of $\mathcal{S}$ in which modus ponens is the only inference rule.

PROOF. The theorem follows immediately from Thms. 1.30 and 1.42. □

2 Fregean Deductive Systems

In this section all deductive systems are 1-deductive except where specifically indicated to the contrary.

In Section 1 we considered only properties of deductive systems that can be defined by sequents of the form $\dfrac{\psi_0,\dots,\psi_{n-1}}{\varphi}$. These can be thought of as "1st-order" sequents. We consider several important properties of deductive systems, the Fregean property being one of them, that are defined by "2nd-order" sequents. These are sequents of the form

$$\frac{\dfrac{\psi^0_0,\dots,\psi^0_{n_0-1}}{\varphi^0},\dots,\dfrac{\psi^{k-1}_0,\dots,\psi^{k-1}_{n_{k-1}-1}}{\varphi^{k-1}}}{\dfrac{\vartheta_0,\dots,\vartheta_{m-1}}{\xi}}. \tag{30}$$

The natural models of 2nd-order sequents are 2nd-order matrices, which we now define.

Definition 2.1 *Let Λ be an arbitrary language type. A* 2nd-order *(or* generalized*)* matrix (over Λ) *is a pair $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C}_{\mathfrak{A}} \rangle$, where $\boldsymbol{A}$ is a Λ-algebra and $\mathcal{C}_{\mathfrak{A}}$ is an (algebraic) closed-set system over A. $\mathcal{C}_{\mathfrak{A}}$ is called the* designated closed-set system *of $\mathfrak{A}$ and $\boldsymbol{A}$ the* underlying algebra of $\mathfrak{A}$. *2nd-order matrices are called* abstract logics *in [8,5] and* generalized matrices *in [3].* □

Let $\mathfrak{A}, \mathfrak{B}$ be 2nd-order matrices over Λ. $\mathfrak{B}$ is a *submatrix* of $\mathfrak{A}$, in symbols $\mathfrak{B} \le \mathfrak{A}$, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $\mathcal{C}_{\mathfrak{B}} = \{\, B \cap F : F \in \mathcal{C}_{\mathfrak{A}} \,\}$. Note that, for every $X \subseteq B$,

$$\mathrm{Clo}_{\mathcal{C}_{\mathfrak{B}}}(X) = \mathrm{Clo}_{\mathcal{C}_{\mathfrak{A}}}(X) \cap B.$$

$\mathfrak{B}$ is a *finitely generated* submatrix of $\mathfrak{A}$ if $\boldsymbol{B}$ is a finitely generated subalgebra of $\boldsymbol{A}$. A homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is called an *matrix homomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$ if $h^{-1}(\mathcal{C}_{\mathfrak{B}}) \subseteq \mathcal{C}_{\mathfrak{A}}$; equivalently, if

$$h\Big(\mathrm{Clo}_{\mathcal{C}_{\mathfrak{A}}}(X)\Big) \subseteq \mathrm{Clo}_{\mathcal{C}_{\mathfrak{B}}}(h(X)) \quad \text{for all } X \subseteq A.$$

h is a *strict matrix homomorphism*, in symbols $h\colon \mathfrak{A} \to_{\mathrm{s}} \mathfrak{B}$, if $h^{-1}(\mathcal{C}_{\mathfrak{B}}) = \mathcal{C}_{\mathfrak{A}}$. If h is also surjective, then

$$h\Big(\mathrm{Clo}_{\mathcal{C}_{\mathfrak{A}}}(X)\Big) = \mathrm{Clo}_{\mathcal{C}_{\mathfrak{B}}}(h(X)) \quad \text{for all } X \subseteq A. \tag{31}$$

In this case we say that $\mathfrak{B}$ is a *strict homomorphic image* of $\mathfrak{A}$, in symbols $\mathfrak{B} \preccurlyeq \mathfrak{A}$. A strict bijective matrix homomorphism is a *matrix isomorphism*; in this case we write $\mathfrak{A} \cong \mathfrak{B}$.

A matrix $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$ is sometimes called a 1*st-order matrix* to contrast it with 2nd-order matrices. For a similar reason, the designated closed-set system $\mathcal{C}_{\mathfrak{A}}$ of a 2nd-order matrix $\mathfrak{A}$ is occasionally referred to as the *designated* 2*nd-order filter* of $\mathfrak{A}$, and the closed sets of $\mathcal{C}_{\mathfrak{A}}$ 1*st-order filters* of $\mathfrak{A}$. The subscripts $_{\mathfrak{A}}$ on $F_{\mathfrak{A}}$ and $\mathcal{C}_{\mathfrak{A}}$ are often omitted if the matrix $\mathfrak{A}$ is clear from context. It is convenient on occasion to identify the 1st-order matrix $\langle \boldsymbol{A}, F \rangle$ with the special 2nd-order matrix $\langle \boldsymbol{A}, \{F, A\} \rangle$.

Definition 2.2 *The* 2*nd-order sequent (30) is* valid *in a* 2*nd-order matrix* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$, *and the matrix is a* model *of the sequent, if for all assignments* $h\colon \mathbf{Fm}_{\Lambda} \to \boldsymbol{A}$ *of elements of* $\boldsymbol{A}$ *to variables,*

$$h(\varphi^i) \in \mathrm{Clo}_{\mathcal{C}}\{\, h(\psi^i_j) : j < n_i \,\} \text{ for each } i < k$$
$$\text{implies} \quad h(\xi) \in \mathrm{Clo}_{\mathcal{C}}\{\, h(\vartheta_j) : j < m \,\}. \qquad \Box$$

The class of all models of a fixed but arbitrary set of 2nd-order sequents is called a 2*nd-order class* or 2*nd-order property* of 2nd-order matrices.

Theorem 2.3 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* 2*nd-order matrices.*

(i) *If* $\mathfrak{B} \leq \mathfrak{A}$, *then every* 2*nd-order sequent that is valid in* $\mathfrak{A}$ *is also valid in* $\mathfrak{B}$.
(ii) *If* $\mathfrak{B} \preccurlyeq \mathfrak{A}$, *then a* 2*nd-order sequent is valid in* $\mathfrak{B}$ *iff it is valid in* $\mathfrak{A}$.

PROOF. Let $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\mathfrak{B} = \langle \boldsymbol{B}, \mathcal{D} \rangle$.

(i). Assume $\boldsymbol{B} \leq \boldsymbol{A}$ and $\mathcal{D} = \{\, F \cap B : F \in \mathcal{C} \,\}$, and that the sequent (30) is valid in $\mathfrak{A}$. Let $h\colon \mathbf{Fm}_{\Lambda} \to \boldsymbol{B}$ be any evaluation in $\boldsymbol{B}$ such that $h(\varphi^i) \in$

$\mathrm{Clo}_{\mathcal{D}}\{\, h(\psi^i_j) : j < n_i \,\}$ for all $i < k$. Then $h(\varphi^i) \in \mathrm{Clo}_{\mathcal{C}}\{\, h(\psi^i_j) : j < n_i \,\}$ for all $i < k$, and hence $h(\xi) \in \mathrm{Clo}_{\mathcal{C}}\{\, h(\vartheta_j) : j < m \,\}$ since (30) is valid in $\mathfrak{A}$. Thus, since $\mathrm{Clo}_{\mathcal{D}}\{\, h(\vartheta_j) : j < m \,\} = \mathrm{Clo}_{\mathcal{C}}\{\, h(\vartheta_j) : j < m \,\} \cap B$ and $h(\xi) \in B$, we have $h(\xi) \in \mathrm{Clo}_{\mathcal{D}}\{\, h(\vartheta_j) : j < m \,\}$. So (30) is valid in $\mathfrak{B}$.

(ii). Let $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ be a surjective algebra homomorphism such that $h^{-1}(\mathcal{D}) = \mathcal{C}$. We prove that the 2nd-order sequent (30) is valid in $\langle \boldsymbol{B}, \mathcal{D}\rangle$ iff it is valid in $\langle \boldsymbol{A}, \mathcal{C}\rangle$. We note first of all that, for every $\Gamma \subseteq \mathrm{Fm}_\Lambda$ and every evaluation $g\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$,

$$h^{-1}\Big(\mathrm{Clo}_{\mathcal{D}}(h \circ g)(\Gamma)\Big) = \mathrm{Clo}_{\mathcal{C}}\, g(\Gamma). \tag{32}$$

By (31) $\mathrm{Clo}_{\mathcal{D}}(h \circ g)(\Gamma) = h\Big(\mathrm{Clo}_{\mathcal{C}}\, g(\Gamma)\Big)$. Thus

$$h^{-1}\Big(\mathrm{Clo}_{\mathcal{D}}(h \circ g)(\Gamma)\Big) = h^{-1}h\Big(\mathrm{Clo}_{\mathcal{C}}\, g(\Gamma)\Big) = \mathrm{Clo}_{\mathcal{C}}\, g(\Gamma).$$

The last equality holds since h is surjective.

Suppose that the sequent (30) is valid in $\langle \boldsymbol{B}, \mathcal{D}\rangle$ and $g\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ is an evaluation in $\boldsymbol{A}$ such that $g(\varphi^i) \in \mathrm{Clo}_{\mathcal{C}}\{g(\psi^i_j) : j < n_i\,\}$ for all $i < k$. Then by (32), $g(\varphi^i) \in h^{-1}\Big(\mathrm{Clo}_{\mathcal{D}}\{\, (h \circ g)(\psi^i_j) : j < n_i \,\}$ for every $i < k$. Thus, since the sequent (30) is valid in $\langle \boldsymbol{B}, \mathcal{D}\rangle$, $(h \circ g)(\xi) \in \mathrm{Clo}_{\mathcal{D}}\{\, (h \circ g)(\vartheta_j) : j < m \,\}$, and hence, by (32) again,

$$g(\xi) \in h^{-1}\Big(\mathrm{Clo}_{\mathcal{D}}\{\, (h \circ g)(\vartheta_j) : j < m \,\}\Big) = \mathrm{Clo}_{\mathcal{C}}\{\, g(\vartheta_j) : j < m \,\}\Big).$$

Thus (30) is also valid in $\langle \boldsymbol{A}, \mathcal{C}\rangle$. The proof of the reverse implication is similar and is omitted. □

The following rather technical lemma of the same kind will also be useful in the sequel.

Lemma 2.4 *A 2nd-order sequent is valid in a 2nd-order matrix $\mathfrak{A}$ if, for every finitely generated $\mathfrak{B} \leq \mathfrak{A}$, there exists a countably generated $\mathfrak{B}'$ such that $\mathfrak{B} \leq \mathfrak{B}' \leq \mathfrak{A}$ and the sequent is valid in $\mathfrak{B}'$.*

PROOF. Let $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ be a 2nd-order matrix, and assume that, for every finitely generated $\mathfrak{B} \leq \mathfrak{A}$, there exists a countably generated $\mathfrak{B}'$ such that $\mathfrak{B} \leq \mathfrak{B}' \leq \mathfrak{A}$ and the sequent (30) is valid in $\mathfrak{B}'$. Let $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ be an evaluation in $\boldsymbol{A}$ such that $h(\varphi^i) \in \mathrm{Clo}_{\mathcal{C}}\{\, h(\psi^i_j) : j < n_i \,\}$ for all $i < k$. Let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{A}$ generated by the h-image of the set of all variables that occur in the sequent (30). Let $\mathfrak{B} = \langle \boldsymbol{B}, \mathcal{D}\rangle$ where $\mathcal{D} = \{\, B \cap F : F \in \mathcal{C} \,\}$. $\mathfrak{B}$ is a finitely generated submatrix of $\mathfrak{A}$, and hence by hypothesis there exists a countably generated $\mathfrak{B}' = \langle \boldsymbol{B}', \mathcal{D}'\rangle$ such that $\mathfrak{B} \leq \mathfrak{B}' \leq \mathfrak{A}$ and (30) is valid in $\mathfrak{B}'$. Then $h(\varphi^i) \in \mathrm{Clo}_{\mathcal{D}'}\{\, h(\psi^i_j) : j < n_i \,\}$ for all $i < k$. Thus by assumption

$h(\xi) \in \mathrm{Clo}_{\mathcal{D}'}\{\, h(\vartheta_j) : j < m \,\} \subseteq \mathrm{Clo}_{\mathcal{C}}\{\, h(\vartheta_j) : j < m \,\}$. So the sequent (30) is also valid in $\mathfrak{A}$. $\square$

A 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C}_{\mathfrak{A}} \rangle$ is *endomorphism-invariant* if $h^{-1}(\mathcal{C}_{\mathfrak{A}}) \subseteq \mathcal{C}_{\mathfrak{A}}$ for every endomorphism $h\colon \boldsymbol{A} \to \boldsymbol{A}$. Recall that by definition a 1-deductive system is an endomorphism-invariant 2nd-order matrix $\mathcal{S} = \langle \mathbf{Fm}_{\Lambda}, \mathrm{Th}\,\mathcal{S} \rangle$ over the formula algebra (Def. 1.1). Thus we can speak of a 2nd-order sequent being valid in a deductive system, and the deductive system being a model of the sequent. We also say in this event that the sequent is a 2nd-order rule of $\mathcal{S}$. In more detail, the sequent (30) is a *2nd-order rule* of a 1-deductive system $\mathcal{S}$ if, for all substitutions σ,

$$\sigma(\psi^i_0), \ldots, \sigma(\psi^i_{n_i-1}) \vdash_{\mathcal{S}} \sigma(\varphi^i) \text{ for all } i < k \text{ implies } \sigma(\vartheta_0), \ldots, \sigma(\vartheta_{m-1}) \vdash_{\mathcal{S}} \sigma(\xi).$$

A class consisting of all deductive systems that are models of some fixed but arbitrary set of 2nd-order sequents is called a *2nd-order class* or *2nd-order property* of deductive systems. 2nd-order sequents can be thought of as Gentzen-style rules, and a 2nd-order class of deductive systems as the class of all deductive systems with a fixed but arbitrary Gentzen-style formalization. [8]

In Gentzen-style formulations a 1st-order sequent $\dfrac{\psi_0, \ldots, \psi_{n-1}}{\varphi}$ if often called simply a *sequent* and denoted fby $\psi_0, \ldots, \psi_{n-1} \Rightarrow \psi$, or by some similar normally linear denotation. Thus in the context of a Gentzen formalization the 2nd-order sequent (30) would be written

$$\frac{\psi^0_0, \ldots, \psi^0_{n_0-1} \Rightarrow \varphi^0; \ldots ; \psi^{k-1}_0, \ldots, \psi^{k-1}_{n_{k-1}-1} \Rightarrow \varphi^{k-1}}{\vartheta_0, \ldots, \vartheta_{m-1} \Rightarrow \xi}.$$

2nd-order matrices were first explicitly used in abstract algebraic logic by Font and Jansana in [15,5]. The theory of 2nd-order classes of deductive systems is systematically developed in [39]. In the present paper we consider only four such classes. The class of all deductive systems over Λ that have the multiterm deduction-detachment theorem with a fixed but arbitrary deduction-detachment system $\Delta(x, y) = \{\, \delta_0(x, y), \ldots, \delta_{n-1}(x, y) \,\}$ of Λ-formulas is a

[8] There is however a subtle but important distinction between these two notions. A Gentzen formalization is normally used to define a single deductive system rather than a whole class of systems, namely the largest model of the Gentzen rules in the sense that it has the largest number of theories.

2nd-order class. It is defined by the set of 2nd-order sequents

$$\frac{\dfrac{\dfrac{\bar z, x}{y}}{\bar z}}{\delta_i(x,y)}, \quad \text{for all } i < n,$$

were $\bar z$ ranges over all finite sequence of variables, together of course with the 1st-order detachment rule for $\Delta(x, y)$. In view of this it makes sense to speak of an arbitrary 2nd-order matrix $\mathfrak{A}$ having the *multiterm deduction-detachment theorem* with deduction-detachment system $\Delta(x, y)$.

