ABSTRACT ALGEBRAIC LOGIC AND THE DEDUCTION THEOREM

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Contents

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1. Introduction

In this paper we outline the basic part of the theory of abstract algebraic logic as developed originally in [13, 14] and apply it to an algebraic study of the deduction theorem. In contrast to traditional algebraic logic, where the focus is on the algebraic forms of specific logical systems, we are concerned here with the process of algebraization itself and how it can be applied to a wide class of logical systems that includes among others all sentential logics and equational logics.

The original motivation for this work was an attempt to find the proper framework in which to study general properties of the deduction theorem from an algebraic point-of-view. The deduction theorem is the formal expression of one of the most important and useful principles of classical logic: to prove that an implication holds between propositions it suffices to give a proof of the conclusion on the basis of the assumption of the antecedent. It is such a familiar part of ordinary logical argumentation that it is hardly recognizable as being something whose use might be problematic. But in fact it is not part of the usual formalizations of the classical propositional logic and must be proved as a metatheorem. Moreover, while the principle remains valid for intuitionistic logic, it is known to fail in other important logics, for instance, certain systems of modal logic. It also fails in the most common formalization of first-order predicate logic, although a weaker version holds that is often not distinguished from the deduction theorem in its usual form. It turns out that there is a close connection between the deduction theorem and the universal algebraic notion of definable principal congruence relations. The abstract theory of algebraizability outlined here was developed in part to provide the proper context in which to formalize this connection precisely. The ultimate goal was to be able to apply the extensive work on the definability of principal congruence relations in universal algebra to answer some important questions about the validity of the deduction theorem in a variety of logical systems.

To motivate our abstract theory of algebraization we consider in Section 2 several different logics whose algebraic counterparts are well-known and whose algebraic metatheory constitute the core of classical algebraic logic. These include the classical and intuitionistic sentential logics as well as two modal systems: the system of Kripke and Gödel's formulation of Lewis's system S5. The construction of the algebra counterpart of each of these logics follows the same pattern: An equivalence relation on formulas is associated with each theory T of the logic. Two formulas φ and ψ are equivalent if there is a proof of the biconditional $\varphi \leftrightarrow \psi$ from the logical axioms and rules of inference of the logic together with the nonlogical axioms that define T. This equivalence relation respects the logical connectives; for example, if φ and ψ are equivalent respectively to φ' and ψ', then $\varphi \wedge \psi$ is equivalent to $\varphi' \wedge \psi'$. Consequently, a quotient algebra can be formed whose elements are equivalence classes of formulas and whose operations are induced by the logical connectives. This classical "Lindenbaum-Tarski process" associates a class of algebras with each logic: Boolean algebras in the case of classical propositional logic, and Heyting algebras, modal algebras, and monadic algebras respectively for the intuitionistic propositional logic, Kripke's modal logic, and Gödel's version of S5. Each class of algebras can be viewed as the algebra counterpart of its corresponding logic in the sense that there is a close correspondence between the deductive theory of the logic and the equational theory of the algebras. (We mean "equational theory" in a somewhat more general sense than it is normally meant in algebra. We will be concerned with the quasi-identities that a class of algebras satisfies rather than just the identities.)

There are logics for which the classical Lindenbaum-Tarski process does not seem to work, and as an example we consider another formalization of S5, due to Carnap. In this case the equivalence relation between formulas defined by the biconditional $\leftrightarrow$ does not preserve the necessity connective $\Box$. It is not clear however if the failure of the classical Lindenbaum-Tarski process to produce an algebra counterpart is due to some inherent deficiency of the logic, or if there is a generalization of the process that will work. It is partly in an attempt to answer questions of this kind that the theory of abstract algebraic logic was developed, and it can be viewed as an attempt to abstract the Lindenbaum-Tarski process.

The logics considered in Section 2 are all examples of what we call 1-*dimensional deductive systems* (1-*deductive systems* for short). Roughly speaking these correspond to sentential logics with a "Hilbert-style" axiomatization. They encompass a wide class of logical systems, many of which are not normally conceived of as sentential logics. For example the first-order predicate logic can be viewed this way when appropriately formalized. Equational logic cannot be naturally formalized as a 1-deductive system. For the purpose of providing a context in which 1-deductive systems and equational logic can be treated uniformly, we introduce in Section 3 the notion of a k-*dimensional deductive* (k-*deductive*) *system* for each nonzero natural number k and discuss its semantics. The equational logic of every quasivariety of algebras can be naturally formalized as a 2-deductive system. We call such systems *algebraic*. k-dimensional deductive systems are sufficiently general to encompass not only all 1-deductive and equational logics, but also a class of logics that are closely related to "Gentzen-style" formalizations.

The k-deductive systems provide a natural universe in which to study abstract algebraic logic, and, in particular, to define precisely what it means for an arbitrary deductive system to be *algebraizable*, that is, to have a natural algebra counterpart. The definition is given in Section 4 and is based on a general notion of *equivalence* of arbitrary deductive systems that is motivated by the case studies in Sections 2 and 3. A k-dimensional deductive system and an l-dimensional deductive system are said to be *equivalent* if the consequence relation of each is formally definable in that of the other, and the two definitions are inverses of one another in a natural sense. We then define a k-deductive system $\mathcal{S}$ to be *algebraizable* if it is equivalent in this sense to an algebraic deductive system, i.e., the formalization of the equational logic of some quasivariety K as a 2-deductive system. The quasivariety K is taken to be the natural algebra counterpart of $\mathcal{S}$. Moreover, we prove that there can be at most one quasivariety in this relation to $\mathcal{S}$, so that the algebra counterpart of a k-deductive system, if it exists, is an intrinsic property of the system. Our definition of being algebraizable is justified by showing that each of the deductive systems considered in Sections 2 and 3, that we knew beforehand to have algebra counterparts, are indeed algebraizable in this sense.

One of the most important aspects of traditional algebraic logic is the way in which it can be used to reformulate metalogical properties of a particular logical system in algebraic terms. Abstract algebraic logic allows us to study such connections in a general context. Known metalogical or algebraic results can then be applied to obtain new results in the other domain. In Section 3 we introduce an abstract version of the deduction theorem applicable to arbitrary k-dimensional deductive systems. In the special case of an algebraic 2-deductive system, i.e., the equational logic of a quasivariety K, the abstract deduction theorem turns out to be equivalent to a property of K that is well-known in universal algebra: that of having *equationally definable principal relative congruences* (EDPRC). In Section 4 we prove that the abstract deduction theorem is preserved under equivalence of deductive systems. Thus an algebraizable k-deductive system has the deduction theorem if and only if its algebra counterpart has EDPRC.

In Section 5 a number of algebraic consequences of EDPRC for quasivarieties of algebras are surveyed and then applied to obtain new results about the deduction theorem. In particular, this method is used to decide, for several well-known logical systems, whether or not they have the abstract deduction theorem.

The principal tool that we use in our investigations of the algebraic character of a k-deductive system $\mathcal{S}$ is the *Leibniz operator* $\boldsymbol{\Omega}_{\mathcal{S}}$. This is the operator that associates with each theory T of $\mathcal{S}$ the largest congruence $\boldsymbol{\Omega}_{\mathcal{S}} T$ on the algebra of formulas with the property that it does not identify any formula in T with one outside of T. (The term "Leibniz operator" was introduced in [13] where it was justified by the obvious connection with Leibniz's well known definition of equality.) We take the Leibniz operator to be the appropriate abstraction of the classical Lindenbaum-Tarski process, and indeed the two coincide for all those deductive systems of Sections 2 and 3 to which the classical Lindenbaum-Tarski process applies. The Leibniz operator is used in Section 4 to obtain a characterization of algebraizability that is intrinsic in the sense that it does not depend on a priori knowledge of either the algebraic counterpart of the system, or of the special formulas that serve to establish the equivalence between the consequence relation of the system and that of its of its algebraic counterpart. For this reason the Leibniz operator is especially useful for proving that a given deductive system fails to be algebraizable; for example it is used to show that Carnap's version of S5 is not algebraizable.

The results on the deduction theorem were first obtained in the early eighties, and were reported on in the manuscripts "the algebraization of logic" (1983) and "the deduction theorem in algebraic logic" (1988), neither of which appeared in print. The second manuscript circulated widely in the form of the 1991 preprint [14] and has been referred to extensively by a number of authors including ourselves. The paper [15] was originally intended as a sequel to "the deduction theorem in algebraic logic" and contains many references to it. The specific results from Sections 1–5 of "the deduction theorem in algebraic logic" referred to in [15] can all be found in the present paper.

1.1. **Connections with other work.** We focus in this paper on the extension to k-deductive systems of the original notion of abstract algebraizability that is presented in [13] for 1-deductive systems. Abstract algebraic logic now comprises a much wider body of work that is not treated in any detail here. In particular, the notion of algebraizability that we adapt from [13] is only one of many that have been put forward in the literature. It has certain properties that can be used to justify our view that it is "first among equals" in this class, but it is impossible to assert that it is the unique proper notion of algebraizability. For example, although, as mentioned above, Carnap's version of S5 is not algebraizable in the sense of this paper, we shall see from the discussion in Section 2 that a reasonable case can be made for its algebraizability in a much wider sense.

At appropriate places in the paper, most often in the subsection of notes that are appended to each section, we discuss briefly other notions of algebraizability, with references, and attempt to place them in the context of the paper. The same applies to the extensive literature on what could be called "subalgebraizable" deductive systems. These are systems that most investigators in abstract algebraic logic would agree are not algebraizable in any reasonable sense, but with properties of a definitely algebraic nature that sets them apart from arbitrary deductive systems. The general *equivalential* and *protoalgebraic* systems discussed in Section 4 fall in this category. The excellent monograph of J. Czelakowski [32] provides a comprehensive exposition of abstract algebraic logic in this more general setting.

Finally, we want to call attention to the fact that an abstract version of the Lindenbaum-Tarski process that greatly predates ours was studied by H. Rasiowa and her collaborators; the book [120] contains a systematic exposition. Some information on the connection between the two notions can be found in the notes for Section 4.

2. Deductive Systems

In this section we define a deductive system and give several examples, some familiar and some not so familiar. We discuss the algebraic semantics of each system and the question of whether or not the deduction theorem holds. We also consider a deductive system, Carnap's version of Lewis's S5, which does not seem to have an algebraic semantics in the usual sense. At the end of the section we introduce the more general notion of *matrix semantics* that proves to be adequate for any deductive system.

2.1. **Preliminaries.** A *sentential language* consists of the set of all formulas built from a countably infinite set Va of sentential variables and a set $\mathcal{L} = \{\, \omega_i : i \in I \,\}$ of finitary connectives. To each symbol in $\mathcal{L}$ is assigned a natural number called its *arity*. $\mathcal{L}$ is called the *type* of the language. In the sequel it is assumed to be fixed but arbitrary.

The sentential language over $\mathcal{L}$ will be denoted by $\mathrm{Fm}_{\mathcal{L}}$, or often by just Fm. The elements of this set are referred to as *$\mathcal{L}$-formulas*, or simply *formulas* when $\mathcal{L}$ is clear from context. Formulas are finite strings of sentential variables and connectives and are defined recursively in the usual way: every variable and connective of rank 0 (i.e., constant symbol) is a formula, and if $\varphi_0, \dots, \varphi_{n-1}$ are formulas and ω is a n-ary connective, then $\omega\varphi_0 \cdots \varphi_{n-1}$ is a formula. In the development of the theory we are thus using the so-called *Polish prefix* notation which obviates the need for punctuation symbols and parentheses to ensure that formulas can be parsed in only one way. In the examples we will revert to the customary notation, using familiar symbols, infix notation and concomitant parentheses.

For $\varphi \in \mathrm{Fm}$ we write $\varphi = \varphi(p_0, \dots, p_{n-1})$ to indicate that the variables of φ are all included in the list $p_0, \dots, p_{n-1}$. Let $\varphi(p_0, \dots, p_{n-1}) \in \mathrm{Fm}$, and let $\psi_0, \dots, \psi_{n-1} \in \mathrm{Fm}$. Then $\varphi(\psi_0, \dots, \psi_{n-1})$ will denote the formula that results from φ by simultaneously substituting, for all $i < n$, ψ_i for all occurrences of p_i in φ. In the sequel we frequently write $\bar{p}$ for the sequence of variables $p_0, \dots, p_{n-1}$ and $\varphi(\bar{p})$ for $\varphi(p_0, \dots, p_{n-1})$. By an *assignment* we mean any mapping $\sigma : \mathrm{Va} \to \mathrm{Fm}_{\mathcal{L}}$ of variables to formulas. σ extends naturally to a map from Fm to itself, also denoted by σ, by setting $\sigma\big(\varphi(p_0, \dots, p_{n-1})\big) = \varphi(\sigma p_0, \dots, \sigma p_{n-1})$. σ is called a *substitution*. By a *substitution instance* of a formula φ we mean a formula of the form $\sigma\varphi$ where σ is any substitution.

2.1.1. *Consequence relations.* By a (*finitary*) *inference rule* over $\mathcal{L}$ we mean any pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas and φ is a single formula. An *axiom* is a rule of the form $\langle \emptyset, \varphi \rangle$, which is usually written as simply φ; an inference rule that is not an axiom is said to be *strict*. We will represent a rule $\langle \Gamma, \varphi \rangle$ pictorially as $\dfrac{\Gamma}{\varphi}$. A formula ψ is *directly derivable* from a set Δ of formulas by the rule $\langle \Gamma, \varphi \rangle$ if there is a substitution σ such that $\sigma\varphi = \psi$ and $\sigma(\Gamma) \subseteq \Delta$ $\big(\sigma(\Gamma) = \{\sigma\theta : \theta \in \Gamma\}\big)$. A *deductive system* $\mathcal{S}$ (*over* $\mathcal{L}$) is defined by a (possibly infinite) set of inference rules and axioms; it consists of the pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is the relation between sets of formulas and individual formulas defined by the following condition: $\Gamma \vdash_{\mathcal{S}} \varphi$ iff (i.e., if and only if) φ is contained in the smallest set of formulas that includes Γ together with all substitution instances of the axioms of $\mathcal{S}$, and is closed under direct derivability by the inference rules of $\mathcal{S}$. We write $\vdash_{\mathcal{S}} \varphi$ for $\emptyset \vdash_{\mathcal{S}} \varphi$. An $\mathcal{S}$*-derivation* of φ from Γ is a finite sequence $\vartheta_0, \dots, \vartheta_{n-1}$ of formulas such that $\vartheta_{n-1} = \varphi$ and, for each $i < n$, ϑ_i is either a member of Γ, a substitution instance of an axiom, or is directly derivable from $\{\vartheta_0, \dots, \vartheta_{i-1}\}$. An $\mathcal{S}$-derivation from $\emptyset$ is called an $\mathcal{S}$*-proof*. Clearly, $\Gamma \vdash_{\mathcal{S}} \varphi$ iff there is an $\mathcal{S}$-derivation of φ from Γ, and $\vdash_{\mathcal{S}} \varphi$ iff there is an $\mathcal{S}$-proof of φ.

In informal remarks we often refer to a deductive system as a *logical system* or simply a *logic*. The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation* of $\mathcal{S}$. It is easily seen to satisfy the following three conditions for all $\Gamma, \Delta \subseteq \mathrm{Fm}$ and $\varphi \in \mathrm{Fm}$.

(i) $\varphi \in \Gamma$ implies $\Gamma \vdash_{\mathcal{S}} \varphi$;
(ii) $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_{\mathcal{S}} \varphi$;
(iii) $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Delta \vdash_{\mathcal{S}} \psi$ for every $\psi \in \Gamma$ implies $\Delta \vdash_{\mathcal{S}} \varphi$.

In addition, $\vdash_{\mathcal{S}}$ is *finitary* in the sense

(iv) $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma' \vdash_{\mathcal{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$,

and it is *structural* in the sense

(v) $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma\varphi$ for every substitution σ.

Conversely, every relation satisfying conditions (i)–(v) is the consequence relation for some deductive system $\mathcal{S}$, i.e., is defined by some set of axioms and rules of inference ([99]). Consequently, in the sequel, by a deductive system we will mean a pair $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a relation between the powerset of $\mathrm{Fm}_{\mathcal{L}}$ and $\mathrm{Fm}_{\mathcal{L}}$ that satisfies conditions (i)–(v); specific set axioms and rules of inference is not assumed. A particular selection of axioms and inference rules that defines a deductive system $\mathcal{S}$ is called a *presentation* of $\mathcal{S}$. Of course, a given deductive system may have many different presentations.

Let $\mathcal{S}$ be a deductive system. If $\varphi \in \mathrm{Fm}$, then φ is called a *theorem* of $\mathcal{S}$ if $\vdash_{\mathcal{S}} \varphi$; by the structurality of $\mathcal{S}$ the set of theorems is closed under substitution. If $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ and Γ is finite, then $\langle \Gamma, \varphi \rangle$ is called a *derived inference rule* of $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$. A set $T \subseteq \mathrm{Fm}$ is called an $\mathcal{S}$-*theory* if $T \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in T$, for all $\varphi \in \mathrm{Fm}$. Observe that the theorems of $\mathcal{S}$ belong to every $\mathcal{S}$-theory.

See Note 2.1 for historical information about deductive systems.

2.1.2. *Algebras and equational logic.* The type $\mathcal{L}$ of our sentential language will also be viewed as the type of an algebraic, or equational, language in which the connectives ω_i are operation symbols. Sentential variables and $\mathcal{L}$-formulas coincide respectively with what are normally called individual variables and terms of the equational language. In an algebraic context we usually refer to elements of Va simply as *variables* and represent them by $x, y, z, \ldots$; also, we occasionally refer to elements of Fm as *terms*. The special formulas of first-order predicate logic of the form

$$\varphi \approx \psi \quad \text{and} \quad \eta_0 \approx \vartheta_0 \,\&\, \cdots \,\&\, \eta_{n-1} \approx \vartheta_{n-1} \Longrightarrow \varphi \approx \psi,$$

where $\eta_0, \vartheta_0, \ldots, \eta_{n-1}, \vartheta_{n-1}, \varphi, \psi \in \mathrm{Fm}$ are called respectively $\mathcal{L}$-*equations* and $\mathcal{L}$-*quasi-equations*; as usual the prefix $\mathcal{L}$ is normally omitted. Equations and quasi-equations play roughly the same role in equational logic that formulas and inference rules play in deductive systems. This connection will be made more precise when we consider k-deductive systems in Sec. 3.

By an $\mathcal{L}$-*algebra* (in the universal algebraic sense, cf. [103]) we mean a structure $\boldsymbol{A} = \big\langle A, \langle \omega^{\boldsymbol{A}} : \omega \in \mathcal{L} \rangle \big\rangle$ where A is a nonempty set, called the *universe* of $\boldsymbol{A}$, and $\omega^{\boldsymbol{A}}$ is an operation on A of rank k for each connective ω of rank k, i.e., a mapping $\omega^{\boldsymbol{A}} : A^k \to A$. We say simply that $\boldsymbol{A}$ is an *algebra* when $\mathcal{L}$ is clear. Let $\varphi(p_0, \ldots, p_{n-1}) \in \mathrm{Fm}$, and let $\boldsymbol{A}$ be an algebra and $a_0, \ldots, a_{n-1} \in A$. Then $\varphi^{\boldsymbol{A}}(a_0, \ldots, a_{n-1})$ denotes the interpretation of φ in $\boldsymbol{A}$ when $p_0, \ldots, p_{n-1}$ are interpreted respectively as $a_0, \ldots, a_{n-1}$. We often write $\bar{a}$ for $a_0, \ldots, a_{n-1}$ and $\varphi^{\boldsymbol{A}}(\bar{a})$ for $\varphi^{\boldsymbol{A}}(a_0, \ldots, a_{n-1})$. $\varphi^{\boldsymbol{A}}(\bar{a})$ is defined recursively on the structure of φ in the natural way: if $\varphi = p_i$, then $\varphi^{\boldsymbol{A}} = a_i$; if $\varphi = \omega$, ω a constant symbol, then $\varphi^{\boldsymbol{A}}(\bar{a}) = \omega^{\boldsymbol{A}}$; if $\varphi = \omega\psi_0 \cdots \psi_{n-1}$, then $\varphi^{\boldsymbol{A}}(\bar{a}) = \omega^{\boldsymbol{A}}\big(\psi_0^{\boldsymbol{A}}(\bar{a}), \ldots, \psi_{n-1}^{\boldsymbol{A}}(\bar{a})\big)$.

Let $\varphi \approx \psi$ be an equation and Δ a set of equations. Let $\bar{p} = p_0, p_1, p_2, \ldots$ be a list (possibly infinite) of the variables that occur in at least one of the equations in $\Delta \cup \{\varphi \approx \psi\}$. Finally, let $\boldsymbol{A}$ be an algebra. We write $\Delta \vDash_{\boldsymbol{A}} \varphi \approx \psi$ if, for all $\bar{a} = a_0, a_1, a_2, \dots \in A$,

$$\eta^{\boldsymbol{A}}(\bar{a}) = \vartheta^{\boldsymbol{A}}(\bar{a}) \text{ for every } \eta \approx \vartheta \in \Delta \text{ implies } \varphi^{\boldsymbol{A}}(\bar{a}) = \psi^{\boldsymbol{A}}(\bar{a}).$$

We write $\vDash_{\boldsymbol{A}} \varphi \approx \psi$ for $\Delta \vDash_{\boldsymbol{A}} \varphi \approx \psi$ if $\Delta = \emptyset$. If K is a class of algebras, then we write $\Delta \vDash_{\mathsf{K}} \varphi \approx \psi$ if $\Delta \vDash_{\boldsymbol{A}} \varphi \approx \psi$ for all $\boldsymbol{A} \in \mathsf{K}$, and we write $\vDash_{\mathsf{K}} \varphi \approx \psi$ if $\vDash_{\boldsymbol{A}} \varphi \approx \psi$ for all $\boldsymbol{A} \in \mathsf{K}$. An equation $\varphi \approx \psi$ is an *identity* of $\boldsymbol{A}$ if $\vDash_{\boldsymbol{A}} \varphi \approx \psi$. Similarly, a quasi-equation $\eta_0 \approx \vartheta_0 \,\&\, \cdots \,\&\, \eta_{n-1} \approx \vartheta_{n-1} \Longrightarrow \varphi \approx \psi$ is a *quasi-identity* of $\boldsymbol{A}$ if $\eta_0 \approx \vartheta_0, \ldots, \eta_{n-1} \approx \vartheta_{n-1} \vDash_{\boldsymbol{A}} \varphi \approx \psi$. The class of all algebras that satisfy a given set of equations and quasi-equations respectively is called a *variety* and a *quasivariety*. A variety (quasivariety) is *trivial* if it contains only one-element algebras. We assume the reader is familiar with the basic notions of universal algebra: subalgebra, homomorphism, congruence, quotient algebra, and direct product. Some knowledge of free algebras and ultraproducts is also helpful but not essential; ultraproducts are used only at two points in Section 5. As reference to these topics we recommend [22], [74], or [103].