The other 2nd-order classes we consider are the self-extensional and the Fregean deductive systems (see Def. 2.16 below), and the weakly normal modal logics (see Sec. 2.5 below).

Let $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ be a 1st-order matrix. A binary relation R on A is *definable over* $\mathfrak{A}$ if there exists a set $\Phi(x, y) = \{\, \varphi_i(x, y) : i \in I \,\}$ of binary 1-formulas such that

$$R = \{\, \langle a, b\rangle \in A^2 : \varphi_i^{\boldsymbol{A}}(a, b) \in F \text{ for all } i \in I \,\}.$$

Lemma 2.5 ([8]) *Let* $\boldsymbol{A}$ *be a* Λ*-algebra, and let* $F \subseteq A$*. Let* $R \subseteq A^2$ *be definable over the matrix* $\langle \boldsymbol{A}, F\rangle$*. If* R *is reflexive, then* $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq R$*.*

PROOF. Assume R is definable by $\Phi(x, y) = \{\, \varphi_i(x, y) : i \in I \,\}$. Suppose $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$. Then, since $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is a congruence relation, for all $i \in I$, $\varphi_i^{\boldsymbol{A}}(a, a) \equiv \varphi_i^{\boldsymbol{A}}(a, b) \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$. But $\varphi_i^{\boldsymbol{A}}(a, a) \in F$ because R is reflexive. So $\varphi_i^{\boldsymbol{A}}(a, b) \in F$ since $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is compatible with F. Since this is true for all $i \in I$, $\langle a, b\rangle \in R$. □

The definition of Leibniz congruence can be extended from 1st- to 2nd-order filters; at the same time we define the closely related notion of the Frege relation of a 2nd-order filter.

Definition 2.6 *Let* A *be a nonempty set and let* $\mathcal{C}$ *be an arbitrary closed-set system over* A*.*

(i) $\boldsymbol{\Lambda}\,\mathcal{C} := \left\{ \langle a, b\rangle \in A^2 : \mathrm{Clo}_{\mathcal{C}}\{a\} = \mathrm{Clo}_{\mathcal{C}}\{b\} \right\}$. $\boldsymbol{\Lambda}\,\mathcal{C}$ *is called the* Frege relation of $\mathcal{C}$*.*
(ii) *Assume now that* A *is the universe of a* Λ*-algebra* $\boldsymbol{A}$*. Let* $\boldsymbol{\Omega}_{\boldsymbol{A}}\mathcal{C}$ *be the set of all* $\langle a, b\rangle \in A^2$ *such that*

$$\mathrm{Clo}_{\mathcal{C}}\{\varphi^{\boldsymbol{A}}(a, \bar c)\} = \mathrm{Clo}_{\mathcal{C}}\{\varphi^{\boldsymbol{A}}(b, \bar c)\}, \text{ for all } \varphi(x, \bar z) \in \mathrm{Fm}_\Lambda \text{ and all } \bar c \in A^{|\bar z|}$$

$\boldsymbol{\Omega_A}\,\mathcal{C}$ *is called the (*2nd-order*)* Leibniz congruence*, or the* Tarski congruence*, of* $\mathcal{C}$*.* $\square$

The Frege relation and the 2nd-order Leibniz congruence were introduced in [15]. Their theory is developed and applied to abstract algebraic logic in [5] (where the term Tarski congruence and the notation $\widetilde{\boldsymbol{\Omega}}\,\mathcal{C}$ are used).

It is easy to see that $\boldsymbol{\Lambda}\,\mathcal{C}$ is an equivalence relation on A; in fact it is the largest equivalence relation on A that is compatible with $\mathcal{C}$ in the sense that is it compatible with each $F \in \mathcal{C}$. Although not as obvious, it is also not difficult to see that $\boldsymbol{\Omega_A}\,\mathcal{C}$ is indeed a congruence relation on $\boldsymbol{A}$, and is in fact the largest congruence compatible with $\mathcal{C}$. It follows that $\boldsymbol{\Omega_A}\,\mathcal{C}$ can also be characterized as the largest congruence that is included in $\boldsymbol{\Lambda}\,\mathcal{C}$.

As mappings from closed-set systems over A into the sets of equivalence and congruence relations on $\boldsymbol{A}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega_A}$ are called respectively the *Frege operator* and the *2nd-order Leibniz*, or *Tarski*, *operator*. The ordinary Leibniz congruence $\boldsymbol{\Omega_A}\,F$ associated with a subset F of $\boldsymbol{A}$ can be obtained by applying the 2nd-order operator $\boldsymbol{\Omega_A}$ to the discrete closed-set system $\{F, A\}$, i.e., $\boldsymbol{\Omega_A}\,F = \boldsymbol{\Omega_A}\{F, A\}$. For contrast we shall refer to the ordinary Leibniz operator, which ranges over subsets of A, as the *1st-order Leibniz operator*.

We can extend the domain of the Frege operator to include subsets of A by defining $\boldsymbol{\Lambda}\,F := \boldsymbol{\Lambda}\{F, A\}$ for each $F \subseteq A$. This gives us 1st- and 2nd-order notions of the Frege operator in analogy to the 1st- and 2nd-order Leibniz operator. Note that $\boldsymbol{\Lambda}\,F$ is simply the equivalence relation associated with the two-element partition $\{F, A \setminus F\}$, so the 1st-order Frege operator is little more than a curiosity, but it is useful for the symmetry between 1st- and 2nd-order notions it provides. For example, for any algebra $\boldsymbol{A}$ and any $F \subseteq A$, we have that a congruence is compatible with F iff it is included in $\boldsymbol{\Lambda}\,F$. Thus $\boldsymbol{\Omega_A}\,F$ is the largest congruence included in $\boldsymbol{\Lambda}\,F$.

For any closed-set system $\mathcal{C}$ over a set A or over the universe A of an algebra $\boldsymbol{A}$ we have,

$$\boldsymbol{\Lambda}\,\mathcal{C} = \textstyle\bigcap_{F \in \mathcal{C}} \boldsymbol{\Lambda}\,F \quad \text{and} \quad \boldsymbol{\Omega_A}\,\mathcal{C} = \textstyle\bigcap_{F \in \mathcal{C}} \boldsymbol{\Omega_A}\,F. \tag{33}$$

The 2nd-order Frege and Leibniz operators on closed-set systems are both antimonotonic in the sense that, if $\mathcal{C}$ and $\mathcal{D}$ are two closed-set systems over A such that $\mathcal{C} \subseteq \mathcal{D}$ (i.e., $F \in \mathcal{C}$ implies $F \in \mathcal{D}$), then $\boldsymbol{\Lambda}\,\mathcal{C} \supseteq \boldsymbol{\Lambda}\,\mathcal{D}$ and $\boldsymbol{\Omega_A}\,\mathcal{C} \supseteq \boldsymbol{\Omega_A}\,\mathcal{D}$. This is an immediate consequence of the characterizations (33). However, neither the 1st-order Frege nor the 1st-order Leibniz operators are either monotonic or antimonotonic on arbitrary subsets of A. As is well-known, the 1st-order Leibniz operator is monotonic in special circumstances, for example, when restricted to the filters of a protoalgebraic deductive system.

In analogy with its 1st-order counterpart (Lem. 1.11) the 2nd-order Leibniz operator commutes with inverse surjective homomorphisms, and the Frege operator has the same property.

Lemma 2.7 *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* Λ*-algebras and* $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ *a surjective homomorphism. Then for any closed-set system* $\mathcal{C}$ *over* $\boldsymbol{B}$*,*

(i) $\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(\mathcal{C})\big) = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}}\,\mathcal{C})$*, and*
(ii) $\boldsymbol{\Lambda}\big(h^{-1}(\mathcal{C})\big) = h^{-1}(\boldsymbol{\Lambda}\,\mathcal{C})$*.*

PROOF. (i):

$$\begin{aligned}
\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(\mathcal{C})\big) &= \bigcap\Big\{\, \boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big) : F \in \mathcal{C} \,\Big\}, && \text{by (33)}\\
&= \bigcap\Big\{\, h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}}\,F) : F \in \mathcal{C} \,\Big\}, && \text{by Lem. 1.11}\\
&= h^{-1}\Big(\bigcap\Big\{\, \boldsymbol{\Omega}_{\boldsymbol{B}}\,F : F \in \mathcal{C} \,\Big\}\Big)\\
&= h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}}\,\mathcal{C}).
\end{aligned}$$

The proof of (ii) is similar. □

In analogy to 1st-order matrices, a 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is *reduced* if $\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C} = \mathrm{Id}_A$. For an arbitrary $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$, $\mathfrak{A}^*$ is the quotient matrix $\langle \boldsymbol{A}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C},\ \mathcal{C}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C} \rangle$, where $\mathcal{C}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C} = \{\, F/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C} : F \in \mathcal{C} \,\}$. By Lem. 2.7(i), $\mathfrak{A}^*$ is always reduced; it is called the *reduction* of $\mathfrak{A}$.

Definition 2.8 *Two* 2*nd-order matrices* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are said to be* reduction-isomorphic *if* $\mathfrak{A}^* \cong \mathfrak{B}^*$*.* □

The natural map from $\boldsymbol{A}$ to $\boldsymbol{A}/\,\boldsymbol{\Omega}_{\boldsymbol{A}}\,\mathcal{C}$ is a strict, surjective matrix homomorphism between $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ and its reduction $\mathfrak{A}^*$, and $\mathfrak{A}$ and $\mathfrak{B}$ are reduction-isomorphic iff there exists a 2nd-order matrix $\mathfrak{C}$ such that $\mathfrak{C} \preccurlyeq \mathfrak{A}$ and $\mathfrak{C} \preccurlyeq \mathfrak{B}$, i.e., there exist strict, surjective matrix homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{C}$.

Corollary 2.9 *The validity of each* 2*nd-order sequent is preserved under reduction isomorphism. I.e., if a* 2*nd-order matrix* $\mathfrak{A}$ *is a model of a* 2*nd-order sequent, then so is any* 2*nd-order matrix* $\mathfrak{B}$ *such that* $\mathfrak{B}^* \cong \mathfrak{A}^*$*.*

PROOF. Assume $\mathfrak{A}$ and $\mathfrak{B}$ are reduction-isomorphic and let $\mathfrak{C}$ be a 2nd-order matrix such that $\mathfrak{C} \preccurlyeq \mathfrak{A}$ and $\mathfrak{C} \preccurlyeq \mathfrak{B}$. Then by Thm. 2.3(ii), a 2nd-order sequent is valid in $\mathfrak{A}$ iff it is valid in $\mathfrak{C}$ iff it is valid in $\mathfrak{B}$. □

Protoalgebraicity, a concept that was defined previously for 1-deductive systems, a special kind of 2nd-order matrix, can be applied to any 2nd-order matrix.

Definition 2.10 *A 2nd-order matrix* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *is said to be* protoalgebraic *if the* 1*st-order Leibniz operator is monotonic on* $\mathcal{C}$*, i.e., for all* $F, G \in \mathcal{C}$*,* $F \subseteq G$ *implies* $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} G$*.* □

This notion is consistent with and extends the notion of a protoalgebraic deductive system; in fact, a 1-deductive system $\mathcal{S}$ is protoalgebraic in the sense of Def. 1.14 if as a 2nd-order matrix it is protoalgebraic in the sense of Def. 2.10.

We localize the 2nd-order Frege and Leibniz operators to individual closed sets F in a closed-set system $\mathcal{C}$ by applying them to the principal filter $[\,F\,)_{\mathcal{C}}$ of $\mathcal{C}$ generated by F. These will turn out to be important notions and we introduce corresponding notation and terminology.

Definition 2.11 *Let* $\mathcal{C}$ *be a closed-set system over a nonempty set* A *or over the universe* A *of a* Λ*-algebra* $\boldsymbol{A}$*. If* $F \in \mathcal{C}$ *we define*

$$\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F := \boldsymbol{\Lambda}[\,F\,)_{\mathcal{C}} \quad \textit{and} \quad \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}}\, F := \boldsymbol{\Omega}_{\boldsymbol{A}}[\,F\,)_{\mathcal{C}}.$$

$\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$ *and* $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}} F$ *are respectively called the* local 2nd-order Frege relation and Leibniz congruence of F over $\mathcal{C}$*.* $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}}\, F$ *is also called the* Suszko congruence of F over $\mathcal{C}$*.* □

Note that

$$\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F = \bigcap\{\, \boldsymbol{\Lambda}\, G : F \subseteq G \in \mathcal{C}\,\} \quad \text{and} \quad \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}}\, F = \bigcap\{\, \boldsymbol{\Omega}_{\boldsymbol{A}}\, G : F \subseteq G \in \mathcal{C}\,\}.$$

Note also that $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}$ and $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}}$, as operators on the closed sets of $\mathcal{C}$, are always monotonic. It is useful to keep in mind that, while the 1st-order Leibniz congruence of F is an absolute notion in the sense that it depends only on F and the underlying algebra $\boldsymbol{A}$, the Suszko congruence of F depends also on the closed-set system of which F is a member.

As special cases of Lem. 2.7 we have, for every surjective homomorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$, every closed set system $\mathcal{C}$ over $\boldsymbol{B}$, and every $F \in \mathcal{C}$,

$$\widetilde{\boldsymbol{\Lambda}}_{h^{-1}(\mathcal{C})}\, h^{-1}(F) = h^{-1}\Big(\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F\Big), \quad \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},h^{-1}(\mathcal{C})}\, h^{-1}(F) = h^{-1}\Big(\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{B},\mathcal{C}}\, F\Big). \tag{34}$$

The Suszko operator can also be thought of as relativized to a 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ by setting $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\, F = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A},\mathcal{C}}\, F$, for every $F \in \mathcal{C}$.

Lemma 2.12 *Let* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *be a 2nd-order matrix. The following conditions are equivalent.*

(i) $\mathfrak{A}$ *is protoalgebraic;*
(ii) $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F = \boldsymbol{\Omega}_{\boldsymbol{A}} F$, *for every* $F \in \mathcal{C}$*;*
(iii) $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$, *for every* $F \in \mathcal{C}$.

PROOF. (i) $\Leftrightarrow$ (ii): If $\mathfrak{A}$ is protoalgebraic, then $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} G$ for every $G \in [\, F\,)_{\mathcal{C}}$. So $\boldsymbol{\Omega}_{\boldsymbol{A}} F = \bigcap \{\, \boldsymbol{\Omega}_{\boldsymbol{A}} G : F \subseteq G \in \mathcal{C}\,\} = \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F$.

Conversely, if $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F = \boldsymbol{\Omega}_{\boldsymbol{A}} F$, then, for each G such that $F \subseteq G \in \mathcal{C}$, $\boldsymbol{\Omega}_{\boldsymbol{A}} F = \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} G$.

(ii)$\Leftrightarrow$(iii): Since $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F$ is the largest congruence included in $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$ and $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is a congruence that includes $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} F$, clearly $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F = \boldsymbol{\Omega}_{\boldsymbol{A}} F$ iff $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$. $\square$

In general, for every 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ and every $F \in \mathcal{C}$, we have $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F$ and $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F \subseteq \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$, while $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ and $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$ are incomparable.

2.1 Self-extensional and Fregean 2nd-order matrices

We define the key notions of the paper.

Definition 2.13 *Let* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *be a 2nd-order matrix.*

(i) $\mathfrak{A}$ *is* self-extensional[9] *if the Frege relation of* $\mathcal{C}$ *is a congruence, i.e., if*

$$\boldsymbol{\Omega}_{\boldsymbol{A}} \mathcal{C} = \boldsymbol{\Lambda}\, \mathcal{C}.$$

(ii) $\mathfrak{A}$ *is* Fregean *if every principal-filter matrix* $\langle \boldsymbol{A}, [\, F\,)_{\mathcal{C}} \rangle$ *of* $\mathfrak{A}$ *is self-extensional, i.e., if*

$$\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F, \quad \textit{for every } F \in \mathcal{C}. \quad \square$$

Note that $\mathfrak{A}$ is self-extensional iff $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} F_0 = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F_0$, where F_0 is the smallest $\mathcal{C}$-filter.