The class of all subalgebras, isomorphic images, homomorphic images, isomorphic images of direct products, and isomorphic images of ultraproducts of members of an arbitrary class K of algebras of type $\mathcal{L}$ is denoted respectively by $\mathbf{S}\mathsf{K}$, $\mathbf{I}\mathsf{K}$, $\mathbf{H}\mathsf{K}$, $\mathbf{P}\mathsf{K}$, $\mathbf{P}_{\mathrm{U}}\mathsf{K}$.

Theorem 2.1 (Birkhoff)**.** *A class* K *of algebras of the same language type is a variety iff it is closed under the formation of homomorphic images, subalgebras, and direct products, i.e.,* $\mathbf{H}\,\mathsf{K}, \mathbf{S}\,\mathsf{K}, \mathbf{P}\,\mathsf{K} \subseteq \mathsf{K}$*. Thus* K *is a variety iff* $\mathbf{HSP}\,\mathsf{K} = \mathsf{K}$*.*

The proof can be found in any textbook on universal algebra, for example [22].

Theorem 2.2 (Mal'cev)**.** K *is a quasivariety iff it is closed under the formation of subalgebras, products, and ultraproducts, i.e.,* $\mathbf{S}\,\mathsf{K}, \mathbf{P}\,\mathsf{K}, \mathbf{P}_{\mathrm{U}}\mathsf{K} \subseteq \mathsf{K}$*. Thus* K *is a quasivariety iff* $\mathbf{SPP}_{\mathrm{U}}\mathsf{K} = \mathsf{K}$*.*

This particular form of the theorem appears in [75]. A proof is given in [22].

It follows from Birkhoff's theorem that if K is any class of algebras of type $\mathcal{L}$, then $\mathbf{HSP}\mathsf{K}$ is the variety generated by K, i.e., the class of all algebras that satisfy every identity satisfied by all members of K. Similarly, as a corollary of Mal'cev's theorem we have that $\mathbf{SPP}_{\mathrm{U}}\mathsf{K}$ is the quasivariety generated by K, i.e., the class of all algebras that satisfy every quasi-identity satisfied by every member of K. If K is finite, then the quasivariety generated by K is $\mathbf{SP}\mathsf{K}$, since $\mathbf{P}_{\mathrm{U}}\mathsf{K} = \mathbf{I}\mathsf{K}$ in this case.

An algebra $\boldsymbol{A}$ is *generated* by a subset C of A if for every $a \in A$ there is a $\varphi(p_0, \ldots, p_{n-1}) \in \mathrm{Fm}$ and $\bar{c} \in C^n$ such that $a = \varphi^{\boldsymbol{A}}(\bar{c})$. Let K be a class of algebras (of type $\mathcal{L}$). Then $\boldsymbol{A}$ is *freely generated* by C *over* K if it is generated by C and, for any finite sequence $c_0, \ldots, c_{n-1}$ without repetitions of elements of C we have

$$\varphi^{\boldsymbol{A}}(c_0, \ldots, c_{n-1}) = \psi^{\boldsymbol{A}}(c_0, \ldots, c_{n-1}) \text{ iff } \vDash_{\mathsf{K}} \varphi \approx \psi,$$

for every equation $\varphi(p_0, \ldots, p_{n-1}) \approx \psi(p_0, \ldots, p_{n-1})$. It can be shown (see [22]) that if K is a quasivariety, in particular a variety, and $\boldsymbol{A}$ is freely generated over K by some $C \subseteq A$, then $\boldsymbol{A} \in \mathsf{K}$. Each such algebra is called a *free algebra of* K. Free algebras of K have the *universal mapping property*: every mapping from the set of free generators of $\boldsymbol{A}$ into an algebra $\boldsymbol{B}$ of K extends uniquely to a homomorphism from $\boldsymbol{A}$ into $\boldsymbol{B}$. Thus any two free algebras of K with the same number of free generators are isomorphic. Every nontrivial quasivariety has a free algebra with any given number of free generators. Note that if $\boldsymbol{A}$ is freely generated by C over the quasivariety of *all* algebras of type $\mathcal{L}$, then $\varphi^{\boldsymbol{A}}(c_0, \ldots, c_{n-1}) = \psi^{\boldsymbol{A}}(c_0, \ldots, c_{n-1})$ iff φ and ψ are lexically equal, i.e., the same formula. Such algebras are said to be *absolutely free*. We denote by $\mathbf{Fm}_{\mathcal{L}}$ the *algebra of* $\mathcal{L}$*-formulas* (*terms*) $\big\langle \mathrm{Fm}_{\mathcal{L}}, \langle \omega^{\mathbf{Fm}_{\mathcal{L}}} : \omega \in \mathcal{L} \rangle \big\rangle$. It has the set of $\mathcal{L}$-formulas (terms) as universe, and for $\omega \in \mathcal{L}$, ω k-ary, and $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{Fm}$, we define $\omega^{\mathbf{Fm}_{\mathcal{L}}}(\varphi_0, \ldots, \varphi_{k-1}) = \omega\varphi_0 \cdots \varphi_{k-1}$. As expected we normally omit the subscript "$\mathcal{L}$" and write simply $\mathbf{Fm}$. Since $\varphi^{\mathbf{Fm}}(p_0, \ldots, p_{n-1}) = \psi^{\mathbf{Fm}}(p_0, \ldots, p_{n-1})$ iff φ and ψ are lexically equal, it is clear that $\mathbf{Fm}$ is absolutely freely generated by Va.

The formula algebra can be used to reformulate the notion of an interpretation of a formula in an algebra in more algebraic terms. Let $\varphi(p_0, \ldots, p_{n-1}) \in \mathrm{Fm}$ and $a_0, \ldots, a_{n-1}$ be elements of an algebra $\boldsymbol{A}$. Let $h : \mathbf{Fm} \to \boldsymbol{A}$ be any homomorphism such that $hp_i = a_i$ for $i < n$. Then $\varphi^{\boldsymbol{A}}(a_0, \ldots, a_{n-1}) = h\varphi$. We can now reformulate the definition of $\Delta \vDash_{\boldsymbol{A}} \varphi \approx \psi$ as follows

$$h\eta = h\vartheta \text{ for every } \eta \approx \vartheta \in \Delta \text{ implies } h\varphi = h\psi, \quad \text{for every } h : \mathbf{Fm} \to \boldsymbol{A}.$$

Substitution is just the particular case of formula evaluation in the formula algebra itself. Thus we can identify substitutions with endomorphisms $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ of the formula algebra.

2.2. **Examples.** We now briefly discuss some axiomatizations of the important deductive systems and their algebraic semantics. p, q, r are distinct sentential variables.

2.2.1. *Classical propositional calculus* (CPC). $\mathcal{L} = \{ \to, \wedge, \vee, \neg, \bot, \top \}$. Axioms:

(A1) $p \to (q \to p)$,
(A2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,
(A3) $(\neg q \to \neg p) \to (p \to q)$.

The remaining axioms serve to define $\wedge$, $\vee$, $\bot$, and $\top$ in terms of $\to$ and $\neg$:

(A4) $p \wedge q \to p$,
(A5) $p \wedge q \to q$,
(A6) $(r \to p) \to \big((r \to q) \to (r \to p \wedge q)\big)$,
(A7) $p \to p \vee q$,
(A8) $q \to p \vee q$,
(A9) $(p \to r) \to \big((q \to r) \to (p \vee q \to r)\big)$,
(A10) $\bot \to p$,
(A11) $p \to \top$.

There is only one rule of inference that is not an axiom:

(R1) $\dfrac{p,\ p \to q}{q}$ (modus ponens).

Let $\mathbf{2} = \langle \{0,1\}, \to^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}}, \neg^{\mathbf{2}}, \bot^{\mathbf{2}}, \top^{\mathbf{2}} \rangle$ be the two-element Boolean algebra where $\bot^{\mathbf{2}} = 0$ and $\top^{\mathbf{2}} = 1$ denote respectively "false" and "true", and $\to^{\mathbf{2}}, \wedge^{\mathbf{2}}, \vee^{\mathbf{2}} : \{0,1\}^2 \to \{0,1\}$ and $\neg^{\mathbf{2}} : \{0,1\} \to \{0,1\}$ are given by the usual truth tables. A formula $\varphi = (p_0, \dots, p_{n-1})$ is a *tautology* if $\vDash_{\mathbf{2}} \varphi$, i.e., if $\varphi^{\mathbf{2}}(a_0, \dots, a_{n-1}) = 1$ for all $a_0, \dots, a_{n-1} \in \{0,1\}$, that is if φ has value "true" at every row of its truth table.

Theorem 2.3 (Weak Logical Completeness Theorem)**.** *For all* $\varphi \in \mathrm{Fm}$*,*

$$\vdash_{\mathrm{CPC}} \varphi \quad \textit{iff} \quad \vDash_{\mathbf{2}} \varphi \approx \top.$$

φ is a *logical consequence* of a set Γ of formulas if $\Gamma \vDash_{\mathbf{2}} \varphi$, i.e., for every sequence $\bar{a} = a_0, a_1, a_2, \dots$ of 0's and 1's, $\psi^{\mathbf{2}}(\bar{a}) = 1$ for all $\psi \in \Gamma$ implies $\varphi^{\mathbf{2}}(\bar{a}) = 1$. If $\Gamma \subseteq \mathrm{Fm}$, we write $\Gamma \approx \top$ as shorthand for the set of equations $\{ \psi \approx \top : \psi \in \Gamma \}$.

Theorem 2.4 (Logical Completeness Theorem)**.** *For all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$*,*

$$\Gamma \vdash_{\mathrm{CPC}} \varphi \quad \textit{iff} \quad \Gamma \approx \top \vDash_{\mathbf{2}} \varphi \approx \top.$$

One of the basic results of the metatheory of CPC is the deduction theorem. It is an immediate consequence of the logical completeness theorem.

Theorem 2.5 (Deduction Theorem)**.** *Let* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$*. Then*

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{CPC}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{CPC}} \varphi \to \psi.$$

See Note 2.2.

The class of Boolean algebras forms an algebraic semantics for CPC. We treat Boolean algebras as algebras of type $\mathcal{L}$, where $\mathcal{L}$ is the language of CPC. A *Boolean algebra* is thus an algebra $\boldsymbol{A} = \langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ such that $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded, complemented, distributive lattice with smallest element $\bot^{\boldsymbol{A}}$, largest element $\top^{\boldsymbol{A}}$, and complementation $\neg^{\boldsymbol{A}}$, while $\to^{\boldsymbol{A}}$ is *relative complementation*, i.e., $a \to^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} b$.

The class of Boolean algebras, which is a variety, is denoted by BA. We also denote by $\mathbf{2}$ the 2-element Boolean algebra $\langle\{0,1\},\to,\wedge,\vee,\neg,0,1\rangle$ with $\to$, $\wedge$, $\vee$, $\neg$ defined as usual.

Boolean algebras arise from CPC by the classical Lindenbaum-Tarski process, which we now describe. Let $\varphi\leftrightarrow\psi$ be an abbreviation for $(\varphi\to\psi)\wedge(\psi\to\varphi)$. Let T be a theory of CPC and let Σ be a set of *nonlogical axioms* for T (i.e., axioms distinct from those of CPC). Define the binary relation $\equiv_T$ on Fm by taking $\varphi\equiv_T\psi$ if $\Sigma\vdash_{\mathrm{CPC}}\varphi\leftrightarrow\psi$. Since $\varphi\leftrightarrow\varphi$, $(\varphi\leftrightarrow\psi)\to(\psi\leftrightarrow\varphi)$, and $(\varphi\leftrightarrow\psi)\to\big((\psi\leftrightarrow\vartheta)\to(\varphi\leftrightarrow\vartheta)\big)$ are all tautologies and hence theorems of CPC, we get that $\equiv_T$ is an equivalence relation on the set Fm of formulas. It follows immediately from the fact that $(\varphi\leftrightarrow\varphi')\to\big((\psi\leftrightarrow\psi')\to((\varphi\wedge\psi)\leftrightarrow(\varphi'\wedge\psi'))\big)$ is a tautology that $\varphi\equiv_T\varphi'$ and $\psi\equiv_T\psi'$ imply $(\varphi\wedge\psi)\equiv_T(\varphi'\wedge\psi')$. Similarly for the other connectives. Thus $\equiv_T$ is a congruence relation on the formula algebra $\mathbf{Fm}$. Let $\boldsymbol{A}_T$ be the quotient algebra $\mathbf{Fm}/{\equiv_T}$. The elements of $\boldsymbol{A}_T$ are the equivalence classes of formulas and the operation $\wedge^{\boldsymbol{A}_T}$ is defined by setting $\bar\varphi\wedge^{\boldsymbol{A}_T}\bar\psi$ equal to $\overline{\varphi\wedge\psi}$ ($\bar\varphi$, $\bar\psi$, and $\overline{\varphi\wedge\psi}$ are the equivalence classes of φ, ψ, and $\varphi\wedge\psi$, respectively). The other operations of $\boldsymbol{A}_T$ are similarly defined. The congruence property assures that these operations are properly defined. $\bar\varphi\le^{\boldsymbol{A}}\bar\psi$ if $\Sigma\vdash_{\mathrm{CPC}}\varphi\to\psi$ defines a partial ordering of the set of equivalence classes. This follows from the definition of $\varphi\leftrightarrow\psi$ and the fact that $\varphi\to\varphi$ and $(\varphi\to\psi)\to\big((\psi\to\vartheta)\to(\varphi\to\vartheta)\big)$ are tautologies. Axioms (A4)–(A11) insure that $\le$ is a bounded lattice ordering with $\bar\varphi\wedge^{\boldsymbol{A}_T}\bar\psi$ and $\bar\varphi\vee^{\boldsymbol{A}_T}\bar\psi$ as the meet and join of $\bar\varphi$ and $\bar\psi$, respectively, and with $\bar\top$ and $\bar\bot$ as upper and lower bound. The following tautologies guarantee that the lattice is distributive and complemented and thus that $\boldsymbol{A}_T$ is a Boolean algebra: $\varphi\wedge(\psi\vee\vartheta)\leftrightarrow(\varphi\wedge\psi)\vee(\varphi\wedge\vartheta)$, $\varphi\wedge\neg\varphi\leftrightarrow\bot$, $\varphi\vee\neg\varphi\leftrightarrow\top$.

It is well known that every countable Boolean algebra is isomorphic to $\boldsymbol{A}_T$ for some theory T of CPC. We sketch the proof. Let $\boldsymbol{B}$ be a countable Boolean algebra and $h:\mathbf{Fm}\twoheadrightarrow\boldsymbol{A}$ any surjective homomorphism (h exists because $\mathbf{Fm}$ is absolutely free). Let $T=h^{-1}(1):=\{\,\varphi\in\mathrm{Fm}:h\varphi=1\,\}$. It is easy to check that T includes all substitution instances of each axiom of CPC. Consider for example (A1). For all $a,b\in B$ we have $a\to(b\to a)=\neg a\vee(b\to a)=\neg a\vee(\neg b\vee a)\ge\neg a\vee a=1$. T is also closed under modus ponens: if $a,a\to b=1$, then $b=0\vee b=\neg a\vee b=a\to b=1$. So T is a theory of CPC. For all $\varphi,\psi\in\mathrm{Fm}$ we have $\varphi\equiv_T\psi$ iff $\varphi\leftrightarrow\psi\in T$ iff $h\varphi\leftrightarrow h\psi=1$ iff $h\varphi\le h\psi$ and $h\psi\le h\varphi$ iff $h\varphi=h\psi$. So the mapping $\varphi/\equiv_T\mapsto h\varphi$ is a bijection between $\boldsymbol{A}_T$ and $\boldsymbol{B}$. Moreover, it clearly preserves all the operations $\to$, $\wedge$, $\vee$, $\neg$, $\bot$ and $\top$, and hence is an isomorphism.

The formula $\varphi\leftrightarrow(\varphi\leftrightarrow\top)$ is a tautology. Thus $\vdash_{\mathrm{CPC}}\varphi$ iff $\vdash_{\mathrm{CPC}}\varphi\leftrightarrow\top$ iff $\varphi\equiv_T\top$ for every theory T iff $\varphi^{\boldsymbol{A}_T}=\top^{\boldsymbol{A}_T}$ for every Boolean algebra $\boldsymbol{A}_T$. This gives the following theorem, which is called the weak algebraic completeness theorem in contrast to Thm. 2.3.

Theorem 2.6 (Weak Completeness Theorem)**.** *For* $\varphi\in\mathrm{Fm}$*,*

$$\vdash_{\mathrm{CPC}}\varphi \quad \textit{iff} \quad \models_{\mathsf{BA}}\varphi\approx\top.$$

Combining the two weak completeness theorems we have that $\models_{\mathsf{BA}}\varphi\approx\top$ iff $\models_{\mathbf{2}}\varphi\approx\top$. This equivalence reflects the fact that the variety of Boolean algebras is generated by the Boolean algebra $\mathbf{2}$, i.e., that $\mathsf{BA}=\mathbf{HSP}(\mathbf{2})$.

The above completeness theorem is weak in the sense that it deals only with theorems of CPC. The (algebraic) completeness theorem in its full strength gives an algebraic characterization of derived inference rules of CPC. If $\mathit{\Gamma} \subseteq \mathrm{Fm}$, we write $\mathit{\Gamma} \leftrightarrow \top$ as shorthand for the set of formulas $\{\, \psi \leftrightarrow \top : \psi \in \mathit{\Gamma} \,\}$.

Note that $\mathit{\Gamma} \vdash_{\mathrm{CPC}} \varphi$ iff $\mathit{\Gamma} \leftrightarrow \top \vdash_{\mathrm{CPC}} \varphi \leftrightarrow \top$ iff $\varphi \equiv_T \top$, where T is the theory defined by the non-logical axioms $\mathit{\Gamma}$. This gives the following equivalence.

Theorem 2.7 (Completeness Theorem). *For $\mathit{\Gamma} \cup \{\varphi\} \subseteq \mathrm{Fm}$,*

$$\mathit{\Gamma} \vdash_{\mathrm{CPC}} \varphi \quad \textit{iff} \quad \mathit{\Gamma} \approx \top \models_{\mathsf{BA}} \varphi \approx \top.$$

Combining this theorem with the logical completeness theorem we get that $\mathit{\Gamma} \approx \top \models_{\mathsf{BA}} \varphi \approx \top$ iff $\mathit{\Gamma} \approx \top \models_{\mathbf{2}} \varphi \approx \top$. This equivalence reflects the fact that the variety of Boolean algebras is also generated by $\mathbf{2}$ as a quasivariety, i.e., $\mathsf{BA} = \mathbf{SP}(\mathbf{2})$.

The completeness theorem formalizes the fact that the deductive apparatus of CPC is faithfully represented in the equational logic of Boolean algebras. This is summarized by saying that BA constitutes a (*purely*) *algebraic semantics* for CPC. Conversely, the equational logic of Boolean algebras is faithfully represented in the consequence relation of CPC. The following theorem makes this claim precise. Its proof relies on the fact that for a Boolean algebra $\boldsymbol{A}$, $\bar{a} \in A^\omega$, $\varphi, \psi \in \mathrm{Fm}$, we have $\varphi^{\boldsymbol{A}}(\bar{a}) = \psi^{\boldsymbol{A}}(\bar{a})$ if and only if $(\varphi \leftrightarrow \psi)^{\boldsymbol{A}}(\bar{a}) = \top^{\boldsymbol{A}}$.

Theorem 2.8 (Inverse Completeness Theorem). *Let $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ for $i \in I$ and $\varphi_0, \varphi_1 \in \mathrm{Fm}$. Then*

$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \models_{\mathsf{BA}} \varphi_0 \approx \varphi_1 \quad \textit{iff} \quad \{\, \psi_0^i \leftrightarrow \psi_1^i : i \in I \,\} \vdash_{\mathrm{CPC}} \varphi_0 \leftrightarrow \varphi_1.$$

Because of this duality between the consequence relation of CPC and the equational consequence relation BA, BA can be thought of as an *equivalent* algebraic semantics for CPC.

Essentially the same classical Lindenbaum-Tarski process generates an equivalent algebraic semantics for a number of different logics. We consider several more here omitting for the most part the proofs.

More details on the Lindenbaum-Tarski process and its history can be found in Note 2.3.

2.2.2. *Intuitionistic propositional calculus* (IPC). The language of IPC is the same as that of CPC. The system IPC is strictly weaker than CPC, i.e., $\vdash_{\mathrm{IPC}} \subset \vdash_{\mathrm{CPC}}$. It can be axiomatized by (A1), (A2), (A4)–(A11), together with

(A12) $\neg p \to (p \to \bot)$,
(A13) $(p \to \bot) \to \neg p$.

Again modus ponens (R1) is the only strict rule of inference. The deduction theorem also holds for IPC. It can be proved by induction on the length of IPC-derivations. Such a proof depends only on the presence of axioms (A1) and (A2), together with the fact that the rule (R1) of modus ponens is the only rule of inference. The deduction theorem therefore holds in fact for any fragment of CPC satisfying these conditions, as well as for any axiomatic expansion of such a system (by new axioms, but not by new inference rule other than modus ponens). This applies also to expansions of such systems, in which new connectives are allowed along with new axioms, but again modus ponens is the only rule of inference.

The deduction theorem for CPC itself is often proved by induction, too, because it is helpful to have it available when proving the logical completeness theorem for CPC.

Theorem 2.9. *For* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$,

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{IPC}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{IPC}} \varphi \to \psi.$$

An example of a formula that is a theorem of CPC, but not of IPC, is the *law of the excluded middle* $\varphi = p \vee \neg p$. Given the completeness theorem for CPC, it is obvious that $\vdash_{\mathrm{CPC}} \varphi$. To see $\nvdash_{\mathrm{IPC}} \varphi$, we can apply the following completeness theorem for IPC that is the analogue of Thm. 2.7. There is also an analogue of the inverse completeness theorem, Thm. 2.8, which we formulate simultaneously.