Self-extensional and Fregean 2nd-order matrices form 2nd-order classes. In fact, it is easy to see that $\mathfrak{A}$ is self-extensional iff the following 2nd-order

[9] Self-extensional 2nd-order matrices are referred to as abstract logics with the *congruence property* in [5].

sequent is valid in $\mathfrak{A}$ for every $\lambda \in \Lambda$, where n is the rank of λ,

$$\frac{\dfrac{x_0}{y_0}, \dfrac{y_0}{x_0}, \ldots, \dfrac{x_{n-1}}{y_{n-1}}, \dfrac{y_{n-1}}{x_{n-1}}}{\dfrac{\lambda x_0 \ldots x_{n-1}}{\lambda y_0 \ldots y_{n-1}}}.$$

The class of Fregean matrices is also a 2nd-order class but to show this requires more work. It depends on the following technical lemma.

A filter F of an algebraic closed-set system $\mathcal{C}$ over a set A is *finitely generated* if $F = \mathrm{Clo}_{\mathcal{C}}\, X$ for some $X \subseteq_{\omega} A$.

Lemma 2.14 *A 2nd-order matrix* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *is Fregean iff* $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$ *for every finitely generated* $F \in \mathcal{C}$.

PROOF. Suppose $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F \subsetneq \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$ for some $F \in \mathcal{C}$. Let $\langle a, b \rangle \in \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F$ such that $\langle a, b \rangle \notin \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F$. Then, for every $G \in [\, F\,)_{\mathcal{C}}$, $a \in G$ iff $b \in G$, i.e.,

$$\mathrm{Clo}_{\mathcal{C}}(F \cup \{a\}) = \mathrm{Clo}_{\mathcal{C}}(F \cup \{b\}).$$

Let X be a finite subset of F such that $\mathrm{Clo}_{\mathcal{C}}(X \cup \{a\}) = \mathrm{Clo}_{\mathcal{C}}(X \cup \{b\})$ (such an X exists because $\mathcal{C}$ is algebraic), and let $F' = \mathrm{Clo}_{\mathcal{C}}(X)$. Then $\langle a, b \rangle \in \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}} F'$, but $\langle a, b \rangle \notin \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F'$ since $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F' \subseteq \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}} F$. □

It follows easily from this lemma that a 2nd-order matrix $\mathfrak{A}$ is Fregean iff

$$\frac{\dfrac{\bar{z}, x_0}{y_0}, \dfrac{\bar{z}, y_0}{x_0}, \ldots, \dfrac{\bar{z}, x_{n-1}}{y_{n-1}}, \dfrac{\bar{z}, y_{n-1}}{x_{n-1}}}{\dfrac{\bar{z}, \lambda x_0 \ldots x_{n-1}}{\lambda y_0 \ldots y_{n-1}}}$$

is valid in $\mathfrak{A}$ for every $\lambda \in \Lambda$ and every finite sequence of variables $\bar{z} = z_0, \ldots, z_{m-1}$.

By the *accumulative form* of the 2nd-order sequent (30) we mean the family of 2nd-order sequents

$$\frac{\dfrac{\bar{z}, \psi^0_0, \ldots, \psi^0_{n_0-1}}{\varphi^0}, \ldots, \dfrac{\bar{z}, \psi^{k-1}_0, \ldots, \psi^{k-1}_{n_{k-1}-1}}{\varphi^{k-1}}}{\dfrac{\bar{z}, \vartheta_0, \ldots, \vartheta_{m-1}}{\xi}},$$

where $\bar{z}$ ranges over all finite sequences of variables. It is easy to see that a 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is a model of the accumulative form of a 2nd-order sequent iff $\langle \boldsymbol{A}, [\,F\,)_{\mathcal{C}}\rangle$ is a model of the sequent itself for every finitely generated $\mathcal{C}$-filter F, and hence, by the obvious generalization of Lem. 2.14, iff $\langle \boldsymbol{A}, [\,F\,)_{\mathcal{C}}\rangle$ is a model of the sequent for every $F \in \mathcal{C}$, without restriction.

A 2nd-order class of 2nd-order matrices is said to be *accumulative* if it is defined by the accumulative forms of a set of 2nd-order sequents. The class of Fregean 2nd-order matrices is accumulative. The 2nd-order matrices that have the deduction-detachment theorem (with a fixed but arbitrary deduction-detachment system $\Delta(x, y)$) form another accumulative class.

We will be especially interested in those 2nd-order matrices $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ that are both protoalgebraic and Fregean. In this case all three of the relations $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F$, $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\, F$, and $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$ coincide for every $F \in \mathcal{C}$, as we now show.

Theorem 2.15 *Let $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ be a 2nd-order matrix.*

(i) *If $\mathfrak{A}$ is protoalgebraic, then $\mathfrak{A}$ is self-extensional iff $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F_0 = \boldsymbol{\Lambda}\, \mathcal{C}$, where F_0 is the smallest $\mathcal{C}$-filter.*
(ii) *$\mathfrak{A}$ is protoalgebraic and Fregean iff $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$ for every $F \in \mathcal{C}$.*

PROOF. (i) is immediate.

(ii). If $\mathfrak{A}$ is protoalgebraic and Fregean, then, for every $F \in \mathcal{C}$, $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F = \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\, F$ (by Lem. 2.12) $= \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$ (by Def. 2.13). Conversely, if $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$, then $\boldsymbol{\Omega}_{\boldsymbol{A}}\, F = \widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\, F$ because $\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\, F$ is the largest congruence included in $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\, F$. Thus $\mathfrak{A}$ is both protoalgebraic and Fregean. □

We now specialize to deductive systems. Let $\mathcal{S}$ be a 1-deductive system. Recall that $\mathcal{S}$ is by definition the 2nd-order formula matrix $\langle \mathbf{Fm}_{\Lambda}, \operatorname{Th} \mathcal{S}\rangle$. We thus have

$$\boldsymbol{\Lambda}\, \operatorname{Th} \mathcal{S} = \{\, \langle \varphi, \psi\rangle \in \mathrm{Fm}^2_{\Lambda} : \varphi \dashv\vdash_{\mathcal{S}} \psi \,\}.$$

More generally, for any $\mathcal{S}$-theory T,

$$\widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}}\, T = \{\, \langle \varphi, \psi\rangle \in \mathrm{Fm}^2_{\Lambda} : T, \varphi \dashv\vdash_{\mathcal{S}} T, \psi \,\}.$$

For obvious reasons the Frege relation $\boldsymbol{\Lambda}\, \operatorname{Th} \mathcal{S}$ is also called the *symmetric consequence relation*, and $\widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}}\, T$ the *symmetric localized consequence relation* (at T). Recall that $\operatorname{Thm} \mathcal{S}$ is the set of theorems of $\mathcal{S}$, the smallest $\mathcal{S}$-theory, and that we write $\boldsymbol{\Omega}$ for $\boldsymbol{\Omega}_{\mathbf{Fm}_{\Lambda}}$.

Definition 2.16 *Let $\mathcal{S}$ be a 1-deductive system.*

(i) $\mathcal{S}$ *is* self-extensional *if* $\mathcal{S} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S} \rangle$ *is self-extensional, i.e., if*

$$\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{S} = \boldsymbol{\Lambda}\,\mathrm{Th}\,\mathcal{S}.$$

(ii) $\mathcal{S}$ *is* Fregean *if* $\langle \mathbf{Fm}_\Lambda, T \rangle$ *is self-extensional for every* $\mathcal{S}$*-theory* T*, i.e., if*

$$\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}\, T = \widetilde{\boldsymbol{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\, T, \quad \textit{for every } T \in \mathrm{Th}\,\mathcal{S}. \qquad \square$$

The notion of a self-extensional deductive system was first considered by Wójcicki; see [3]. Fregean logics first explicitly occurred in the literature in [15], although protoalgebraic Fregean logics were independently considered by the present authors about the same time in unpublished notes; see [40]. A closely related algebraic notion can be found in [16]. For additional references see [5], pages 64 and 66.

A characteristic property of Fregean deductive systems is the coalescence of the notions of protoalgebraicity and equivalentialness.

Lemma 2.17 *Let* $\mathcal{S}$ *be a protoalgebraic 1-deductive system with at least one theorem, and let* $\Delta(x, y)$ *be a (necessarily nonempty) protoequivalence system for* $\mathcal{S}$*. If* $\mathcal{S}$ *is Fregean, then* $\Delta(x, y) \cup \Delta(y, x)$ *is an equivalence system for* $\mathcal{S}$*.*

PROOF. Assume $\mathcal{S}$ is protoalgebraic and Fregean. Let

$$\Delta(x, y) = \{\, \delta_0(x, y), \ldots, \delta_{n-1}(x, y) \,\}$$

be a protoequivalence system for $\mathcal{S}$, and set $E(x, y) = \Delta(x, y) \cup \Delta(y, x)$. To show that $E(x, y)$ is an equivalence system for $\mathcal{S}$, we first show that $E(x, y)$ defines a subrelation of the Frege relation. Let $T \in \mathrm{Th}\,\mathcal{S}$ and let $R = \{\, \langle \varphi, \psi \rangle \in \mathrm{Fm}_\Lambda^2 : E(\varphi, \psi) \subseteq T \,\}$. Suppose $\langle \varphi, \psi \rangle \in R$. Then since $\varphi, \Delta(\varphi, \psi) \vdash_{\mathcal{S}} \psi$, by the detachment rule for Δ, and $\Delta(\varphi, \psi) \subseteq E(\varphi, \psi) \subseteq T$, we have $\varphi, T \vdash_{\mathcal{S}} \psi$. Similarly, from $\psi, \Delta(\psi, \varphi) \vdash_{\mathcal{S}} \varphi$ and $\Delta(\psi, \varphi) \subseteq E(\varphi, \psi) \subseteq T$, we get $\psi, T \vdash_{\mathcal{S}} \varphi$. Hence $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\, T$. Thus $R \subseteq \widetilde{\boldsymbol{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\, T = \boldsymbol{\Omega}\, T$, since $\mathcal{S}$ is protoalgebraic and Fregean. But R is a reflexive relation, because of the reflexivity condition for protoequivalence systems (14), and it is definable by $E(x, y)$ over the matrix $\langle \mathbf{Fm}_\Lambda, T \rangle$. Thus by Lem. 2.5 $\boldsymbol{\Omega}\, T \subseteq R$. So $R = \boldsymbol{\Omega}\, T$. This shows that $E(x, y)$ is an equivalence system for $\mathcal{S}$ by Thm. 1.19. $\square$

The following theorem was proved independently by the present authors (see [40]) and Font [15].

Theorem 2.18 *Every protoalgebraic Fregean 1-deductive system with at least one theorem is regularly algebraizable.*

PROOF. Let $\mathcal{S}$ be a protoalgebraic and Fregean deductive system with at least one theorem, and let $\Delta(x, y)$ be a protoequivalence system. By the preceding lemma, $E(x, y) = \Delta(x, y) \cup \Delta(y, x)$ is an equivalence system for $\mathcal{S}$. It remains only to show that the E-G-rule is a rule of $\mathcal{S}$. Let $T \in \operatorname{Th} \mathcal{S}$ and suppose $\varphi, \psi \in T$. Then trivially $T, \varphi \vdash_{\mathcal{S}} \psi$ and $T, \psi \vdash_{\mathcal{S}} \varphi$. Thus $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}} T$, and hence $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega} T$ because $\mathcal{S}$ is protoalgebraic and Fregean. So, by Thm. 1.19, $\varphi, \psi \in T$ implies $E(\varphi, \psi) \subseteq T$, for every $T \in \operatorname{Th} \mathcal{S}$, i.e., the E-G-rule is a rule of $\mathcal{S}$. □

In Lem. 2.17 we proved that, if a protoalgebraic deductive system $\mathcal{S}$ with at least one theorem is Fregean, then $\Delta(x, y) \cup \Delta(y, x)$ is an equivalence system for every protoequivalence system $\Delta(x, y)$. Clearly the converse fails. Any equivalential system that fails to be Fregean, for example almost every normal modal logic, provides a counterexample.

But in the next lemma we prove the converse under stronger assumptions about the system $\Delta(x, y)$.

2.2 *The deduction theorem and conjunction*

Recall the definition of a deduction-detachment system given in Def. 1.38, and of the notions of the uniterm and multiterm deduction-detachment theorem.

A nonempty system $\Delta(x, y)$ of binary formulas is a deduction-detachment system for $\mathcal{S}$ if it defines the localized consequence relation in the sense that, for every $T \in \operatorname{Th} \mathcal{S}$, $T, \varphi \vdash_{\mathcal{S}} \psi$ iff $\Delta(\varphi, \psi) \subseteq T$. Consequently, $\Delta(x, y) \cup \Delta(y, x)$ defines the symmetric localized consequence relation $\widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}} T$. This observation leads directly to the following result.

Lemma 2.19 *Let $\mathcal{S}$ be a deductive system with a deduction-detachment system $\Delta(x, y) = \{ \delta_i(x, y) : i \in I \}$. Then $\mathcal{S}$ is Fregean iff $\Delta(x, y) \cup \Delta(y, x)$ is an equivalence system for $\mathcal{S}$.*

PROOF. Since $\Delta(x, y)$ defines the localized consequence relation, $\Delta(x, y) \cup \Delta(y, x)$ defines the symmetric localized consequence relation $\widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}} T$, for every $T \in \operatorname{Th} \mathcal{S}$. If $\Delta(x, y) \cup \Delta(y, x)$ is an equivalence system, then it also defines $\boldsymbol{\Omega} T$ by Thm. 1.19. Thus $\boldsymbol{\Omega} T = \widetilde{\boldsymbol{\Lambda}}_{\operatorname{Th} \mathcal{S}} T$, for all $T \in \operatorname{Th} \mathcal{S}$, and so $\mathcal{S}$ is Fregean. Conversely, if $\mathcal{S}$ is Fregean, then $\Delta(x, y) \cup \Delta(y, x)$ defines $\boldsymbol{\Omega} T$ and hence is an equivalence system. □

Using this lemma we can obtain a very satisfactory characterization of Fregean deductive systems with the uniterm deduction-detachment theorem. In the next theorem, and in the sequel, $\varphi_0 \to \varphi_1 \to \cdots \to \varphi_{n-1} \to \varphi_n$ will be shorthand for $\varphi_0 \to (\varphi_1 \to (\cdots \to (\varphi_{n-1} \to \varphi_n) \cdots))$.

Theorem 2.20 *Let Λ be an arbitrary language type. Let $\mathcal{S}$ be a Fregean deductive system over Λ with the uniterm deduction-detachment theorem. Let $x \to y$ be a single deduction-detachment formula for $\mathcal{S}$. Then $\mathcal{S}$ is an axiomatic extension of the deductive system presented by the axioms*

$$x \to y \to x, \tag{35}$$

$$(x \to y \to z) \to (x \to y) \to (x \to z), \tag{36}$$

$$\begin{aligned}(x_0 \to y_0) \to (y_0 \to x_0) \to \cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1})\\ \to \lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1},\\ \textit{for each } \lambda \in \Lambda \ (n \textit{ is the rank of } \lambda),\end{aligned} \tag{37}$$

and the single inference rule

$$\frac{x, x \to y}{y} \qquad \textit{Modus Ponens.} \tag{MP$_\to$}$$

Conversely, every axiomatic extension of this deductive system is Fregean and has the uniterm deduction-detachment theorem with $x \to y$ as the deduction-detachment formula.

PROOF. That (35) and (36) are theorems of $\mathcal{S}$ and (MP$_\to$) is an inference rule are immediate consequences of the assumption that $x \to y$ is a deduction-detachment formula for $\mathcal{S}$. To see this note that, from the trivial entailment $x, y \vdash_{\mathcal{S}} x$, we get $\vdash_{\mathcal{S}} x \to y \to x$ by the deduction property. We have that x, $x \to y$, $x \to y \to z \vdash_{\mathcal{S}} z$ by three applications of detachment, and then three applications of the deduction property give $\vdash_{\mathcal{S}} (x \to y \to z) \to (x \to y) \to (x \to z)$.

By Lem. 2.19, $E(x, y) = \{x \to y, y \to x\}$ is an equivalence system for $\mathcal{S}$. So by the replacement rule for equivalence systems (23) we have

$$\begin{aligned}(x_0 \to y_0) \to (y_0 \to x_0) \to \cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1})\\ \vdash_{\mathcal{S}} \lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1}.\end{aligned}$$

Hence multiple applications of the deduction property give

$$\begin{aligned}\vdash_{\mathcal{S}} (x_0 \to y_0) \to (y_0 \to x_0) \to \cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1})\\ \to \lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1}.\end{aligned}$$

So (37) is a theorem of $\mathcal{S}$ for every $\lambda \in \Lambda$. Finally, by Thm. 1.42 we know that a presentation of $\mathcal{S}$ can be obtained by adjoining additional axioms to

(35)–(37) and ($\mathrm{MP}_{\rightarrow}$).

For the converse, let $\mathcal{S}$ be any axiomatic extension of the deductive system with presentation (35)–(37) and ($\mathrm{MP}_{\rightarrow}$). Since ($\mathrm{MP}_{\rightarrow}$) is the only inference rule, the deduction property for $x \rightarrow y$ can be proved by induction on the length of the derivation using axioms (35) and (36) in exactly the same way it is for the classical and intuitionistic propositional calculi. We omit the details.

That $\mathcal{S}$ is Fregean will then follow by Lem. 2.19 from the fact that $E(x, y) := \{x \rightarrow y, y \rightarrow x\}$ is an equivalence system for $\mathcal{S}$. This is easily verified. For example, we have $x, x \rightarrow y, y \rightarrow z \vdash_{\mathcal{S}} z$ by detachment, and hence $E(x, y), E(y, z) \vdash_{\mathcal{S}} E(x, z)$ by the deduction property. So the transitivity rule (22) holds. We get the replacement rule from the axioms (37) together with the deduction and detachment properties. □

This theorem can be reformulated in a way that shows more clearly the relation between Fregean deductive systems with the uniterm deduction-detachment theorem and the intuitionistic propositional calculus IPC. Let $\mathrm{IPC}^{\rightarrow,\top}$ be the $\{\rightarrow, \top\}$-fragment of IPC. Let Λ be an arbitrary language type with $\Lambda \cap \{\rightarrow, \top\} = \emptyset$, and let $\mathrm{IPC}_{\Lambda}^{\rightarrow,\top}$ be the expansion (see Sec. 1) of $\mathrm{IPC}^{\rightarrow,\top}$ to the language type $\{\rightarrow, \top\} \cup \Lambda$ obtained by adding the axioms (37). The new connectives are said to be *extensional* over $\mathrm{IPC}^{\rightarrow,\top}$, and $\mathrm{IPC}_{\Lambda}^{\rightarrow,\top}$ and its axiomatic extensions are collectively referred to as the *expansions of* $\mathrm{IPC}^{\rightarrow,\top}$ *by extensional connectives.*

Let $\mathcal{S}$ be any deductive system over the language type Λ. By a $\{\rightarrow, \top\}$-*definitional expansion of* $\mathcal{S}$ we mean an conservative expansion of $\mathcal{S}$ to the language type $\Lambda \cup \{\rightarrow, \top\}$ such that $x \rightarrow y \equiv \varphi(x, y) \pmod{\boldsymbol{\Omega}\, \mathrm{Th}\, \mathcal{S}'}$ for some binary formula $\varphi(x, y)$ over Λ and $\top \equiv \psi \pmod{\boldsymbol{\Omega}\, \mathrm{Th}\, \mathcal{S}'}$ for some theorem of ψ of $\mathcal{S}$.

Theorem 2.20 can now be reformulated in the following way. A deductive system over the language type Λ is Fregean with the uniterm deduction-detachment theorem iff it has a definitional expansion by the connectives $\rightarrow$ and $\top$ that is an axiomatic extension of $\mathrm{IPC}_{\Lambda}^{\rightarrow,\top}$; in summary, every Fregean system with the uniterm deduction-detachment theorem is (up to definitional expansion) an expansion of $\mathrm{IPC}^{\rightarrow,\top}$ by extensional connectives.

Since conjunction, disjunction, and negation are all extensional over $\mathrm{IPC}^{\rightarrow,\top}$, the (full) intuitionistic propositional calculus IPC itself and all its axiomatic extensions, in particular CPC, fall within the scope of Theorem 2.20, as well any expansions of these logics by additional extensional connectives. In the case of CPC or course there are no such connectives (up to definitional expansion) because of the functional completeness of the two-element Boolean algebra. We know of no connectives that are extensional over IPC apart from the fundamental ones and those definable in their terms. The nontrivial modal operators all fail to be extensional, and hence Theorem 2.20 does not apply to modal logics, at least the so-called *strong normal* modal logics. We will return to the subject of modal logics below in Section 2.5.

There are many examples of protoalgebraic Fregean deductive systems without even the multiterm deduction-detachment theorem. The paradigm for this class of deductive systems is $\mathrm{IPC}^{\leftrightarrow,\top}$, the biconditional fragment of intuitionistic propositional logic; see [19]. However, every Fregean deductive system with conjunction that has at least one theorem has a uniterm deduction-detachment system, as we next show.

A single formula $\kappa(x,y)$ is a *conjunction formula* for $\mathcal{S}$ if the following sequents are rules of $\mathcal{S}$.

$$\frac{x,y}{\kappa(x,y)},\quad \frac{\kappa(x,y)}{x},\quad \text{and}\quad \frac{\kappa(x,y)}{y}.$$

A deductive system is said to be *conjunctive* if it has a conjunction formula. In the sequel $x \wedge y$ will represent an arbitrary conjunction formula. When writing iterated conjunctions we omit parenthesis and assume association is to the right.

Note that, if $x\wedge y$ is a conjunction formula, then, for all $\varphi,\psi \in \mathrm{Fm}_\Lambda$ and every $T \in \mathrm{Th}\,\mathcal{S}$,

$$T,\varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad T,\varphi \dashv\vdash_{\mathcal{S}} T,\varphi\wedge\psi \quad \text{iff} \quad \langle \varphi,\, \varphi\wedge\psi\rangle \in \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\, T.$$

Theorem 2.21 *Assume $\mathcal{S}$ is a 1-deductive system with a conjunction formula $x \wedge y$. If $\mathcal{S}$ is protoalgebraic and Fregean with at least one theorem, then it has the uniterm deduction-detachment theorem. More precisely, if $\Delta(x,y) = \{\delta_0(x,y),\dots,\delta_{n-1}(x,y)\}$ is a protoequivalence system for $\mathcal{S}$, then*

$$x \to y := \delta_0(x, x\wedge y) \wedge \cdots \wedge \delta_{n-1}(x, x \wedge y)$$

is a single deduction-detachment formula for $\mathcal{S}$; moreover,

$$x \leftrightarrow y := (x \to y) \wedge (y \to x)$$

is a single equivalence formula.

PROOF. For all $\varphi,\psi \in \mathrm{Fm}_{\mathcal{S}}$ and every $T \in \mathrm{Th}\,\mathcal{S}$,

$$\begin{aligned}
&T,\varphi \vdash_{\mathcal{S}} \psi\\
&\quad \text{iff} \quad T,\varphi \dashv\vdash_{\mathcal{S}} T,\varphi\wedge\psi\\
&\quad \text{iff} \quad \Big\langle \varphi,\, \varphi\wedge\psi\Big\rangle \in \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\, T\\
&\quad \text{iff} \quad \varphi \equiv \varphi\wedge\psi \pmod{\mathbf{\Omega}\, T}, \quad \text{since } \mathcal{S} \text{ is Fregean and protoalgebraic.}
\end{aligned}$$

Consequently, $\delta_i(\varphi, \varphi) \equiv \delta_i(\varphi, \varphi \wedge \psi) \pmod{\boldsymbol{\Omega} T}$ for each $i < n$. Thus, since $\delta_i(\varphi, \varphi) \in T$ (because $\Delta(x, y)$ is a protoequivalence system) and $\boldsymbol{\Omega} T$ is compatible with T, we have $\delta_i(\varphi, \varphi \wedge \psi) \in T$ for each $i < n$. Hence $T \vdash_{\mathcal{S}} \delta_0(x, x \wedge y) \wedge \cdots \wedge \delta_{n-1}(x, x \wedge y)$. Conversely, if $T \vdash_{\mathcal{S}} \delta_0(x, x \wedge y) \wedge \cdots \wedge \delta_{n-1}(x, x \wedge y)$, then $T, \varphi \vdash_{\mathcal{S}} \psi$ since $\{\delta_0(x, y), \ldots, \delta_{n-1}(x, y)\}$ is a protoequivalence system. So $x \to y$ is a deduction-detachment formula for $\mathcal{S}$.

$x \to y$ is clearly a single protoequivalence formula for $\mathcal{S}$. Thus $x \leftrightarrow y$ is a single equivalence formula by Lem. 2.17. □

Theorem 2.22 *Let Λ be an arbitrary language type. Let $\mathcal{S}$ be a protoalgebraic Fregean 1-deductive system over Λ with conjunction, and let $x \wedge y$ be a conjunction formula. If $\mathcal{S}$ has at least one theorem, then there exists a binary formula $x \to y$ such that $\mathcal{S}$ is an axiomatic extension of the deductive system presented by the axioms* (35)–(37) *and the single inference rule* ($\mathrm{MP}_{\to}$) *of Thm.* 2.20, *along with the following additional axioms.*

$$x \wedge y \to x, \tag{38}$$

$$x \wedge y \to y, \tag{39}$$

$$x \to y \to x \wedge y. \tag{40}$$

Conversely, every axiomatic extension of this deductive system is protoalgebraic and Fregean and has conjunction with $x \wedge y$ as conjunction formula.

PROOF. Let $\mathcal{S}$ be a protoalgebraic Fregean deductive system over Λ with a conjunction and with at least one theorem. Let $x \wedge y$ be a conjunction formula. By Thm. 2.21 $\mathcal{S}$ has a single deduction-detachment formula $x \to y$. Thus by Thm. 2.20 $\mathcal{S}$ is an axiomatic extension of the system presented by (35)–(37) and ($\mathrm{MP}_{\to}$). Clearly (38)–(40) are theorems of $\mathcal{S}$. The converse follows easily from Thm. 2.20. □

This theorem can be reformulated as Theorem 2.20 was reformulated in the remarks that followed it: every Fregean system with conjunction is (up to definitional expansion) an expansion of $\mathrm{IPC}^{\to,\wedge,\top}$ by extensional connectives. We omit the details.

2.3 Strongly algebraizable Fregean deductive systems

Every protoalgebraic Fregean deductive system is algebraizable and in fact regularly algebraizable by Theorem 2.18. One of the more interesting challenges in abstract algebraic logic has been to try to explain why almost all the traditional deductive systems are strongly algebraizable, that is, they are finitely algebraizable and their equivalent semantics is a variety; in the broader context of abstract algebraic logic the equivalent semantics of a finitely algebraizable system is normally a proper quasivariety.

At least for those logics that have a large enough fragment of the classical or intuitionistic propositional calculi at their core, the key to the puzzle appears to be the Fregean property or, more significantly, self-extensionality; both these properties can be shown to induce strong algebraizability under a variety of different circumstances. The first result of this kind presented here is a direct consequence of the characterization of Fregean deductive systems with the uniterm deduction-detachment theorem presented in Theorem 2.20.

As is well-known, $\mathrm{IPC}^{\to,\top}$, the $\{\to,\top\}$-fragment of the intuitionistic propositional calculus, is strongly, regularly algebraizable with equivalent semantics the variety of Hilbert algebras (Diego [41]). Hilbert algebras are defined by the following four identities ([41], p. 7). (Recall that by convention multiple occurrences of $\to$ in a formula are associated to the right.)

$$x \to x \approx \top; \tag{41}$$
$$\top \to x \approx x; \tag{42}$$
$$x \to y \to z \approx (x \to y) \to x \to z; \tag{43}$$
$$(x \to y) \to (y \to x) \to x \approx (x \to y) \to (y \to x) \to y. \tag{44}$$

The first part of the following theorem was originally proved by Font and Jansana [5, Proposition 4.49].

Theorem 2.23 *Every Fregean deductive system with the uniterm deduction-detachment theorem is strongly, regularly algebraizable. More precisely, let Λ be an arbitrary language type and let $\mathcal{S}$ be a Fregean deductive system over Λ with a single deduction-detachment formula $x \to y$. Then $\mathsf{Alg\,Mod}^*\mathcal{S}$ is a subvariety of the variety defined by the four identities* (41)–(44) *of Hilbert algebras, together with the identity*

$$\begin{aligned}(x_0 \to y_0) \to &(y_0 \to x_0) \to \cdots \\ &\cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1}) \to \lambda x_0 \dots x_{n-1} \\ \approx (x_0 \to y_0) \to &(y_0 \to x_0) \to \cdots \\ &\cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1}) \to \lambda y_0 \dots y_{n-1}\end{aligned} \tag{45}$$

for each $\lambda \in \Lambda$ (n is the rank of λ).

PROOF. By Thm. 2.20, $\mathcal{S}$ is an axiomatic extension of the deductive system presented by the following axioms and inference rules.

$$x \to y \to x,$$
$$(x \to y \to z) \to (x \to y) \to x \to z,$$
$$(x_0 \to y_0) \to (y_0 \to x_0) \to \cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1}) \\ \to \lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1}, \\ \text{for each } \lambda \in \Lambda \ (n \text{ is the rank of } \lambda),$$

and the single inference rule

$$\frac{x, x \to y}{y}$$

Applying Thm. 1.43 directly, with $E(x, y) = \{x \to y, y \to x\}$, we have that $\mathsf{Alg\,Mod}^* \mathcal{S}$ is defined by adjoining some set of identities of the form $\varphi \approx \top$ to the following identities and quasi-identities.

$$x \to x \approx \top,$$
$$\frac{x \approx \top,\ x \to y \approx \top}{y \approx \top}, \tag{46}$$
$$\frac{x \to y \approx \top,\ y \to x \approx \top}{x \approx y}, \tag{47}$$
$$x \to y \to x \approx \top, \tag{48}$$
$$(x \to y \to z) \to (x \to y) \to (x \to z) \approx \top, \tag{49}$$

$$(x_0 \to y_0) \to (y_0 \to x_0) \to \cdots \to (x_{n-1} \to y_{n-1}) \to (y_{n-1} \to x_{n-1}) \\ \to \lambda x_0 \ldots x_{n-1} \to \lambda y_0 \ldots y_{n-1} \approx \top, \\ \text{for each } \lambda \in \Lambda \ (n \text{ is the rank of } \lambda). \tag{50}$$

It is proved in [41], p. 7, that the quasi-identity (47) and the two identities (48) and (49) together define Hilbert algebras, i.e., are logically equivalent to the four identities (41)–(44). (46) is an immediate consequence of (42), and (45) and (50) are easily shown to be equivalent in the presence of (46), (47), and (49). □

Let Λ be an arbitrary language type disjoint from $\{\to, \top\}$. Let HI_Λ be the variety of algebras of type $\{\to, \top\} \cup \Lambda$ defined by the four identities of Hilbert algebras together with the identity (45) for each $\lambda \in \Lambda$ (n is the rank of λ). HI_Λ is called the *the variety of Hilbert algebras with compatible operations* over Λ. Theorem 2.23 says that, up to termwise definitional equivalence, the subvarieties of HI_Λ are exactly the equivalent quasivarieties of Fregean deductive systems with the uniterm deduction-detachment theorem.