A *Heyting algebra* is an algebra $\boldsymbol{A} = \langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}}\rangle$ such that$\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}}\rangle$ is a bounded, distributive lattice, and, for $a, b \in A$, $a \to^{\boldsymbol{A}} b$ is the largest element c (with respect to the lattice order) such that $a \wedge^{\boldsymbol{A}} c \leq b$, and $\neg^{\boldsymbol{A}} a = a \to^{\boldsymbol{A}} \bot^{\boldsymbol{A}}$. Thus $\to^{\boldsymbol{A}}$ is a binary operation with the property, for all $a, b, c \in A$,

$$c \leq a \to^{\boldsymbol{A}} b \quad \text{iff} \quad a \wedge^{\boldsymbol{A}} c \leq b.$$

The operation $\to^{\boldsymbol{A}}$ is called *relative pseudo-complementation*. Each finite distributive lattice admits a unique relative pseudo-complementation operation. Hence every finite distributive lattice is the reduct of a unique Heyting algebra. Although it is not immediately obvious, the class of Heyting algebras can be defined by identities alone and thus forms a variety, which we denote by HA; see [7]. We have the following combined completeness and inverse completeness theorem for IPC.

Theorem 2.10. (i) *For* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{IPC}} \varphi$ *iff* $\Gamma \approx \top \models_{\mathsf{HA}} \varphi \approx \top$.

(ii) *Let* $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ *for* $i \in I$ *and* $\varphi_0, \varphi_1 \in \mathrm{Fm}$. *Then*

$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \models_{\mathsf{HA}} \varphi_0 \approx \varphi_1 \quad \text{iff} \quad \{\, \psi_0^i \leftrightarrow \psi_1^i : i \in I \,\} \vdash_{\mathrm{IPC}} \varphi_0 \leftrightarrow \varphi_1.$$

Using the theorem, we see $\nvdash_{\mathrm{IPC}} p \vee \neg p$. Indeed, let $\boldsymbol{A}$ be the 3-element linearly ordered Heyting algebra on $0 < 1 < 2$. Then $\neg^{\boldsymbol{A}} 1 = \neg^{\boldsymbol{A}} 2 = 0$, $\neg^{\boldsymbol{A}} 0 = 2$, $\bot^{\boldsymbol{A}} = 0$, and $\top^{\boldsymbol{A}} = 2$. Thus $1 \vee^{\boldsymbol{A}} \neg^{\boldsymbol{A}} 1 = 1 \vee^{\boldsymbol{A}} 0 = 1 \neq 2 = \top^{\boldsymbol{A}}$. Hence $\not\models_{\boldsymbol{A}} p \vee \neg p$, $\not\models_{\mathsf{HA}} p \vee \neg p$, and therefore $\nvdash_{\mathrm{IPC}} p \vee \neg p$.

There is no single finite Heyting algebra $\boldsymbol{A}$ whose tautologies (i.e., the formulas that universally evaluate to $\top^{\boldsymbol{A}}$) coincide with the theorems of IPC. Algebraically, this means that HA, in contrast with BA, is not generated (either as a variety or as a quasivariety) by a finite algebra.

Note 2.4 contains additional information on the completeness theorems for CPC and IPC and their proofs.

2.2.3. *Łukasiewicz's many-valued logic* (MVL). (To be written)

2.2.4. B-C-K *logic* (BCK). There are a number of weaker deductive systems than IPC that have been recently received increased attention recently in the study of the logic of computation; *linear logic* is the prime example. Some of these have equivalent algebraic semantics. We consider one here, B-C-K *logic*, BCK for short, first formulated by Meredith (see [116]); its name refers to the combinators B, C, and K of combinatory logic. It is of interest for algebraic logic because it is an example of a relatively simple logic with an equivalent algebraic semantics that is a proper quasivariety (i.e., not a variety).

We will first define a slightly richer logic that is in some sense better behaved. This logic does not have a standard name, but might be called B-C-K *logic with fusion*, and will be denoted BCK^*.

Let $\mathcal{L} = \{*, \to, \top\}$.

Axioms:

(B) $(q \to r) \to \big((p \to q) \to (p \to r)\big)$,
(C) $\big(p \to (q \to r)\big) \to \big(q \to (p \to r)\big)$,
(K) $p \to (q \to p)$,
(A1$*$) $\big(p \to (q \to p * q)\big)$,
(A2$*$) $\big(p \to (q \to r)\big) \to \big((p * q) \to r\big)$,
(A3$*$) $p \to \top$.

The only rule of inference is modus ponens

(R1) $\dfrac{p,\, p \to q}{q}$.

The $\{\wedge, \to, \top\}$-fragment of IPC is an axiomatic extension of BCK^*; it is obtained by adding the axiom $p \to (p * p)$ to those of BCK^*.

B-C-K logic is the $\to$-fragment of the logic BCK^* defined above, and axiomatized by (B), (C), (K) together with the rule of modus ponens.

The deduction theorem does not hold for either system (see 5.6.2), but we do have what we will call a *local deduction theorem*. We will formulate the theorem for BCK only. For each $k < \omega$, let

$$\varphi \xrightarrow{k} \psi = \underbrace{\varphi \to (\varphi \to \dots (\varphi}_{k \text{ times}} \to \psi) \dots).$$

Theorem 2.11 (Local Deduction Theorem)**.** *For $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$,*

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{BCK}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{BCK}} \varphi \xrightarrow{k} \psi, \quad \textit{for some } k < \omega.$$

See Note 2.5

A structure $\boldsymbol{A} = \langle A, \cdot, 1, \leq \rangle$ is a *partially ordered monoid* if $\langle A, \cdot, 1 \rangle$ is a monoid, $\leq$ is a partial order on A, and for all $x, y, z \in A$, if $x \leq y$, then $x * z \leq y * z$ and $z * x \leq z * y$. $\boldsymbol{A}$ is *integral* if $x \leq 1$, for all $x \in A$. Finally, $\boldsymbol{A}$ is *residuated* if for all $x, y \in A$ the set $\{\, z : x{*}z \leq y \,\}$ contains a largest element, called the *residual of* x relative to y, and denoted by $x \to y$. A partially ordered commutative, residuated and integral monoid $\langle A, *, 1, \leq \rangle$ can be treated as an algebra $\langle A, *, \to, 1 \rangle$, since the partial order $\leq$ can be recovered via $x \leq y$ iff $x \to y = 1$; we refer to such an algebra by the acronym *pocrim*.

The class PO of all pocrims is a quasivariety definable by:

(PO1) $x * 1 \approx x$,
(PO2) $x * y \approx y * x$,
(PO3) $x \to 1 \approx 1$,
(PO4) $1 \to x \approx x$,
(PO5) $(z \to x) \to ((x \to y) \to (z \to y)) \approx 1$,

(PO6) $x \to (y \to z) \approx (x * y) \to z$,
(PO7) $x \to y \approx 1 \quad \& \quad y \to x \approx 1 \Longrightarrow x \approx y$.

BCK algebras can be defined as the class of algebras $\boldsymbol{A} = \langle A, \to, 1 \rangle$ satisfying (PO3), (PO4), (PO5) and (PO7) together with

(PO8) $x \to x \approx 1$
(PO9) $x \to (y \to z) \approx y \to (x \to z)$.

The class of all BCK algebras is denoted by BCK. By (PO8) the constant 1 is definable and may be eliminated. Since (PO8) and (PO9) are satisfied in PO, the class of $\{\to, 1\}$-subreducts of algebras in PO consists of BCK algebras. The reverse inclusion also holds:

Theorem 2.12. *Every* BCK *algebra is a subalgebra of the* $\{\to, \top\}$*-reduct of a pocrim.*

The quasi-identity (PO7) cannot be replaced by identities:

Theorem 2.13. *The quasivarieties* PO *and* BCK *fail to be varieties.*

The class BCK of BCK algebras is the equivalent algebraic semantics for B-C-K logic in the same sense that Boolean algebras form the algebraic semantics for the classical propositional calculus. (The identity $x \to x \approx 1$ is derivable from the above axioms; so the constant 1 can be omitted from the language.) In particular, every countable BCK algebra is isomorphic to an algebra of the form $\mathbf{Fm}/{\equiv_T}$ for some BCK-theory T. In this case $\varphi \equiv_T \psi$ iff $\Sigma \vdash_{\mathrm{BCK}} \varphi \to \psi$ and $\Sigma \vdash_{\mathrm{BCK}} \psi \to \varphi$, where Σ is a set of nonlogical axioms for T. This gives the following algebraic completeness and inverse completeness theorem for BCK.

Theorem 2.14. (i) *For* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{BCK}} \varphi$ *iff* $\Gamma \approx \top \models_{\mathsf{BCK}} \varphi \approx \top$.
(ii) *Let* $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ *for* $i \in I$ *and* $\varphi_0, \varphi_1 \in \mathrm{Fm}$. *Then*

$$(1) \quad \{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \models_{\mathsf{BCK}} \varphi_0 \approx \varphi_1$$
$$\text{iff} \quad \{\, \psi_0^i \to \psi_1^i : i \in I \,\} \cup \{\, \psi_1^i \to \psi_0^i : i \in I \,\} \vdash_{\mathrm{BCK}} \varphi_0 \to \varphi_1 \quad \text{and}$$
$$\{\, \psi_0^i \to \psi_1^i : i \in I \,\} \cup \{\, \psi_1^i \to \psi_0^i : i \in I \,\} \vdash_{\mathrm{BCK}} \varphi_1 \to \varphi_0.$$

A similar completeness and inverse completeness theorem holds for BCK^*-logic with respect to the quasivariety PO of pocrims.

For additional information on B-C-K logic and BCK algebras see Note 2.6.

We now look at an important class of logics that are obtained by adjoining a unary connective to CPC.

2.2.5. *Modal logic* (K*,* $\mathrm{S5^G}$*,* $\mathrm{S5^C}$). Modal logic is the logic of "necessity" and "possibility". Its language is the extension of the language of CPC by one new unary connective $\Box$, i.e., $\mathcal{L} = \{\to, \wedge, \vee, \neg, \bot, \top, \Box\}$. For a formula φ, $\Box\,\varphi$ is to be interpreted as "it is necessary that φ", and the formula $\neg\,\Box\,\neg\varphi$, which is abbreviated by $\Diamond\,\varphi$, is interpreted as "it is possible that φ".

The standard system K (after Kripke) of modal logic is axiomatized by:

(A14) φ, φ any classical tautology,
(A15) $\Box(p \to q) \to (\Box\,p \to \Box\,q)$,
(R1) $\dfrac{p,\ p \to q}{q}$,

(R2) $\dfrac{p}{\Box\, p}$ (necessitation).

Instead of taking all classical tautologies, we could replace (A14) by the eleven axioms (A1)–(A11) of CPC. Also, the axiom (A15) can be replaced by the pair of axioms

(A15a) $\Box(p \wedge q) \to \Box\, p \wedge \Box\, q$,
(A15b) $\Box\, p \wedge \Box\, q \to \Box(p \wedge q)$.

The logic $\mathrm{S5^G}$ (*Gödel style* S5) is axiomatized by adding to the axioms and rules of K the following axioms:

(A16) $\Box\, p \to p$,
(A17) $\Box\, p \to \Box\,\Box\, p$,
(A18) $\Diamond\, p \to \Box\,\Diamond\, p$.

The theorems of $\mathrm{S5^G}$ coincide with those of Lewis's original S5. While the axioms and rules of the standard system embody some weak properties of "necessity" and "possibility", those captured by $\mathrm{S5^G}$ are in better agreement with the intuitive notions of these modalities. For instance, (A16) asserts that if a proposition is necessarily true, then it must be true.