By the theorem, all the intermediate logics, i.e., all the the axiomatic extensions of IPC, are strongly algebraizable, and their equivalent varieties are the subvarieties of the variety of Heyting algebras.

Theorem 2.23 can be given a more algebraic formulation. Recall the definition of the assertional logic of a pointed quasivariety (Def. 1.32),

Corollary 2.24 *Every relatively point-regular quasivariety* Q *whose assertional logic is Fregean and has the uniterm deduction-detachment theorem is a variety. More precisely,* Q *is termwise definitionally equivalent to a subvariety of* HI_Λ*, where* Λ *is the language type of* Q*.*

PROOF. Let Q be a relatively point-regular quasivariety and assume that its assertional logic $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ is both Fregean and has the uniterm deduction-detachment theorem. By Thm. 2.23 $\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ is strongly, regularly algebraizable and its equivalent quasivariety $\mathsf{Alg}\,\mathsf{Mod}^*\,\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ is termwise definitionally equivalent to a subvariety of HI_Λ. But $\mathsf{Q} = \mathsf{Alg}\,\mathsf{Mod}^*\,\boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$ (see Thm. 1.34 and the following remarks). □

Every protoalgebraic Fregean deductive system with conjunction and at least one theorem has the uniterm deduction-detachment theorem (Theorem 2.21), and hence, as a immediate consequence of Theorem 2.23, is strongly, regularly algebraizable. The precise situation is described in the following:

Theorem 2.25 *Let* Λ *be an arbitrary language type and let* $\mathcal{S}$ *be a protoalgebraic Fregean 1-deductive system over* Λ *with a conjunction formula* $x \wedge y$ *and at least one theorem. Let* $x \to y$ *be any deduction-detachment term for* $\mathcal{S}$*. Then* $\mathsf{Alg}\,\mathsf{Mod}^*\,\mathcal{S}$ *is a subvariety of the variety defined by the identities* (41)–(44) *of Hilbert algebras, the identity* (45) *for each* $\lambda \in \Lambda$ *(*n *is the rank of* λ*), together with the three identities*

$$x \wedge y \to x \approx \top, \tag{51}$$

$$x \wedge y \to y \approx \top, \tag{52}$$

$$x \to y \to x \wedge y \approx \top. \tag{53}$$

PROOF. Similar to the proof of Thm. 2.23 but with Thm. 2.22 in place of Thm.2.20. □

This theorem provides an alternative way of showing that all the intermediate logics are strongly algebraizable.

Theorems 2.23 and 2.25 do not comprehend all strongly algebraizable Fregean logics however. For example, $\mathrm{IPC}^{\leftrightarrow,\top}$, the biconditional fragment of intuitionistic propositional logic, is Fregean and strongly algebraizble (see [19]), but it has neither the deduction-detachment theorem nor conjunction.

Let Λ be an arbitrary language type disjoint from $\{\to, \wedge, \top\}$. Let BS_Λ be the variety of algebras of type $\{\to, \wedge, \top\} \cup \Lambda$ defined by the the four identities of Hilbert algebras, the identity (45) for each $\lambda \in \{\wedge\} \cup \Lambda$, and the three identities (51)–(53). BS_Λ is the *variety of Brouwerian semilattices with compatible operations* over Λ. (See [42].) Theorem 2.25 says that, up to termwise definitional equivalence, the subvarieties of BS_Λ are exactly the equivalent quasivarieties of protoalgebraic Fregean deductive systems, with at least one theorem, that have conjunction. The theorem can also be used to show that all the intermediate logics are strongly algebraizable.

Like Theorem 2.23, Theorem 2.25 can be given a more algebraic formulation. Note that, if a quasivariety is point-regular, its assertional logic must have at least one theorem.

Corollary 2.26 ([16, Theorem 4.5]) *Every relatively point-regular quasivariety* Q *whose assertional logic is protoalgebraic, Fregean, and has conjunction is a variety. More precisely,* Q *is termwise definitionally equivalent to a subvariety of* BS_Λ*, where* Λ *is the language type of* Q*.*

PROOF. This is an easy consequence of Thms. 2.22 and 2.23. □

2.4 Strong algebraizability and self-extensionality

Theorems 2.23 and 2.25 give a good account of the strong algebraizability of the classical and intuitionistic calculi and their various fragments, but they do not account for the strong algebraizability of normal modal logic, which is not Fregean. However, recent results of Font and Jansana [5] on the relationship between self-extensionality and strong algebraizability do encompass modal logic, and, moreover, they go a long way towards giving us a clear picture of why strong algebraizability is such a common phenomenon in classical algebraic logic. For some of their results we now present new proofs that generalize the proofs of Theorems 2.23 and 2.25, which themselves generalize the methods used in classical algebraic logic to establish strong algebraizability. This leads to some refinements that may shed further light on this problem.

Let $x \to y$ be a binary Λ-formula. For all $\Gamma \cup \{\varphi\} \subseteq_{\boldsymbol{\omega}} \mathrm{Fm}_\Lambda$, we denote by

$\Gamma \to \varphi$ the formula

$$\psi_0 \to \big(\psi_1 \to (\cdots \to (\psi_{n-1} \to \varphi) \cdots)\big),$$

where $\psi_0, \ldots, \psi_{n-1}$ is a fixed but arbitrary ordering of Γ. Furthermore, if $\Gamma_0, \ldots, \Gamma_{m-1}$ is any finite sequence of finite subsets of Fm_Λ, then

$$\Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_{m-1} \to \varphi$$

will stand for $\Gamma_0 \to \big(\Gamma_1 \to (\cdots \to (\Gamma_{m-1} \to \varphi) \cdots)\big)$.

Recall that every deductive system $\mathcal{S}$ with the multiterm, and in particular the uniterm, deduction-detachment theorem is protoalgebraic. Indeed, if $x \to y$ is a single deduction-detachment formula for $\mathcal{S}$, then $\Delta(x, y) = \{x \to y\}$ is a protoequivalence system and hence (by Thm. 1.18)

$$\begin{aligned} E(x, y, \bar{u}) := \ & \Big\{\, \xi(x, \bar{u}) \to \xi(y, \bar{u}) : \xi(x, \bar{u}) \in \mathrm{Fm}_\Lambda(\mathrm{Va} \setminus \{y\}) \,\Big\} \\ \cup & \Big\{\, \xi(y, \bar{u}) \to \xi(x, \bar{u}) : \xi(x, \bar{u}) \in \mathrm{Fm}_\Lambda(\mathrm{Va} \setminus \{y\}) \,\Big\} \end{aligned} \tag{54}$$

is an equivalence system with parameters for $\mathcal{S}$.

Lemma 2.27 *Let $\mathcal{S}$ be a self-extensional 1-deductive system over a countable language type with the uniterm deduction-detachment theorem. Let $x \to y$ be a single deduction-detachment formula. Then for all $\Gamma \cup \{\varphi, \psi\} \subseteq_{\boldsymbol{\omega}} \mathrm{Fm}_\Lambda$, $\Gamma, \varphi \dashv\vdash_{\mathcal{S}} \Gamma, \psi$ iff $\Gamma \to \varphi \approx \Gamma \to \psi$ is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$.*

PROOF. Assume $\Gamma, \varphi \dashv\vdash_{\mathcal{S}} \Gamma, \psi$. Then $\Gamma, \Gamma \to \varphi \vdash_{\mathcal{S}} \psi$ by the detachment property of $x \to y$. So $\Gamma \to \varphi \vdash_{\mathcal{S}} \Gamma \to \psi$ by the deduction property of $x \to y$. By symmetry, $\Gamma \to \psi \vdash_{\mathcal{S}} \Gamma \to \varphi$. Thus $\Gamma \to \varphi \dashv\vdash_{\mathcal{S}} \Gamma \to \psi$. So $\langle \Gamma \to \varphi, \Gamma \to \psi \rangle \in \boldsymbol{\Lambda}\,\mathrm{Th}\,\mathcal{S}$. Since $\mathcal{S}$ is self-extensional, $\boldsymbol{\Lambda}\,\mathrm{Th}\,\mathcal{S} = \boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{S}$. So $\Gamma \to \varphi \approx \Gamma \to \psi$ is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$ by Cor. 1.22. (Note that $\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{S} = \boldsymbol{\Omega}(\mathrm{Thm}\,\mathcal{S})$ since $\mathcal{S}$ is protoalgebraic).

Conversely, if $\Gamma \to \varphi \approx \Gamma \to \psi$ is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$, then, again by Thm. 1.22, $\langle \Gamma \to \varphi, \Gamma \to \psi \rangle \in \boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{S} \subseteq \boldsymbol{\Lambda}\,\mathrm{Th}\,\mathcal{S}$, and hence $\Gamma \to \varphi \dashv\vdash_{\mathcal{S}} \Gamma \to \psi$. We get $\Gamma, \varphi \vdash_{\mathcal{S}} \Gamma \to \varphi$ by the deduction property of $x \to y$. Similarly, $\Gamma, \psi \vdash_{\mathcal{S}} \Gamma \to \psi$. So, by detachment, $\Gamma, \varphi \dashv\vdash_{\mathcal{S}} \Gamma, \psi$. (Self-extensionality is not needed for this direction.) □

For the next lemma recall that $\forall \bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}) := \Big\{ \varepsilon_i(\varphi, \psi, \bar{\vartheta}) : i \in I, \bar{\vartheta} \in \mathrm{Fm}_\Lambda^{|\bar{u}|} \Big\}$.

Lemma 2.28 *Let $\mathcal{S}$ be a self-extensional 1-deductive system with the uniterm deduction-detachment theorem. Let $x \to y$ be a single deduction-detachment* *formula for $\mathcal{S}$, and let $E(x, y, \bar{u})$ be the equivalential system for $\mathcal{S}$ with parameters that is formed from $x \to y$ as in* (54). *Then the following are identities of* $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$.

(i) $(y \to y) \to x \approx x$, *and*
(ii) $x \to y \to y \approx y \to y$.
(iii) *For every quasi-identity* $\dfrac{\varphi_0 \approx \psi_0,\ \varphi_1 \approx \psi_1, \ldots, \varphi_{n-1} \approx \psi_{n-1}}{\xi \approx \eta}$ *of* $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$, *there is, for each $i < n$, a finite subset $F_i(\varphi_i, \psi_i)$ of $\forall \bar{\vartheta}\, E(\varphi_i, \psi_i, \bar{\vartheta})$ such that*

$$\begin{aligned} &F_0(\varphi_0, \psi_0) \to F_1(\varphi_1, \psi_1) \to \cdots \to F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \to \xi \\ \approx\ &F_0(\varphi_0, \psi_0) \to F_1(\varphi_1, \psi_1) \to \cdots \to F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \to \eta \end{aligned}$$

is an identity of $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$.

PROOF. (i). $\vdash_{\mathcal{S}} y \to y$. Thus $(y \to y) \to x \vdash_{\mathcal{S}} x$ by the detachment property of $x \to y$. On the other hand, $x \vdash_{\mathcal{S}} (y \to y) \to x$ by the deduction property of $x \to y$. So $(y \to y) \to x \dashv\vdash_{\mathcal{S}} x$, and hence $(y \to y) \to x \equiv x \pmod{\boldsymbol{\Omega}\,\mathrm{Th}\,\mathcal{S}}$ by self-extensionality. Thus $(y \to y) \to x \approx x$ is an identity of $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ by Cor. 1.22.

(ii). From $\vdash_{\mathcal{S}} y \to y$ we get $x \vdash_{\mathcal{S}} y \to y$ and hence $\vdash_{\mathcal{S}} x \to y \to y$ by the deduction property of $x \to y$. So $y \to y \vdash_{\mathcal{S}} x \to y \to y$ Clearly $x \to y \to y \vdash_{\mathcal{S}} y \to y$. So $x \to y \to y \approx y \to y$ is an identity of $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$ by self-extensionality and Cor. 1.22.

(iii). Assume

$$\frac{\varphi_0 \approx \psi_0, \ldots, \varphi_{n-1} \approx \psi_{n-1}}{\xi \approx \eta} \tag{55}$$

is a quasi-identity of $\mathsf{Alg}\,\mathsf{Mod}^* \mathcal{S}$. Then, by Thm. 1.21,

$$\forall \bar{\vartheta}\, E(\varphi_0, \psi_0, \bar{\vartheta}), \ldots, \forall \bar{\vartheta}\, E(\varphi_{n-1}, \psi_{n-1}, \bar{\vartheta}) \vdash_{\mathcal{S}} \forall \bar{\vartheta}\, E(\xi, \eta, \bar{\vartheta}),$$

and hence by $E(x, y, \bar{u})$-detachment

$$\forall \bar{\vartheta}\, E(\varphi_0, \psi_0, \bar{\vartheta}), \ldots, \forall \bar{\vartheta}\, E(\varphi_{n-1}, \psi_{n-1}, \bar{\vartheta}),\ \xi \vdash_{\mathcal{S}} \eta.$$

By symmetry, $\forall \bar{\vartheta}\, E(\varphi_0, \psi_0, \bar{\vartheta}), \ldots, \forall \bar{\vartheta}\, E(\varphi_{n-1}, \psi_{n-1}, \bar{\vartheta}),\ \eta \vdash_{\mathcal{S}} \xi$. Thus, since $\mathcal{S}$ is finitary, there is, for each $i < n$, a finite subset $F_i(\varphi_i, \psi_i)$ of $\forall \bar{\vartheta}\, E(\varphi_i, \psi_i, \bar{\vartheta})$ such that

$$F_0(\varphi_0, \psi_0), \ldots, F_{n-1}(\varphi_{n-1}, \psi_{n-1}),\ \xi \dashv\vdash_{\mathcal{S}} F_0(\varphi_0, \psi_0), \ldots, F_{n-1}(\varphi_{n-1}, \psi_{n-1}),\ \eta.$$

The conclusion of (iii) is now an immediate consequence of Lem. 2.27. □

For any class K of Λ-algebras, $\mathsf{Qv}\,\mathsf{K}$ will denote the quasivariety generated by K.

Theorem 2.29 ([5], Theorem 4.45) *Let $\mathcal{S}$ be a 1-deductive system over a countable language type. If $\mathcal{S}$ is self-extensional with the uniterm deduction-detachment theorem, then $\mathsf{Qv}\,\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$ is a variety.*

PROOF. To prove $\mathsf{Qv}\,\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$ is a variety it suffices to show that every quasi-identity of $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$ is a logical consequence of the identities of $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$. Let (55) be a quasi-identity of $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$. Let $x \to y$ be a single deduction-detachment formula for $\mathcal{S}$, and let $E(x, y, \bar{u})$ be the equivalence system with parameters for $\mathcal{S}$ that is constructed from $x \to y$ in (54).