We will later see in Sec. 5.6.2 that K does not have the deduction theorem, although it does have a local deduction theorem like BCK ([15]). On the other hand, $\mathrm{S5^G}$ does have the deduction theorem, but in a somewhat different form than the corresponding theorems for CPC and IPC.

Theorem 2.15. *Let* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$. *Then*

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{S5^G}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{S5^G}} \Box\,\varphi \to \psi.$$

The equivalent algebraic semantics for the system K consists of the class MA of *modal algebras*. These are algebras $\boldsymbol{A} = \langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \Box^{\boldsymbol{A}} \rangle$ such that $\langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a Boolean algebra, and $\Box^{\boldsymbol{A}}$ is a unary operation satisfying the identities:

(i) $\Box\,\top \approx \top$ (normality),
(ii) $\Box(x \wedge y) \approx \Box\, x \wedge \Box\, y$ (multiplicativity).

We have a completeness and inverse completeness theorem for K.

Theorem 2.16. (i) *For* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{K}} \varphi$ *iff* $\Gamma \approx \top \vDash_{\mathsf{MA}} \varphi \approx \top$.
(ii) *Let* $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ *for* $i \in I$ *and* $\varphi_0, \varphi_1 \in \mathrm{Fm}$. *Then*

$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \vDash_{\mathsf{MA}} \varphi_0 \approx \varphi_1 \quad \text{iff} \quad \{\, \psi_0^i \leftrightarrow \psi_1^i : i \in I \,\} \vdash_{K} \varphi_0 \leftrightarrow \varphi_1.$$

An equivalent algebraic semantics for $\mathrm{S5^G}$ is obtained by restricting to the class of *monadic algebras*. These are the modal algebras satisfying the additional identities:

(iii) $\Box\, x \wedge x \approx \Box\, x$,
(iv) $\Box\,\Box\, x \approx \Box\, x$,
(v) $\Box\,\Diamond\, x \approx \Diamond\, x$.

Let MO denote the class of monadic algebras. The completeness theorem for $\mathrm{S5^G}$ and its inverse take the following form.

Theorem 2.17. (i) *For* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{S5^G}} \varphi$ *iff* $\Gamma \approx \top \vDash_{\mathsf{MO}} \varphi \approx \top$.

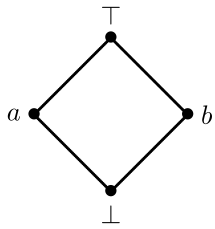

FIGURE 1

(ii) *Let* $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ *for* $i \in I$ *and* $\varphi_0, \varphi_1 \in \mathrm{Fm}$. *Then*
$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \vDash_{\mathsf{MO}} \varphi_0 \approx \varphi_1 \quad \textit{iff} \quad \{\, \psi_0^i \leftrightarrow \psi_1^i : i \in I \,\} \vdash_{\mathrm{S5^G}} \varphi_0 \leftrightarrow \varphi_1.$$

The deductive $\mathrm{S5^C}$ (*Carnap style* S5) has the same theorems as Lewis's S5, and consequently the same theorems as the deductive system $\mathrm{S5^G}$, but its only inference rule is modus ponens, thus assuring that the deduction theorem holds in the standard form. It is axiomatized as follows:

(A16) $\Box p \to p$,
(A14′) $\Box \varphi$, φ any classical tautology,
(A15′) $\Box\big((\Box(p \to q) \to (\Box p \to \Box q)\big)$,
(A16′) $\Box\big(\Box p \to p\big)$,
(A17′) $\Box\big(\Box p \to \Box\Box p\big)$,
(A18′) $\Box\big(\Diamond p \to \Box \Diamond p\big)$,
(R1) $\dfrac{p,\ p \to q}{q}$.

Because the only rule of inference of $\mathrm{S5^C}$ is modus ponens, we have the familiar form of the deduction theorem, as already observed in 2.2.2:

Theorem 2.18. *For* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$,
$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{S5^C}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{S5^C}} \varphi \to \psi.$$

Although $\mathrm{S5^C}$ has the same theorems as $\mathrm{S5^G}$, it does not have the same derived inference rules. In particular the rule of necessitation (R2) is not a derived rule of $\mathrm{S5^C}$. To see this, let $\boldsymbol{A}$ be the 4-element monadic algebra with the underlying Boolean algebra shown in Figure 1 and the necessity operator $\Box$ defined by $\Box^{\boldsymbol{A}} \top^{\boldsymbol{A}} = \top^{\boldsymbol{A}}$, $\Box^{\boldsymbol{A}} x = \bot^{\boldsymbol{A}}$ if $x \neq \top^{\boldsymbol{A}}$. Let $h : \mathbf{Fm} \to \boldsymbol{A}$ be the homomorphism satisfying $hp = a$ for every $p \in \mathrm{Va}$.

It is not difficult to verify that $T = h^{-1}(\{a, \top^{\boldsymbol{A}}\})$ is an $\mathrm{S5^C}$-theory: it contains all the $\mathrm{S5^C}$-axioms and is closed under (R1). But T is not an $\mathrm{S5^G}$-theory since, if p is a variable, then $p \in T$ but $\Box p \notin T$. So T is not closed under the rule of inference (R2).

The following weak completeness result for $\mathrm{S5^C}$ is a trivial consequence of the fact that the theorems of $\mathrm{S5^C}$ and $\mathrm{S5^G}$ coincide: for $\varphi \in \mathrm{Fm}$, $\vdash_{\mathrm{S5^C}} \varphi$ iff $\vDash_{\mathsf{MO}} \varphi \approx \top$. However, because the two systems fail to have the same derived inference rules, the strong completeness result for $\mathrm{S5^G}$ cannot carry over to $\mathrm{S5^C}$ (at least not with MO as the algebraic semantics) and in fact it is easy to see that the classical Lindenbaum-Tarski process does not work in the case of $\mathrm{S5^C}$. For suppose T is the $\mathrm{S5^C}$-theory generated by the single nonlogical axiom p, and as in the case of CPC let $\equiv_T$ be defined by $\varphi \equiv_T \psi$ if $p \vdash_{\mathrm{S5^C}} \varphi \leftrightarrow \psi$. Because all classical tautologies are theorems of $\mathrm{S5^C}$, we see that $\equiv_T$ is an equivalence relation on Fm that is a congruence with respect to the classical connectives. It is not however a congruence relation on **Fm** because, while $p \equiv_T \top$ holds, it is not the case that $\Box\, p \equiv_T \Box\, \top$ since this equivalence implies $\Box\, p \equiv_T \top$ (because $\Box\, \top \equiv_T \top$), and this equivalence can hold only if $p \vdash_{\mathrm{S5^C}} \Box\, p$, as is easily verified.

This observation does not of course preclude the existence of some class of algebras that has all the essential properties of an equivalent algebraic semantics for $\mathrm{S5^C}$ as exemplified by the strong completeness theorems and their inverses that we have been considering. We shall see in Sec. 4.4.3 however that there is in fact no equivalent algebraic semantics for $\mathrm{S5^C}$ in this sense. This is a consequence of the fact that $\mathrm{S5^C}$ is not algebraizable. In order to obtain a strong completeness theorem for $\mathrm{S5^C}$, and distinguish between it and $\mathrm{S5^G}$, we must use richer semantical structures than plain algebras.

A natural semantics for $\mathrm{S5^C}$ can be found among structures we call *filtered modal algebras* (cf. [8]). These are structures $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is a modal algebra and $F \subseteq A$ is a *filter* of $\boldsymbol{A}$, i.e., a subset of A satisfying:

(i) $\top^{\boldsymbol{A}} \in F$,
(ii) if $a, a \to b \in F$, then $b \in F$.

A filter F of $\boldsymbol{A}$ is called *open* if, for all $a \in F$, $\Box\, a \in F$ as well. A filtered modal algebra $\langle \boldsymbol{A}, F\rangle$ is *reduced* if the only open filter included in F is the trivial filter $\{\top^{\boldsymbol{A}}\}$.

For $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ and $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ a filtered modal algebra, we define $\Gamma \vDash_{\mathfrak{A}} \varphi$ if, for every homomorphism $h : \mathrm{Fm} \to \boldsymbol{A}$, if $h\psi \in F$ for every $\psi \in \Gamma$, then $h\varphi \in F$. Let MO^* be the class of all *reduced filtered monadic algebras*, i.e., all reduced filtered modal algebras $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ such that $\boldsymbol{A}$ is a monadic algebra. The completeness theorem for $\mathrm{S5^C}$ now takes the following form:

Theorem 2.19. *For $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ we have*

$$\Gamma \vdash_{\mathrm{S5^C}} \varphi \quad \textit{iff} \quad \Gamma \vDash_{\mathsf{MO}^*} \varphi.$$

As noted earlier, we will see below in Sec. 4.4.3 that the class MO^* cannot be replaced by a suitable class of algebras.

Additional remarks on modal logics and the connection between the two versions of S5 can be found in Note 2.7.

2.2.6. *Predicate logic* (PR_ω). The predicate calculus, in its standard formalization, is not a deductive system in the sense we have defined and for this reason its algebraic semantics is not so easy to determine. Before it can be algebraized by the classical Lindenbaum-Tarski process in the way we have been discussing, its standard formalization has to be transformed into a deductive system in the present sense. The language type of the transformed system $\mathrm{PR}_{\boldsymbol{\omega}}$ contains the usual sentential connectives $\vee$, $\wedge$, $\neg$, $\to$, $\top$, and $\bot$. It also has an infinite sequence of primitive unary connectives $\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \ldots$. Finally, there is a doubly infinite sequence of constant symbols (connectives of rank 0) $\mathsf{d}_{00}, \mathsf{d}_{10}, \mathsf{d}_{01}, \mathsf{d}_{20}, \mathsf{d}_{11}, \mathsf{d}_{02}, \ldots$.

$\mathrm{PR}_{\boldsymbol{\omega}}$ is defined by the following axioms and rules of inference. As an aid in understanding the connection between $\mathrm{PR}_{\boldsymbol{\omega}}$ and the standard predicate calculus, we use $\exists v_n$ and $v_m \approx v_n$, respectively, as aliases for c_n and d_{mn}. We also use $\forall v_n$ as an abbreviation for $\neg\exists v_n \neg$, and take m, n, and k as metavariables ranging over natural numbers. We emphasize however that the individual variable symbols $v_0, v_1, v_2, \dots$ do not actually occur in the formulas of $\mathrm{PR}_{\boldsymbol{\omega}}$.

(PR1) φ, for every classical tautology φ,
(PR2) $\forall v_n(p \to q) \to (\forall v_n p \to \forall v_n q)$,
(PR3) $\forall v_n p \to p$,
(PR4) $\forall v_n p \to \forall v_n \forall v_n p$,
(PR5) $\exists v_n p \to \forall v_n \exists v_n p$,
(PR6) $\forall v_m \forall v_n p \to \forall v_n \forall v_m p$,
(PR7) $v_n \approx v_n$,
(PR8) $\exists v_m (v_m \approx v_n)$,
(PR9) $v_m \approx v_n \wedge v_m \approx v_k \to v_n \approx v_k$,
(PR10) $\big(v_m \approx v_n \wedge \exists v_m (v_m \approx v_n \wedge p)\big) \to p, \quad$ if $m \neq n$,
(PR11) $\dfrac{p,\ p \to q}{q}$,
(PR12) $\dfrac{p}{\forall v_n p}$ (generalization).

Note the similarity of axioms (PR1)–(PR5) and the generalization rule (PR12) with axioms (A14)–(A18) and the rule of necessitation (R2) of $\mathrm{S5^G}$. In fact, for any $n < \omega$, the fragment of $\mathrm{PR}_{\boldsymbol{\omega}}$ that is obtained by restriction to the sublanguage $\{\to, \wedge, \vee, \neg, \bot, \top, \forall v_n\}$ is the same as $\mathrm{S5^G}$ in the following sense: $\Gamma \vdash_{\mathrm{S5^G}} \varphi$ iff $\Gamma' \vdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \varphi'$ where Γ' and φ' are obtained from Γ and φ by replacing $\Box$ everywhere by $\forall v_n$.

$\mathrm{PR}_{\boldsymbol{\omega}}$ does not have the deduction theorem in the sense of Thm. 2.5, but it does have a local deduction theorem.

Theorem 2.20. *Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$. Then*

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \forall v_0 \forall v_1 \dots \forall v_{n-1} \varphi \to \psi \quad \textit{for some } v_0, \dots, v_{n-1}.$$

A *cylindric algebra* (*of dimension* ω) is an algebra

$$\boldsymbol{A} = \langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \mathsf{c}_m^{\boldsymbol{A}}, \mathsf{d}_{mn}^{\boldsymbol{A}} \rangle_{m,n<\omega}$$

such that $\langle A, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, *\neg^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a Boolean algebra, and the following identities are satisfied for all $m, n, k < \omega$ (c_m is a unary operation and d_{mn} is a distinguished element).

(C1) $\mathsf{c}_n \bot \approx \bot$,
(C2) $x \vee (ssc_n x \approx \mathsf{c}_n x)$,
(C3) $\mathsf{c}_n(x \wedge \mathsf{c}_n y) \approx \mathsf{c}_n x \wedge \mathsf{c}_n y$,
(C4) $\mathsf{c}_m \mathsf{c}_n x \approx \mathsf{c}_n \mathsf{c}_m x$,
(C5) $\mathsf{d}_{mm} \approx \top$,
(C6) $\mathsf{d}_{mn} \approx \mathsf{c}_k(\mathsf{d}_{mk} \wedge \mathsf{d}_{kn})$, if $k \neq m, n$,
(C7) $\mathsf{c}_m(\mathsf{d}_{mn} \wedge x) \wedge \mathsf{c}_m(\mathsf{d}_{mn} \wedge \neg x) \approx \bot$, if $m \neq n$.

The variety of cylindric algebras of dimension ω is denoted by $\mathsf{CA}_{\boldsymbol{\omega}}$.

Theorem 2.21. (i) *For $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \varphi$ iff $\Gamma \approx \top \vDash_{\mathsf{CA}_{\omega}} \varphi \approx \top$.*

(ii) *Let $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ for $i \in I$ and $\varphi_0, \varphi_1 \in \mathrm{Fm}$. Then*

$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \vDash_{\mathsf{CA}_{\omega}} \varphi_0 \approx \varphi_1 \quad \textit{iff} \quad \{\, \psi_0^i \leftrightarrow \psi_1^i : i \in I \,\} \vdash_{\mathrm{PR}_{\omega}} \varphi_0 \leftrightarrow \varphi_1.$$

If in the definition of PR_ω we allow only the connectives c_i and d_{ij} with $i, j < n$ for some fixed, finite n, we get the deductive system PR_n of predicate logic over n variables. The variety CA_n of *cylindric algebras of dimension* n is similarly defined. For PR_n we get a deduction theorem similar to that for $\mathrm{S5}^{\mathrm{G}}$ (Thm. 2.15). We leave its formulation to the reader. The strong completeness theorem also holds for PR_n and CA_n.

For more detail on the algebraization of predicate logic see Note 2.8.

2.2.7. *Group theory* (GR). Group theory is normally treated as an equational logic, but it can also be formalized as a deductive system in the present sense; we denote it by GR. $\mathcal{L} = \{\cdot, ^{-1}, e\}$ where $\cdot$ is a binary connective, $^{-1}$ is a unary connective, and e is a constant. The axioms and rules of GR:

(GR1) $\big((p \cdot q) \cdot r\big) \cdot \big(p \cdot (q \cdot r)\big)^{-1}$,
(GR2) $(p \cdot e) \cdot p^{-1}$,
(GR3) $(e \cdot p) \cdot p^{-1}$,
(GR4) $p \cdot p^{-1}$,
(GR5) $p^{-1} \cdot p$,

(GR6) $\dfrac{p \cdot q^{-1}}{q \cdot p^{-1}}$, (GR7) $\dfrac{p \cdot q^{-1},\ q \cdot r^{-1}}{p \cdot r^{-1}}$,

(GR8) $\dfrac{p \cdot q^{-1},\ r \cdot s^{-1}}{p \cdot r \cdot (q \cdot s)^{-1}}$, (GR9) $\dfrac{p \cdot q^{-1}}{p^{-1} \cdot q^{-1-1}}$,

(GR10) $\dfrac{p}{p \cdot e^{-1}}$, (GR11) $\dfrac{p \cdot e^{-1}}{p}$.

We shall see below in Sec. 5.6 that GR does not have the deduction theorem in any reasonable sense. The algebraic counterpart of GR is, not surprisingly, the variety GR of groups, and we have a strong completeness theorem and its inverse.

Theorem 2.22. (i) *For* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathrm{GR}} \varphi$ *iff* $\Gamma \approx e \vDash_{\mathsf{GR}} \varphi \approx e$.

(ii) *Let* $\psi_0^i, \psi_1^i \in \mathrm{Fm}$ *for* $i \in I$ *and* $\varphi_0, \varphi_1 \in \mathrm{Fm}$. *Then*

$$\{\, \psi_0^i \approx \psi_1^i : i \in I \,\} \vDash_{\mathsf{GR}} \varphi_0 \approx \varphi_1 \quad \textit{iff} \quad \{\, \psi_0^i \cdot (\psi_1^i)^{-1} : i \in I \,\} \vdash_{\mathrm{GR}} \varphi_0 \cdot \varphi_1^{-1}.$$

It follows immediately from the theorem that $p, q \vdash_{\mathrm{GR}} p \cdot q$, and thus that there is a proof in GR of $p \cdot q$ from p and q. Here is one such proof: p, $p \cdot e^{-1}$, q, $q \cdot e^{-1}$, $(p \cdot q) \cdot (e \cdot e)^{-1}$, $(e \cdot e) \cdot e^{-1}$, $(p \cdot q) \cdot e^{-1}$, $p \cdot q$.

GR is a curiosity; it is of interest only because it is the deductive counterpart of the variety of groups. See Note 2.9.

2.3. **Matrix semantics.** Our discussion of $\mathrm{S5}^{\mathrm{C}}$ above shows that, when looking for a semantics for an arbitrary deductive system, it may be natural to go beyond the purely algebraic semantics of traditional algebraic logic and consider algebras enriched by a unary predicate. This leads to the general notion of a logical matrix.

A (*logical*) $\mathcal{L}$-*matrix* is a pair $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and F is an arbitrary subset of A; the elements of F are called the *designated elements* of A. Let $\mathfrak{A}$ be a matrix, and let $\vDash_{\mathfrak{A}}$ be the relation that holds between a (possibly infinite) set Γ of formulas and a single formula φ, in symbols $\Gamma \vDash_{\mathfrak{A}} \varphi$, if every interpretation of φ in $\boldsymbol{A}$ *holds in* $\mathfrak{A}$ (i.e., is one of the designated elements) provided each $\psi \in \Gamma$ holds in $\mathfrak{A}$ under the same interpretation.

More formally, $\Gamma \vDash_{\mathfrak{A}} \varphi$ if

(2) $h\psi \in F$ for every $\psi \in \Gamma$ implies $h\varphi \in F$, for every $h : \mathbf{Fm} \to \boldsymbol{A}$.

If M is a class of matrices, then $\Gamma \vDash_{\mathsf{M}} \varphi$ if $\Gamma \vDash_{\mathfrak{A}} \varphi$ for every $\mathfrak{A} \in \mathsf{M}$.

A matrix $\mathfrak{A}$ is called a *matrix model* of a deductive system $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vDash_{\mathfrak{A}} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$. A subset F of $\boldsymbol{A}$ is called an *$\mathcal{S}$-filter*, or simply a *filter* when $\mathcal{S}$ is clear from context, if the matrix $\langle \boldsymbol{A}, F\rangle$ is a matrix model of $\mathcal{S}$. Thus F is an $\mathcal{S}$-filter iff F contains all interpretations of the logical axioms of $\mathcal{S}$ and is closed under each inference rule $\langle \Gamma, \varphi\rangle$ in the sense of (2). The $\mathcal{S}$-filters on the formula algebra are exactly the $\mathcal{S}$-theories defined earlier; this will be established in a more general context below in Lem. 3.3. We call the corresponding matrix models $\langle \mathbf{Fm}, T\rangle$ the *formula matrix models* or also *Lindenbaum matrices* of $\mathcal{S}$.

For any algebra $\boldsymbol{A}$ of the language type of CPC, a subset F of A is by definition a CPC-filter of $\boldsymbol{A}$ if it contains the image of each tautology under every interpretation of the formulas in $\boldsymbol{A}$ and is closed under the only inference rule of CPC, modus ponens, in the sense that, if a and $a \to^{\boldsymbol{A}} b$ are both in F, then b is in F. If $\boldsymbol{A}$ is a Boolean algebra, a member of the equivalent algebraic semantics of CPC, then every tautology evaluates to $\top^{\boldsymbol{A}}$, so F is a CPC-filter iff it contains $\top^{\boldsymbol{A}}$ and is closed under modus ponens; it is easy to check that it is closed under modus ponens iff $a \in F$ and $a \leq^{\boldsymbol{A}} b$ imply $b \in F$ and $a, b \in F$ imply $a \wedge^{\boldsymbol{A}} b \in F$. So the CPC-filters on a Boolean algebra coincide with the familiar (Boolean) filters. More generally, F is a CPC-filter of an arbitrary algebra $\boldsymbol{A}$ (over the appropriate language) iff it is the inverse image under some homomorphism of a Boolean filter, that is, iff there is a Boolean algebra $\boldsymbol{B}$ and a homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ such that $F = h^{-1}(G)$ for some Boolean filter G of $\boldsymbol{B}$. Later we shall see that every CPC-filter is in fact the inverse image of the trivial one-element filter $\{\top^{\boldsymbol{A}}\}$ for some Boolean algebra $\boldsymbol{A}$. (See the remarks following Thm. 3.8)

There are similar characterizations of $\mathcal{S}$-filters for the other deductive systems we have considered. For example, if $\boldsymbol{A}$ is a Heyting algebra, then F is an IPC-filter iff it is a Heyting filter ($\top^{\boldsymbol{A}} \in F$, $a, a \to^{\boldsymbol{A}} b \in F$ imply $b \in F$), and, in general, F is an IPC-filter iff it is the inverse image of some Heyting filter. If $\boldsymbol{A}$ is a group, then F is a GR-filter iff it is a normal subgroup of $\boldsymbol{A}$, and, in general, iff it is the inverse image of some normal subgroup. If $\boldsymbol{A}$ is a monadic algebra, then F is a filter in the sense of Sec. 2.2.5 iff it is a $\mathrm{S5^C}$-filter and it is an open filter in the sense of Sec. 2.2.5 iff it is a $\mathrm{S5^G}$-filter. These facts are easily verified.