By Lem. 2.28(iii), there exists, for each $i < n$, a finite subset $F_i(\varphi_i, \psi_i)$ of $\forall\bar{\vartheta}\, E(\varphi_i, \psi_i, \bar{\vartheta})$ such that

$$\begin{aligned} &F_0(\varphi_0, \psi_0) \to F_1(\varphi_1, \psi_1) \to \cdots \to F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \to \xi \\ &\quad\approx\; F_0(\varphi_0, \psi_0) \to F_1(\varphi_1, \psi_1) \to \cdots \to F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \to \eta \end{aligned} \tag{56}$$

is an identity of $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$. Recall that each formula in $\forall\bar{\vartheta}E(\varphi_i, \psi_i, \bar{\vartheta})$ is of the form $\zeta(\varphi_i, \bar{\vartheta}) \to \zeta(\psi_i, \bar{\vartheta})$ or $\zeta(\psi_i, \bar{\vartheta}) \to \zeta(\varphi_i, \bar{\vartheta})$ with $\zeta(x, \bar{u}) \in \mathrm{Fm}_\Lambda(\mathrm{Va} \setminus \{y\})$ and $\bar{\vartheta} \in \mathrm{Fm}_\Lambda^{|\bar{u}|}$ (see (54)). Let $\boldsymbol{A}$ be any member of the variety generated by $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$, and let $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ be an evaluation in $\boldsymbol{A}$ such that $h(\varphi_i) = h(\psi_i)$ for all $i < n$. Then, by Lem. 2.28(i). for every $a \in A$,

$$\begin{aligned} &\Big(\zeta^{\boldsymbol{A}}(h(\varphi_i),\, h(\bar{\vartheta})) \;\to^{\boldsymbol{A}}\; \zeta^{\boldsymbol{A}}(h(\psi_i),\, h(\bar{\vartheta}))\Big) \;\to^{\boldsymbol{A}}\; a \;=\; a \quad \text{and} \\ &\Big(\zeta^{\boldsymbol{A}}(h(\psi_i),\, h(\bar{\vartheta})) \;\to^{\boldsymbol{A}}\; \zeta^{\boldsymbol{A}}(h(\varphi_i),\, h(\bar{\vartheta}))\Big) \;\to^{\boldsymbol{A}}\; a \;=\; a. \end{aligned}$$

Therefore,

$$\begin{aligned} &h(\xi) \\ &\quad = F_0^{\boldsymbol{A}}(h(\varphi_0),\, h(\psi_0)) \to^{\boldsymbol{A}} \cdots \to^{\boldsymbol{A}} F_{n-1}^{\boldsymbol{A}}(h(\varphi_{n-1}),\, h(\psi_{n-1})) \to^{\boldsymbol{A}} h(\xi) \\ &\quad = F_0^{\boldsymbol{A}}(h(\varphi_0),\, h(\psi_0)) \to^{\boldsymbol{A}} \cdots \to^{\boldsymbol{A}} F_{n-1}^{\boldsymbol{A}}(h(\varphi_{n-1}),\, h(\psi_{n-1})) \to^{\boldsymbol{A}} h(\eta), \text{ by (56)} \\ &\quad = h(\eta). \end{aligned}$$

So the quasi-identity (55) is a logical consequence of the identities (56) and $(y \to y) \to x \approx x$ of $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$. □

In [5] a stronger result is obtained, namely that $\mathsf{Alg}\,\mathsf{Mod}^*\mathcal{S}$ itself is a variety, and without the restriction on the cardinality of language type.

As a corollary we have that every algebraizable self-extensional deductive system with the uniterm deduction-detachment theorem is strongly algebraizable.

In particular, every Fregean deductive system with the uniterm deduction-detachment theorem is strongly algebraizable, so we get an alternative proof of the first part of Thm. 2.23. In the case of algebraizable logics the proof of Thm. 2.29 gives an algorithm for actually generating a set of defining identities for the equivalent variety of $\mathcal{S}$ from any given presentation of $\mathcal{S}$ by axioms and rules of inference:

Theorem 2.30 *Every algebraizable self-extensional deductive system $\mathcal{S}$ with the uniterm deduction-detachment theorem is strongly algebraizable.*

Let $E(x,y)$ be a finite equivalence system and $K(x) \approx L(y)$ a finite set of defining equations for $\mathcal{S}$. Let $x \to y$ be a single deduction-detachment formula for $\mathcal{S}$. Then, for each presentation of $\mathcal{S}$ by axioms Ax *and inference rules* Ru*, the variety $\mathsf{Alg\,Mod}^* \mathcal{S}$ is defined by the following identities.*

(i) $(y \to y) \to x \approx x;$

(ii) $K(\varphi) \approx L(\varphi)$ *for each* $\varphi \in \mathrm{Ax}$;

(iii) $K(E(x,y)) \to L(E(x,y)) \to x \approx K(E(x,y)) \to L(E(x,y)) \to y;$

(iv) $$E\Big(K(\psi_0),\, L(\psi_0)\Big) \to \cdots \to E\Big(K(\psi_{n-1}),\, L(\psi_{n-1})\Big) \to K(\varphi)$$
$$\approx E\Big(K(\psi_0),\, L(\psi_0)\Big) \to \cdots \to E\Big(K(\psi_{n-1}),\, L(\psi_{n-1})\Big) \to L(\varphi),$$
for each sequent $\dfrac{\psi_0,\ldots,\psi_{n-1}}{\varphi}$ *in* Ru.

PROOF. (i) is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$ by Lem. 2.28. By Thm. 1.13, the identities (iv) together with the quasi-identities

$$\frac{K\Big(E(x,y)\Big) \approx L\Big(E(x,y\Big)}{x \approx y} \quad \text{and}$$
$$\frac{K(\psi_0) \approx L(\psi_0), \ldots, K(\psi_{n-1}) \approx L(\psi_{n-1})}{K(\varphi) \approx L(\varphi)}, \quad \text{for each } \frac{\psi_0,\ldots,\psi_{n-1}}{\varphi} \text{ in Ru},$$

define $\mathsf{Alg\,Mod}^* \mathcal{S}$. From the proofs of Lem. 2.28 and Thm. 2.29 we see that, in the presence of the identity (i), these quasi-equations are interderivable from the equations (iii) and (iv), respectively. □

By Thm. 2.25, every protoalgebraic, Fregean deductive system with conjunction, that has at least one theorem, is strongly, regularly algebraizable. This is proved using the fact that each such system has the uniterm deduction-detachment theorem (Thm. 2.21). If the assumption that the system is Fregean is weakened to self-extensionality, then the uniterm deduction-detachment theorem need no longer hold and hence this argument cannot be used. But, as we shall see, a conjunction seems to be just as effective as a deduction-detachment formula in forcing strong algebraizability in the presence of self-extensionality.

A conjunction formula will always be denoted by $x \wedge y$. For every nonempty $\Gamma \subseteq_{\omega} \mathrm{Fm}_{\Lambda}$, we denote by $\bigwedge \Gamma$ the formula

$$\psi_0 \wedge (\psi_1 \wedge (\cdots \wedge (\psi_{n-2} \wedge \psi_{n-1}) \cdots)),$$

where $\psi_0, \ldots, \psi_{n-1}$ is a fixed but arbitrary ordering of the formulas of Γ. If $\Gamma_0, \ldots, \Gamma_{n-1}$ is any finite, nonempty sequence of finite, nonempty subsets of Fm_{Λ}, then

$$\bigwedge \Gamma_0 \wedge \bigwedge \Gamma_1 \wedge \cdots \wedge \bigwedge \Gamma_{n-2} \wedge \bigwedge \Gamma_{n-1}$$

denotes

$$\bigwedge \Gamma_0 \wedge (\bigwedge \Gamma_1 \wedge (\cdots \wedge (\bigwedge \Gamma_{n-2} \wedge \bigwedge \Gamma_{n-1}) \cdots)).$$

Let $\mathcal{S}$ be a self-extensional and protoalgebraic 1-deductive system with conjunction $x \wedge y$. Let $\Gamma, \Delta \subseteq_{\omega} \mathrm{Fm}_{\Lambda}$. If $\Gamma \dashv\vdash_{\mathcal{S}} \Delta$, then $\bigwedge \Gamma \dashv\vdash_{\mathcal{S}} \bigwedge \Delta$, and hence $\bigwedge \Gamma \equiv \bigwedge \Delta \pmod{\boldsymbol{\Omega}(\mathrm{Thm}\,\mathcal{S})}$ by self-extensionality. Thus $\bigwedge \Gamma \approx \bigwedge \Delta$ is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$ by Cor. 1.22, provided the language type of $\mathcal{S}$ is countable.

Lemma 2.31 *Let $\mathcal{S}$ be a self-extensional and protoalgebraic 1-deductive system with conjunction and with at least one theorem. Assume in addition that the language type of $\mathcal{S}$ is countable. Let $\Delta(x, y)$ be a protoequivalence system for $\mathcal{S}$ and let $E(x, y, \bar{u})$ be the the equivalence system with parameters constructed from $\Delta(x, y)$ in Thm.* 1.18. *Finally, let $x \wedge y$ be a conjunction formula for $\mathcal{S}$.*

(i) $\bigwedge F(y, y) \wedge x \approx x$ *is an identity of* $\mathsf{Alg\,Mod}^* \mathcal{S}$ *for every* $F(y, y) \subseteq_{\omega} \forall \bar{\vartheta}\, E(y, y, \bar{\vartheta})$.

(ii) *For every quasi-identity* $\dfrac{\varphi_0 \approx \psi_0,\ \varphi_1 \approx \psi_1, \ldots, \varphi_{n-1} \approx \psi_{n-1}}{\xi \approx \eta}$ *of* $\mathsf{Alg\,Mod}^* \mathcal{S}$, *there is, for each $i < n$, a finite subset $F_i(\varphi_i, \psi_i)$ of $\forall \bar{\vartheta}\, E(\varphi_i, \psi_i, \bar{\vartheta})$ such that*

$$\begin{aligned} &\bigwedge F_0(\varphi_0, \psi_0) \wedge \cdots \wedge \bigwedge F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \wedge \xi \\ &\qquad \approx \bigwedge F_0(\varphi_0, \psi_0) \wedge \cdots \wedge \bigwedge F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \wedge \eta \end{aligned}$$

is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$.

PROOF. (i). $\vdash_{\mathcal{S}} F(y, y)$ and hence $F(y, y),\, x \dashv\vdash_{\mathcal{S}} x$. Thus $\bigwedge F(y, y) \wedge x \approx x$ is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$ by self-extensionality.

(ii). If $\dfrac{\varphi_0 \approx \psi_0, \ldots, \varphi_{n-1} \approx \psi_{n-1}}{\xi \approx \eta}$ is a quasi-identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$, then by

Thm. 1.21 and $E(x, y, \bar{u})$-detachment,

$$\forall\bar{\vartheta}\, E(\varphi_0, \psi_0, \bar{\vartheta}), \ldots, \forall\bar{\vartheta}\, E(\varphi_{n-1}, \psi_{n-1}, \bar{\vartheta}),\ \xi \\ \dashv\vdash_{\mathcal{S}}\ \forall\bar{\vartheta}\, E(\varphi_0, \psi_0, \bar{\vartheta}), \ldots, \forall\bar{\vartheta}\, E(\varphi_{n-1}, \psi_{n-1}, \bar{\vartheta}),\ \eta.$$

Since $\mathcal{S}$ is finitary, there is, for each $i < n$, a finite subset $F_i(\varphi_i, \psi_i)$ of $\forall\bar{\vartheta}\, E(\varphi_i, \psi_i, \bar{\vartheta})$ such that

$$\bigwedge F_0(\varphi_0, \psi_0) \wedge \cdots \wedge \bigwedge F_{n-1}(\varphi_{n-1}\, \psi_{n-1}) \wedge \xi \\ \dashv\vdash_{\mathcal{S}}\ \bigwedge F_0(\varphi_0, \psi_0) \wedge \cdots \wedge \bigwedge F_{n-1}(\varphi_{n-1}, \psi_{n-1}) \wedge \eta.$$

Thus the equation in (iii) is an identity of $\mathsf{Alg\,Mod}^* \mathcal{S}$ by self-extensionality. □

Theorem 2.32 ([5], Theorem 4.27) *Let $\mathcal{S}$ be a protoalgebraic 1-deductive system with at least one theorem over a countable language type. If $\mathcal{S}$ is self-extensional with conjunction, then* $\mathsf{Qv\,Alg\,Mod}^* \mathcal{S}$ *is a variety.*

PROOF. Similar to the proof of Thm. 2.29 with Lem. 2.31 in place of Lem. 2.28. □

As in the case of self-extensional systems with the uniterm deduction-detachment theorem, the stronger result that $\mathsf{Alg\,Mod}^* \mathcal{S}$ is a variety is obtained in [5, Theorem 4.27], without the restriction on the language type, and also without the requirement that $\mathcal{S}$ be protoalgebraic But as before, in the case $\mathcal{S}$ is algebraizable, the proof of Thm. 2.32 gives an algorithm for actually generating a set of defining identities for the equivalent variety of $\mathcal{S}$ from any given presentation of $\mathcal{S}$ by axioms and rules of inference. We omit the details.

2.5 Strong algebraizability and modal logics

By a *normal modal logic* we will mean a 1-deductive system $\mathcal{M}$ over the language type $\Lambda = \{\wedge, \vee, \neg, \to, \top, \Box\}$ whose set of theorems contains $\Box(x \to y) \to \Box x \to \Box y$, in addition to all classical tautologies, and is closed under modus ponens and necessitation, i.e., the following "nonaccumulative" forms of 2*nd-order modus ponens* and *necessitation* are rules of $\mathcal{M}$.

$$\frac{\dfrac{\emptyset}{x}\ ,\ \dfrac{\emptyset}{x \to y}}{\dfrac{\emptyset}{y}} \quad \text{and} \quad \frac{\dfrac{\emptyset}{x}}{\dfrac{\emptyset}{\Box x}}.$$

The accumulative forms of these sequents are easily seen to be equivalent respectively to the 1st-order rules of modus ponens and necessitation: $\dfrac{x, x \to y}{y}$ and $\dfrac{x}{\Box x}$. Note that the defining properties of normal modal logics involve only the set of theorems and give no information about the consequence relation of the system. Each of the standard modal logics $\mathcal{M}$, thought of in this way simply as a set of formulas, can be presented in two different ways as a deductive system. The "weak" system $\mathcal{M}_{\mathrm{w}}$ has (1st-order) modus ponens as its only proper rule of inference, while the "strong" system $\mathcal{M}_{\mathrm{s}}$ has both modus ponens and the 1st-order rule of necessitation. The weak system, $\mathcal{M}_{\mathrm{w}}$, is self-extensional and has the uniterm deduction-detachment theorem, with the classical implication $x \to y$ as the single deduction-detachment formula. Thus $\mathsf{Qv\,Alg\,Mod}^* \mathcal{M}_{\mathrm{w}}$ is a variety by Thm. 2.29; in fact, as noted, $\mathsf{Alg\,Mod}^* \mathcal{M}_{\mathrm{w}}$ itself is a variety. $\mathcal{M}_{\mathrm{w}}$ is however not in general algebraizable. On the other hand, the strong system $\mathcal{M}_{\mathrm{s}}$ is generally not self-extensional and, while in many cases it has the multiterm and even the uniterm deduction-detachment system, the classical implication is a deduction-detachment formula only in very special cases. $\mathcal{M}_{\mathrm{s}}$ is however always algebraizable. Moreover, we have $\mathsf{Alg\,Mod}^* \mathcal{M}_{\mathrm{s}} = \mathsf{Alg\,Mod}^* \mathcal{M}_{\mathrm{w}}$ (this is shown in [5]). So $\mathcal{M}_{\mathrm{s}}$ is always strongly algebraizable. For a discussion of the algebraizability of the weak and strong systems for the modal logic S5 of Lewis, see [21]. For more detailed discussion of the algebraizability of modal logics see [5] and the additional references cited there.

There are other algebraizable deductive systems $\mathcal{S}$ that, like the normal modal logics, have an associated weak form $\mathcal{S}_{\mathrm{w}}$ that is self-extensional with the uniterm deduction-detachment theorem, or that is protoalgebraic and self-extensional with conjunction, and such that the equivalent quasivariety of $\mathcal{S}$ coincides with $\mathsf{Alg\,Mod}^* \mathcal{S}_{\mathrm{w}}$. The strong algebraizability of $\mathcal{S}$ then follows from either [5, Theorem 4.27] or [5, Theorem 4.45]. The first-order predicate logic has this property, after first being reformulated as a deductive system in the sense of this paper; see [22]. Each of the finitely valued Łukasiewicz logics also has this property ([43]); quantum logic is another example ([44]). There are a number of strongly algebraizable logics however that do not fit this paradigm. Among the so-called substructural logics, the relevance logic R is strongly algebraizable ([45]) but not self-extensional ([46]); it has a weak version WR that is self-extensional but not protoalgebraic. Another example of this kind is the infinitely valued Łukasiewicz logic ([44]). The attempt to find a comprehensive theory explaining the phenomenon of strong algebraizability is an ongoing project.

3 Matrix Semantics for Fregean Deductive Systems

The notion of a full 2nd-order model of a 1-deductive system was introduced by Font and Jansana in [5]. It turns out to be a very useful device for studying the 2nd-order properties of deductive systems, in particular Fregean systems. In this section we show that, if a deductive system is protoalgebraic, any accumulative 2nd-order property, that is a property defined by the accumulative forms of some set of 2nd-order sequents, is inherited by its full 2nd-order models. Something even stronger than the converse holds: any 2nd-order property common to all members of a family of 2nd-order matrices is inherited by the deductive system that they define. This latter result gives us a convenient method for constructing Fregean systems with various properties. As an application we construct a regularly algebraizable Fregean deductive system that is not strongly algebraizable.

We will consider only 1-deductive systems in this section.

Definition 3.1 *A 2nd-order matrix* $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ *is said to be a* (2nd-order) model *of a 1-deductive system* $\mathcal{S}$ *if* $\mathcal{C} \subseteq \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$*, i.e.,* $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}\, \mathcal{S}$ *for each* $F \in \mathcal{C}$*. By a* full (2nd-order) model *of* $\mathcal{S}$ *we mean a 2nd-order model that is reduction-isomorphic to one of the form* $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A} \rangle$*; full models of the latter kind are called* basic full models*.* $\square$

A reduced full 2nd-order model of $\mathcal{S}$ is always basic. Thus $\mathfrak{A}$ is a full model of $\mathcal{S}$ iff $\langle \boldsymbol{B}, \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{B} \rangle \preccurlyeq \mathfrak{A}$ for some Λ-algebra $\boldsymbol{B}$.

For the purpose of proving the next theorem we need to consider formula algebras over uncountable sets of variable symbols. For each infinite cardinal α, let $\mathrm{Fm}_{\Lambda,\alpha}$ be the set of formulas over a set Va_α of α variable symbols, and let $\mathbf{Fm}_{\Lambda,\alpha}$ be the corresponding formula algebra. We assume $\mathrm{Va}_{\boldsymbol{\omega}} = \mathrm{Va}$ and thus $\mathbf{Fm}_{\Lambda,\boldsymbol{\omega}} = \mathbf{Fm}_{\Lambda}$. The only property of $\mathbf{Fm}_{\Lambda,\alpha}$ we use is that it is absolutely freely generated by Va_α, i.e., for every Λ-algebra $\boldsymbol{A}$ and every mapping $h\colon \mathrm{Va}_\alpha \to A$, h has a unique extension extension to a homomorphism $h^*\colon \mathbf{Fm}_{\Lambda,\alpha} \to \boldsymbol{A}$.

Lemma 3.2 *Let* $\mathcal{S}$ *be 1-deductive system and let* α *be an infinite cardinal. A 2nd-order sequent is valid in* $\mathfrak{F}_{\Lambda,\alpha} = \langle \mathbf{Fm}_{\Lambda,\alpha},\, \mathrm{Fi}_{\mathcal{S}}\, \mathbf{Fm}_{\Lambda,\alpha} \rangle$ *iff it is a 2nd-order rule of* $\mathcal{S}$*.*

PROOF. Consider any denumerable subset X of Va_α and let $\mathbf{Fm}_{\Lambda,\alpha}(X)$ be the subalgebra of $\mathbf{Fm}_{\Lambda,\alpha}$ generated by X. We claim that the submatrix

$$\mathfrak{Fm}_{\Lambda,\alpha}(X) = \Big\langle \mathbf{Fm}_{\Lambda,\alpha}(X),\ \{\, F \cap \mathrm{Fm}_{\Lambda,\alpha}(X) : F \in \mathrm{Fi}_{\mathcal{S}}\, \mathbf{Fm}_{\Lambda,\alpha} \,\} \Big\rangle$$

of $\mathfrak{Fm}_{\Lambda,\alpha}(X)$ is isomorphic to $\mathcal{S}$ (as the 2nd-order matrix $\langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S}\rangle$. Clearly $\langle \mathbf{Fm}_{\Lambda,\alpha}(X),\ \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}(X)\rangle \cong \mathcal{S}$ and

$$\mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}(X) \supseteq \{\, F \cap \mathrm{Fm}_{\Lambda,\alpha}(X) : F \in \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}\,\}.$$

So we only have to show the inclusion in the opposite direction. Let $F \in \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}(X)$ and let $r\colon \mathbf{Fm}_{\Lambda,\alpha} \twoheadrightarrow \mathbf{Fm}_{\Lambda,\alpha}(X)$ be a retraction, i.e., a surjective homomorphism such that $r \circ r = r$; a retraction exists because $\mathbf{Fm}_{\Lambda,\alpha}$ is absolutely freely generated Va_α. Thus $F \subseteq r^{-1}(F)$ and hence $F = r^{-1}(F) \cap \mathbf{Fm}_{\Lambda,\alpha}(X)$. Since $r^{-1}(F) \in \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}$, this establishes the claim. It follows immediately that a 2nd-order sequent is valid in $\mathfrak{Fm}_{\Lambda,\alpha}(X)$ iff it is a rule of $\mathcal{S}$.

Consider any finitely generated submatrix $\mathfrak{B}$ of $\mathfrak{Fm}_{\Lambda,\alpha}$. Let X be a denumerable subset of Va_α that includes all the variables occurring in the generators of $\mathfrak{B}$. Then $\mathfrak{B} \le \mathfrak{Fm}_{\Lambda,\alpha}(X) \le \mathfrak{Fm}_{\Lambda,\alpha}$ and $\mathfrak{Fm}_{\Lambda,\alpha}(X)$ is countably generated. Furthermore, a 2nd-order sequent is valid in $\mathfrak{Fm}_{\Lambda,\alpha}(X)$ iff it is a rule of $\mathcal{S}$. Thus by Lem. 2.4 the 2nd-order sequents valid in $\mathfrak{Fm}_{\Lambda,\alpha}$ are exactly the 2nd-order rules of $\mathcal{S}$. □

Let P be a 2nd-order property of 2nd-order matrices, i.e., a class of 2nd-order matrices that is defined by some set of 2nd-order sequents. For instance, P can be the property of having the multiterm deduction-detachment theorem with a fixed deduction-detachment system, or the properties of being self-extensional or Fregean. Recall that P is said to be accumulative if it is defined by the accumulative forms of a set of 2nd-order sequents. The deduction-detachment and Fregean properties are accumulative; self-extensionality is not. A 2nd-order property P of a 1-deductive system $\mathcal{S}$ is said to *transfer to full models* if every full 2nd-order model of $\mathcal{S}$ has P.

Theorem 3.3 *Let* P *be any accumulative 2nd-order property, and let* $\mathcal{S}$ *be a 1-deductive system with* P*. If* $\mathcal{S}$ *is protoalgebraic, then* P *transfers to the full models of* $\mathcal{S}$*.*

PROOF. To prove that every full model of $\mathcal{S}$ has P it suffices by Cor. 2.9 to prove that every basic full model of $\mathcal{S}$ has P. Take $\langle \boldsymbol{A}, \mathrm{Fi}_\mathcal{S}\,\boldsymbol{A}\rangle$ to be such a model. Let α be the least infinite cardinal greater or equal to $|A|$, and let $h\colon \mathbf{Fm}_{\Lambda,\alpha} \twoheadrightarrow \boldsymbol{A}$ be a surjective homomorphism. Since $\mathcal{S}$ is protoalgebraic, we have by Lem. 1.15 that

$$h^{-1}(\mathrm{Fi}_\mathcal{S}\,\boldsymbol{A}) = [\, h^{-1}(F_0)\,)_{\mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}},$$

were $F_0 = \bigcap \mathrm{Fi}_\mathcal{S}\,\boldsymbol{A}$, the smallest $\mathcal{S}$-filter on $\boldsymbol{A}$. Thus, by Thm. 2.3, $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{A}}\,\mathcal{S}\rangle \in \mathsf{P}$ iff $\left\langle \mathbf{Fm}_{\Lambda,\alpha},\ [\, h^{-1}(F_0)\,)_{\mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}} \right\rangle \in \mathsf{P}$. But $\langle \mathbf{Fm}_{\Lambda,\alpha},\ \mathrm{Fi}_\mathcal{S}\,\mathbf{Fm}_{\Lambda,\alpha}\rangle \in \mathsf{P}$ by Lem. 3.2 and the assumption $\mathcal{S} = \langle \mathbf{Fm}_{\Lambda,\alpha}, \mathrm{Th}\,\mathcal{S}\rangle \in \mathsf{P}$. Hence we have $\Big\langle \mathbf{Fm}_{\Lambda,\alpha},$

$[\,h^{-1}(F_0)\,)_{\mathrm{Fi}_{\mathcal{S}}\,\mathbf{Fm}_{\Lambda,\alpha}}\big\rangle \in \mathsf{P}$ because P is defined by the accumulative forms of a set of 2nd-order sequents (see the remarks following Lem. 2.14). □

Corollary 3.4 ([36], Theorem 2.2) *The property of having the multiterm (uniterm) deduction-detachment theorem transfers to full models. More precisely, if $\Delta(x, y)$ is a deduction-detachment system for a 1-deductive system $\mathcal{S}$, then it is also a deduction-detachment system for every full model of $\mathcal{S}$.*

PROOF. Assume $\Delta(x, y)$ is a deduction-detachment system for $\mathcal{S}$. Then by Cor. 1.39 $\mathcal{S}$ is protoalgebraic and $\Delta(x, y)$ is a protoequivalence system for $\mathcal{S}$. Then, since the multiterm deduction-detachment theorem is an accumulative 2nd-order property, the conclusion of the corollary follows immediately from Thm. 3.3. □

The following corollary is an improvement of Proposition 3.19 of [5].

Corollary 3.5 *For protoalgebraic 1-deductive systems, the property of being Fregean transfers to full models.* □

For some time it was an open problem if the properties of being Fregean and self-extensional transfer to full models in general. But this has recently been shown to fail in both cases by Babyonyshev [47] and, independently, by Bou [48].

We now turn to what can be viewed as the reverse problem, that is, inferring a 2nd-order property of a deductive system from the assumption that a class of 2nd-order models that defines the system has the property.

Every class of 2nd-order matrices defines a deductive system in the natural way.

Definition 3.6 *Let K be any class of 2nd-order matrices. We denote by $\boldsymbol{\mathcal{S}}\,\mathsf{K}$ the largest 1-deductive system (under set-theoretic inclusion of algebraic closed-set systems on Fm_Λ) such that each matrix in K is a 2nd-order model of $\boldsymbol{\mathcal{S}}\,\mathsf{K}$.*

Alternatively, $\boldsymbol{\mathcal{S}}\,\mathsf{K}$ is defined by the condition that, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_\Lambda$, $\Gamma \vdash_{\boldsymbol{\mathcal{S}}\mathsf{K}} \varphi$ iff there are finitely many $\psi_0, \ldots, \psi_{n-1} \in \Gamma$ such that, for every $\langle \boldsymbol{A}, \mathcal{C}\rangle \in \mathsf{K}$ and every evaluation $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$,

$$h(\varphi) \in \mathrm{Clo}_{\mathcal{C}}\Big\{h(\psi_0), \ldots, h(\psi_{n-1})\Big\}.$$

This is the same finitary deductive system that is determined in the usual way by the class of 1st-order matrices $\big\{\,\langle \boldsymbol{A}, F\rangle : \langle \boldsymbol{A}, \mathcal{C}\rangle \in \mathsf{K}, F \in \mathcal{C}\,\big\}$. $\boldsymbol{\mathcal{S}}\,\mathsf{K}$ is said *to be* finitarily determined by K. $\square$

Theorem 3.7 *Let* P *be any 2nd-order property, and let* K *be any class of 2nd-order matrices. If* $\mathsf{K} \subseteq \mathsf{P}$ *then* $\boldsymbol{\mathcal{S}}\,\mathsf{K} \in \mathsf{P}$.

PROOF. Suppose that the 2nd-order sequent (30) is valid in each matrix of K. Let $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$ be a substitution such that $\sigma(\vartheta_0), \dots, \sigma(\vartheta_{k-1}) \nvdash_{\boldsymbol{\mathcal{S}}\mathsf{K}} \sigma\,\xi$. Then there exists a $\langle \boldsymbol{A}, \mathcal{C} \rangle \in \mathsf{K}$ and an evaluation $h \colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ such that $h(\sigma(\xi)) \notin \mathrm{Clo}_{\mathcal{C}}\big\{h(\sigma(\vartheta_0)), \dots, h(\sigma(\vartheta_{k-1}))\big\}$. Since (30) is valid in $\langle \boldsymbol{A}, \mathcal{C} \rangle$, for some $i < m$, $h(\sigma(\varphi^i)) \notin \mathrm{Clo}_{\mathcal{C}}\big\{h(\sigma(\psi^i_0)), \dots, h(\sigma(\psi^i_{n_i-1}))\big\}$. Thus, for some $i < m$, $\sigma(\psi^i)_0, \dots, \sigma(\psi^i_{n_i-1}) \nvdash_{\boldsymbol{\mathcal{S}}\mathsf{K}} \sigma(\varphi^i)$. Hence the sequent (30) is valid in $\boldsymbol{\mathcal{S}}\,\mathsf{K}$. $\square$

Corollary 3.8 *Let* K *be any class of 2nd-order matrices over* Λ. *If each member of* K *is Fregean, so is* $\boldsymbol{\mathcal{S}}\,\mathsf{K}$. $\square$

Similarly, if each member of a class K of 2nd-order matrices is self-extensional, then so is $\boldsymbol{\mathcal{S}}\,\mathsf{K}$. A closely related result can be found in [3], Theorem 5.6.11.

This corollary can be used to construct examples of Fregean deductive systems with various properties. We will use it now to construct a protoalgebraic Fregean deductive system that is not strongly algebraizable.

Some special terminology will prove to be useful. Let $\mathcal{C}$ be a closed-set system over a nonempty set A. Let $F \in \mathcal{C}$. Elements $a, b \in A$ are said to be *$\mathcal{C}$-separable over F* if there exists a G such that $F \subseteq G \in \mathcal{C}$ and either $a \in G$ and $b \notin G$ or vice versa. Clearly a and b are $\mathcal{C}$-separable over F iff $\langle a, b \rangle \notin \widetilde{\boldsymbol{\Lambda}}_{\mathcal{C}}\,F$. Thus a 2nd-order matrix $\mathfrak{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is Fregean iff, for every $F \in \mathcal{C}$, $a \not\equiv b \pmod{\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\,F}$ implies a and b are $\mathcal{C}$-separable over F.

Let $\big\langle \{0, 1, a, b\},\ +, \cdot, -, 0, 1 \big\rangle$ be the 4-element Boolean algebra. Let $\Lambda = \{\to, \top, \mathbf{a}, \mathbf{b}\}$, where $\mathbf{a}, \mathbf{b}$ are constant symbols, and let $\boldsymbol{A}$ be the Λ-algebra

$$\boldsymbol{A} = \Big\langle \{1, a, b\}, \to^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \mathbf{a}^{\boldsymbol{A}}, \mathbf{b}^{\boldsymbol{A}} \Big\rangle,$$

where $x \to^{\boldsymbol{A}} y = -x + y$ for all $x, y \in \{1, a, b\}$, $\top^{\boldsymbol{A}} = 1$, $\mathbf{a}^{\boldsymbol{A}} = a$, and $\mathbf{b}^{\boldsymbol{A}} = b$. We note that the $\{\to, \top\}$-reduct of $\boldsymbol{A}$, $\Big\langle \{1, a, b\}, \to^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \Big\rangle$, is the $\{\to, \top\}$-subreduct of a Boolean algebra and hence is a Hilbert algebra (in fact a Tarski algebra). In the sequel we omit the superscript on $\to^{\boldsymbol{A}}$. Note that $1 \to x = x$ and $x \to 1 = x \to x = 1$, for all $x \in \{1, a, b\}$. These three equalities completely describe the multiplication table for $\to$ with the two exceptions $a \to b$ and its converse. $a \to b = b$ and $b \to a = a$.

Let $\mathfrak{A}$ be the 2nd-order matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$, where

$$\mathcal{C} := \Big\{ \{1\},\ \{1,b\},\ \{1,a,b\} \Big\}.$$

$\mathcal{C}$ is obviously an (algebraic) closed-set system over $\{1, a, b\}$.

Lemma 3.9 $\mathfrak{A}$ *is Fregean.*

PROOF. We first determine the Leibniz congruence for each of the three filters of $\mathcal{C}$. Since $\boldsymbol{\Omega}_{\boldsymbol{A}}\{1\}$ is compatible with $\{1\}$, if it is not the identity relation, $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}}\{1\}}$. But then $1 = a \to a \equiv a \to b = b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}}\{1\}}$, contradicting compatibility with $\{1\}$. So $\boldsymbol{\Omega}_{\boldsymbol{A}}\{1\} = \mathrm{Id}_A$.

Let $\varPhi$ be the equivalence relation whose partition is $\big\{\{1,b\},\{a\}\big\}$. We verify that $\varPhi$ is a congruence. We first observe that $b \to x = x$ and $x \to b = b$ for all $x \neq b$. If $x \neq b$, then

$$1 \to x = b \to x = x \quad \text{and} \quad x \to 1 = 1 \equiv_{\varPhi} b = x \to b.$$

If $x = b$, then

$$1 \to x = b \equiv_{\varPhi} 1 = b \to x \quad \text{and} \quad x \to 1 = x \to b = 1.$$

So $\varPhi$ is a congruence and is clearly the largest congruence compatible with $\{1, b\}$. So $\boldsymbol{\Omega}_{\boldsymbol{A}}\{1,b\} = \varPhi$. Finally, it is obvious that $\boldsymbol{\Omega}_{\boldsymbol{A}}\{1,a,b\} = A^2$, the universal relation. Thus

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\{1\} &= \boldsymbol{\Omega}_{\boldsymbol{A}}\{1\} \cap \boldsymbol{\Omega}_{\boldsymbol{A}}\{1,b\} \cap \boldsymbol{\Omega}_{\boldsymbol{A}}\{1,a,b\} = \mathrm{Id}_A,\\
\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\{1,a\} &= \boldsymbol{\Omega}_{\boldsymbol{A}}\{1,a\} \cap \boldsymbol{\Omega}_{\boldsymbol{A}}\{1,a,b\} = \varPhi,\\
\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\{1,a,b\} &= \boldsymbol{\Omega}_{\boldsymbol{A}}\{1,a,b\} = A^2.
\end{aligned}$$

The pairs $\langle 1, a\rangle$, $\langle 1, b\rangle$, and $\langle a, b\rangle$ are all clearly $\mathcal{C}$-separable over $\{1\}$. $\langle 1, a\rangle$ is $\mathcal{C}$-separable over $\{1, b\}$ since $a \notin \{1, b\}$. Finally, that each pair $\langle x, y\rangle$ such that $x \not\equiv y \pmod{\widetilde{\boldsymbol{\Omega}}_{\mathfrak{A}}\{1,a,b\}}$ is $\mathcal{C}$-separable over $\{1, a, b\}$ holds vacuously. Thus $\mathfrak{A}$ is Fregean. □

It follows from Cor. 3.8 that the deductive system $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$ (i.e., $\boldsymbol{\mathcal{S}}\{\mathfrak{A}\}$) (finitarily) determined by $\{\mathfrak{A}\}$ is Fregean. $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$ is also clearly protoalgebraic with single protoequivalence formula $x \to y$. So $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$ is regularly algebraizable by Thm. 2.18. Let $\mathsf{Q} = \mathsf{Alg}\,\mathsf{Mod}^{*}\,\boldsymbol{\mathcal{S}}\,\mathfrak{A}$, the equivalent quasivariety of $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$. Q is relatively point-regular and $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$ is its assertional logic by Thm. 1.34. (See also the remark following Thm. 1.34.)

It is a trivial matter to verify that the 1st-order sequent $\frac{\mathbf{a}}{\mathbf{b}}$ is valid in $\mathfrak{A}$, i.e., in each of the three 1st-order matrices $\langle \boldsymbol{A}, \{1\}\rangle$, $\langle \boldsymbol{A}, \{1, b\}\rangle$, and $\langle \boldsymbol{A}, \{1, a, b\}\rangle$. Thus the quasi-equation

$$\frac{\mathbf{a} \approx \top}{\mathbf{b} \approx \top} \tag{57}$$

is a quasi-identity of Q. $\langle \boldsymbol{A}, \{1\}\rangle$ is reduced, so $\boldsymbol{A} \in \mathsf{Q}$. Let $\varTheta$ be the equivalence relation on A whose partition is $\big\{\{1, a\}, \{b\}\big\}$. $\varTheta$ is a congruence on $\boldsymbol{A}$. (It is the image of the congruence $\varPhi$ with partition $\big\{\{1, b\}, \{a\}\big\}$ under the automorphism of the $\{\to, \top\}$-reduct of $\boldsymbol{A}$ that interchanges a and b.) But the quasi-equation (57) is not valid in $\boldsymbol{A}/\varTheta$, so $\boldsymbol{A}/\varTheta \notin \mathsf{Q}$. Hence Q is not a variety and thus $\boldsymbol{\mathcal{S}}\,\mathfrak{A}$ is a regularly algebraizable, Fregean deductive system that is not strongly algebraizable.

The first example of a regularly algebraizable, Fregean deductive system that is not strongly algebraizable was found by P. Idziak; the example just presented is essentially Idziak's and is included here with his kind permission. Subsequently it was discovered that the equivalence/negation fragment of the intuitionistic propositional calculus also has these properties; this result can be extracted without difficulty from Kabziński, Porębska, and Wroński [49]. We briefly outline now how this is done.

Let $\mathrm{IPC}_{\leftrightarrow\neg}$ be the $\{\leftrightarrow, \neg\}$-fragment of the intuitionistic propositional calculus IPC. $\mathrm{IPC}_{\leftrightarrow\neg}$ clearly inherits the property of being Fregean from IPC, and hence it is algebraizable since it is evidently protoalgebraic. A presentation of $\mathrm{IPC}_{\leftrightarrow\neg}$ is obtained from a presentation of the $\{\leftrightarrow\}$-fragment $\mathrm{IPC}_{\leftrightarrow}$ of IPC by adjoining two new axioms and the following rule of inference (see [49, Theorem 2]).

$$\frac{\neg x \leftrightarrow x}{y}. \tag{58}$$

Let $\mathrm{IPC}^{-}_{\leftrightarrow\neg}$ be the deductive system obtained by just adjoining the two axioms but not the rule (58). Then $\mathrm{IPC}^{-}_{\leftrightarrow\neg}$ is an axiomatic extension of $\mathrm{IPC}_{\leftrightarrow}$ and hence is strongly algebraizable because $\mathrm{IPC}_{\leftrightarrow}$ is. $\mathrm{IPC}_{\leftrightarrow\neg}$ is not an axiomatic extension of $\mathrm{IPC}^{-}_{\leftrightarrow\neg}$, i.e., the rule (58) cannot be replaced by any set of axioms (this follows easily from [49, Lemma 7]). Hence $\mathrm{IPC}_{\leftrightarrow\neg}$ is not strongly algebraizable.

We may conclude that the behavior of protoalgebraic, Fregean deductive systems that fail to have the uniterm deduction-detachment theorem differs strikingly from those that do. A protoalgebraic, Fregean deductive system may be either strongly algebraizable or not, but the uniterm deduction-detachment theorem guarantees strong algebraizability in this context by Thm. 2.23. The fact that the deduction-detachment system is uniterm is essential. An example of a Fregean deductive system with the multiterm deduction-detachment theorem that is not strongly algebraizable is given in [6].

In the last part of this section we investigate some of the properties of the equivalent quasivarieties of protoalgebraic Fregean deductive systems.

Definition 3.10 ([19]) *A pointed quasivariety* Q *is* congruence-orderable *if, for every* $\boldsymbol{A} \in \mathsf{Q}$ *and all* $a, b \in A$,

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a, \top^{\boldsymbol{A}}) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(b, \top^{\boldsymbol{A}}) \quad \textit{implies} \quad a = b. \tag{59}$$

Q *is* Fregean *if it is both relatively point-regular and relatively congruence-orderable.* □

Let Q be any pointed quasivariety. Then the condition

$$a \leq^{\boldsymbol{A}}_{\mathsf{Q}} b \quad \text{if} \quad \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a, \top^{\boldsymbol{A}}) \supseteq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(b, \top^{\boldsymbol{A}})$$

obviously defines a quasi-ordering on any Λ-algebra $\boldsymbol{A}$. It is called the Q-*congruence order* on A. Q is congruence-orderable iff its congruence order is a partial order.

Theorem 3.11 *A pointed quasivariety is Fregean iff its assertional logic is protoalgebraic and Fregean, with at least one theorem.*

PROOF. Let $\mathcal{S} = \boldsymbol{\mathcal{S}}^{\text{ASL}}\,\mathsf{Q}$. Assume $\mathcal{S}$ is protoalgebraic and Fregean, with at least one theorem. Then $\mathcal{S}$ is regularly algebraizable by Thm. 2.18, and hence Q is relatively point-regular by Thm. 1.34. Let $\boldsymbol{A} \in \mathsf{Q}$. Then by Thm. 1.37, $\{\top^{\boldsymbol{A}}\} \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and $\boldsymbol{\Omega}_{\boldsymbol{A}}\{\top^{\boldsymbol{A}}\} = \mathrm{Id}_A$. Let $a, b \in A$ and suppose

$$\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a, \top^{\boldsymbol{A}}) = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(b, \top^{\boldsymbol{A}}). \tag{60}$$

By the correspondence between $\mathcal{S}$-filters and $\top^{\boldsymbol{A}}$-congruence classes, (60) is equivalent to the condition that $\langle a, b\rangle \in \boldsymbol{\Lambda}(\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A})$. Since $\mathcal{S}$ is protoalgebraic and Fregean, the 2nd-order matrix $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}\rangle$ is also Fregean by Cor. 3.5. Thus $\boldsymbol{\Lambda}(\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}) = \boldsymbol{\Omega}_{\boldsymbol{A}}(\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}) = \boldsymbol{\Omega}_{\boldsymbol{A}}\{\top^{\boldsymbol{A}}\} = \mathrm{Id}_A$. So $a = b$, and hence Q is congruence-orderable.

Conversely, assume that Q is relatively point-regular and congruence-orderable, i.e., that $\boldsymbol{\Lambda}(\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}) = \mathrm{Id}_A$ for every $\boldsymbol{A} \in \mathsf{Q}$. Consider any $\boldsymbol{A} \in \mathsf{Q}$. Then $\boldsymbol{\Lambda}(\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}}\,F) = \mathrm{Id}_{A/\boldsymbol{\Omega}_{\boldsymbol{A}}\,F}$ for each $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$. So by Lem. 2.7, $\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}}\,F = \boldsymbol{\Omega}_{\boldsymbol{A}}\,F$ for every $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$. This shows that $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}\rangle$ is a Fregean 2nd-order matrix for every $\boldsymbol{A} \in \mathsf{Q}$. Since $\mathcal{S}$ is clearly the deductive system (finitarily) determined by this class of 2nd-order matrices, we conclude by Cor. 3.8 that $\mathcal{S}$ is Fregean. □

A pointed variety is defined in [16] to be Fregean if it is congruence-orderable. In light of this theorem it seems more appropriate to reserve the term *Fregean* for congruence-orderable quasivarieties that are relatively point-regular.

A nontrivial member of a quasivariety Q is said to be *relatively subdirectly irreducible* if it cannot be isomorphically represented as a subdirect product of any system of members of Q unless it is isomorphic to at least one of the factors. It can be shown that every algebra in Q is isomorphic to a subdirect product of a system of relatively subdirectly irreducible members of Q, and that $\boldsymbol{A} \in \mathsf{Q}$ is relatively subdirectly irreducible iff the set of all nonidentity Q-congruences of $\boldsymbol{A}$ has a smallest member. The following characterization of the relatively subdirectly irreducible members of a Fregean quasivariety was obtained independently in [29] and [19].

Theorem 3.12 *Let Q be a Fregean quasivariety, and let $\boldsymbol{A} \in \mathsf{Q}$. Then $\boldsymbol{A}$ is relatively subdirectly irreducible iff there exists an element $\star \in A \setminus \{\top^{\boldsymbol{A}}\}$ such that $a \leq^{\boldsymbol{A}}_{\mathsf{Q}} \star$ for all $a \in A \setminus \{\top^{\boldsymbol{A}}\}$.*

PROOF. Suppose $\boldsymbol{A} \in \mathsf{Q}$ is relatively subdirectly irreducible. There exists a smallest nontrivial Q-congruence Θ on $\boldsymbol{A}$. Since Q is relatively point-regular, we have $\star \equiv \top^{\boldsymbol{A}} \pmod{\Theta}$ for some element $\star$ of $A \setminus \{\top^{\boldsymbol{A}}\}$. Clearly $\Theta = \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(\star, \top^{\boldsymbol{A}})$. Then, for every $a \in A \setminus \{\top^{\boldsymbol{A}}\}$, we have $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a, \top^{\boldsymbol{A}}) \supseteq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(\star, \top^{\boldsymbol{A}})$. So $a \leq^{\boldsymbol{A}}_{\mathsf{Q}} \star$.

Conversely, suppose $\star$ is the largest element of $A \setminus \{\top^{\boldsymbol{A}}\}$ under the Q-congruence order, i.e., $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(a, \top^{\boldsymbol{A}}) \supseteq \mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(\star, \top^{\boldsymbol{A}})$ for every $a \in A \setminus \{\top^{\boldsymbol{A}}\}$. Then $\mathrm{Cg}^{\boldsymbol{A}}_{\mathsf{Q}}(\star, \top^{\boldsymbol{A}})$ is the smallest nontrivial congruence of $\boldsymbol{A}$. □

The algebras of Q which satisfy the condition of Thm. 3.12 are customarily called *strongly compact* in the sense of Q.

Corollary 3.13 *Let Q be a relatively point-regular and congruence-orderable quasivariety, and let $\boldsymbol{A} \in \mathsf{Q}$ be strongly compact (in the sense of Q). Then the $(\boldsymbol{\mathcal{S}}^{\mathrm{ASL}}\,\mathsf{Q})$-filter on $\boldsymbol{A}$ generated by $\star$ is $\{\star, \top^{\boldsymbol{A}}\}$.* □

Thm 3.12 shows that the familiar and useful characterization of subdirectly irreducible Heyting algebras, in terms of their natural order, derives from the fact that the intuitionistic propositional calculus, the assertional logic of Heyting algebras, is Fregean.

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