Definition 2.23. Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a deductive system and M a class of $\mathcal{L}$-matrices. M is called a *matrix semantics* of $\mathcal{S}$ if, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\Gamma \vDash_{\mathsf{M}} \varphi$.

A class of matrices that forms a matrix semantics of $\mathcal{S}$ in this sense is said to be *strongly adequate* for $\mathcal{S}$; see [138].

Theorem 2.24. *Let $\mathcal{S}$ be a deductive system. The class of all matrix models forms a matrix semantics for $\mathcal{S}$, and so does the class of all formula matrix models.*

Additional remarks on the history of logical matrices can be found in Note 2.10. We will consider the notion of a logical matrix in more detail and in a more general context in the next section.

2.4. **Notes.**

Note 2.1. The formal notion of a deductive system, as an abstract *consequence operator* originated with Tarski ([128, 129]). (The consequence operator associated with the deductive system $\mathcal{S}$ consequence relation is the mapping $\mathrm{Cn}_{\mathcal{S}}$: from the power set of Fm into itself defined by $\mathrm{Cn}_{\mathcal{S}}\,\Gamma = \{\,\varphi : \Gamma \vdash_{\mathcal{S}} \varphi\,\}$. One easily passes back-and-forth between these two forms of deductive system without difficulty.) The notion has been refined over the years by a number of different authors; [21, 99, 124, 125, 126] is a partial list. The last three papers deal with a generalized form of consequence relation in which multiple formulas are allowed in the conclusion. For even broader conceptions of deductive systems, in a categorical context, see [55, 73, 92, 93]. The monographs [32, 138, 140] contain detailed discussions of consequence relations and operators. For a collection of papers on deductive systems viewed in a variety of ways see [68].

Wójcicki [138] calls a deductive system (viewed as a consequence relation or operator) the "inferential" conception of logic as opposed to the "formulaic", where a logic is identified with the collection of its "theorems". We will have more to say about this distinction in connection with the deduction theorem (Note 2.4) and with two different conceptions of the modal logic S5 (Note 2.9).

Systems such as these whose basic syntactic component is a formula are called 1-dimensional deductive systems as opposed to the multi-dimensional systems considered in the next section. 1-dimensional systems are often referred to as *Hilbert-style* systems, while the multi-dimensional systems, in particular the $\boldsymbol{\omega}$-dimensional systems are called *Gentzen-style* or *sequent-based* systems; see Note 3.2. The papers 1- or multi-dimensional deductive systems are usually presented by a set of axioms and inference rules. In this form they are also called *logistic systems*, although this term is most often used in a broader sense that includes logical systems like the predicate calculus in its usual formalization. The collection [132] contains most of Tarski's early papers on deductive papers and general metamathematics; for a surveys of this work see [11, 38].

Note 2.2. Normally the deduction theorem for CPC refers only to the implication from $\Gamma, \varphi \vdash_{\mathrm{CPC}} \psi$ to $\Gamma \vdash_{\mathrm{CPC}} \varphi \to \psi$, since the implication in the opposite direction is an immediate consequence (and in fact is equivalent) to modus ponens. In abstract algebraic logic it is useful however to consider the two properties in tandem, and to emphasize this we shall refer to their conjunction as the *deduction-detachment theorem*.

The deduction theorem, in its application to the formalism of *Principia Mathematica*, was established by Tarski as early as 1921. It was proved independently by Herbrand who published it without proof in [80]. However, Tarski was the first to recognize its importance for general metamathematical investigations [130, 131].

The investigation of the deduction theorem has been closely bound to the dichotomy between the "formulaic" and "inferential" conceptions of logic that was mentioned in the preceding note. Because of the deduction theorem we have that $\psi_0, \dots, \psi_{n-1} \vdash_{\mathrm{CPC}} \varphi$ iff $\vdash_{\mathrm{CPC}} \psi_0 \to (\psi_1(\cdots \to (\psi_{n-1} \to \varphi)\cdots))$. Thus the consequence relation of CPC is expressible in terms of its theorems (the tautologies). Under what circumstances are the derived inferences of a given deductive system expressible in terms of its theorems? This is called "deduction problem" in [138] and is one of the central topics of abstract algebraic logic. It involves first giving a precise meaning to the idea of the derived inference rules of a system being "expressible" in terms of it theorems, which we do in Section 3.5 below. For a survey of results on the deduction theorem before 1980 see [114]. For a brief survey of more recent results in the context of abstract algebraic logic see Notes 3.6 and 5.1.

The proof of the deduction theorem for CPC (Thm. 2.5) can be found in almost every textbook on mathematical logic; see for instance [105]. It is normally proved by induction on the length of CPC-derivations rather than appealing to the completeness theorem because it is useful to have it available for the proof of the completeness theorem. The same proof works for IPC (Thm. 2.9) and $\mathrm{S5}^{\mathrm{C}}$ (Thm. 2.18). The proof of the deduction theorem for $\mathrm{S5}^{\mathrm{G}}$ (Thm. 2.15) can be found in most books on modal logic, for example in [25].

Note 2.3. By the *classical Lindenbaum-process* we mean the process of forming the quotient of the formula algebra by the relation of logical equivalence. In the case of CPC, IPC and all the other deductive systems considered in this section two formulas φ and ψ are logically equivalent when $\psi \leftrightarrow \varphi$ is a theorem (or equivalently when $\varphi \to \psi$ and $\psi \to \varphi$ are both theorems.) Tarski was the first to use this method [130, 131] to give the first precise formulation of the connection between CPC and Boolean algebra, but other people were aware of essentially the same process about the same time; see [26, pp. 103–104]. This appears to be the first occurrence in the literature of what has become known as the *Lindenbaum* or *Lindebaum-Tarski algebra*; see the remark in [107] in reference to the origin of these names. For additional history of the subject see [127].

It is now common practice to refer also to the formation of the congruence relation on the formula algebra, as separate from forming the quotient, as the Lindenbaum-Tarski process, and to apply it to arbitrary theories. In abstract algebraic logic this process is further abstracted in such a way that it can be applied to arbitrary deductive systems, regardless of whether there is a biconditional or conditional present. This form of abstract algebraic logic, where the algebraic counterpart of a logical system is constructed directly from the consequence relation of the system by the abstract Lindenbaum-Tarski process, is called *logistic abstract algebraic logic*. There is another branch of the subject in which both the deductive system and its algebraic counterpart are defined in terms of an abstract semantics. Although we deal almost exclusively with the logistic aspects of abstract algebraic logic in this paper, the semantical approach is an important part of the subject. For some purposes, for example certain parts of the theory of explicit versus implicit definability, a semantical approach is necessary for a complete understanding of the algebraic aspects of the topic (see [84, 85]). For a selection of papers of a general character semantically based abstract algebraic logic we recommend [3, 4, 5, 6]. For a comparison of the two methods see [58].

Note 2.4. The proofs of the logical completeness theorems for CPC (Thms. 2.3, 2.4) can be found in any textbook on mathematical logic, but generally a slightly different set of axioms is used in each case. For example in the system found in [105] (A3) is replaced by $(\neg p \to q) \to ((\neg p \to \neg q) \to p))$. The axiom system we give here is due to J. Łukasiewicz.

The algebraic completeness theorems for CPC (Thms. 2.6 and 2.7) are normally proved by combining the logical completeness theorems with the fact that **2** generates the Boolean algebras both as a variety (i.e., $\mathsf{BA} = \mathbf{HSP}\{\mathbf{2}\}$) and as a quasivariety ($\mathsf{BA} = \mathbf{SPP}_{\mathrm{U}}\{\mathbf{2}\}$);

these are corollaries of the Stone representation theorem for Boolean algebras (see for instance [78]). From the first of these facts it follows easily that $\vdash_{\mathsf{BA}} \varphi \approx \psi$ iff $\vdash_{\{\mathbf{2}\}} \varphi \approx \psi$ and from the second that $\vdash_{\{\mathbf{2}\}} = \vdash_{\mathsf{BA}}$. For a direct proof of the two algebraic completeness theorems see [120].

The proof of the completeness theorem for IPC is a little harder to find; [49] and [120] are two sources. There is no analog of the logical completeness theorems for IPC in the sense that there is no single finite Heyting algebra whose tautologies (i.e., the formulas that universally evaluate to $\top$) coincide with the theorems of IPC (this was first shown by Gödel [72]). Algebraically, this means that HA is not generated (as either a variety or quasivariety) by a finite Heyting algebra as BA is generated by the two-element Boolean algebra.

The proof of the inverse completeness theorem for IPC (Thm. 2.10(ii)) is similar to the proof we sketched for CPC. The key is the simple observation that in a Heyting algebra $a = b$ iff $a \to b = b \to a = 1$. All the inverse completeness theorems we listed here are proved in essentially the same way. Nowhere in the literature will one find these theorems put on essentially the same level as the completeness theorems as we do here. But they play an important role in the definition of algebraizability as we shall see in the sequel.

Note 2.5. An arbitrary deductive system $\mathcal{S}$ has the *local deduction theorem* if, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$, there exists a finite set of formulas $\{\vartheta_0(p,q), \dots, \vartheta_{n-1}(p,q)\}$, in two variables, depending on Γ, φ, and ψ such that $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ iff $\Gamma \vdash_{\mathcal{S}} \vartheta_i(\varphi, \psi)$ for all $i < n$. The local deduction theorem for arbitrary deductive systems is investigated in [15, 31]. A proof of the local deduction theorems for BCK and K can be found in [15]. Using the fact that $\forall v_n$ has the properties of the modal operator $\Box$ it is easy to see how to adapt the proof of the deduction theorem for $\mathrm{S5}^{\mathrm{G}}$, and that of the local deduction theorem for K, to obtain the corresponding theorems for $\mathrm{PR}_{\boldsymbol{\omega}}$ and PR_n, respectively. The details of the proof can be found in [15]. A more detailed discussion of the local deduction theorem and its algebraic theory can be found in Note 5.1 below.

Note 2.6. B-C-K logic originated with Meredith [116]; its name comes from its connection with combinatory logic, in particular the combinators B, C, and K, but this connection has played almost no role in its recent development. Iseki [86] introduced the class BCK of BCK *algebras* in the late sixties, intending them to provide an algebraic semantics for B-C-K logic. The axiomatization of BCK given here, which can be found in [19], differs from Iséki's orginal one. Since (PO8) and (PO9) are satisfied in PO, the class of $\{\to, 1\}$-subreducts of algebras in PO consists of BCK algebras. Theorem 2.12 was proved independently by Pałasiński [113], Ono and Komori [112], and Fleischer [56]. That PO and BCK form proper quasivarieties (Thm. 2.13) was shown independently by Wroński [139] for BCK and by Higgs [83] for PO. It turns out that for those deductive systems that are algebraizable in the sense of Sec. 5 below, the equivalent algebraic semantics is always a quasivariety, and whether or not it is a variety is in general a difficult question that seems to be connected with the deduction theorem in a complicated way. See Notes 3.6 and 4.3. Note 3.6Note 4.3.

The completeness theorem for BCK and BCK^* (Thm. 2.14) is folklore, but in its essential parts it was shown by Kabziński [88].

BCK* and hence BCK are closely related to the so-called substructural logics that have recently received considerable attention, largely because of their application to computer science. These logics are most naturally presented as Gentzen-style, or sequent-based, logical systems rather than the Hilbert-style systems that we consider here. The term "substructural" refers to the fact that in their formalization some of the standard structural rules that are common to the Gentzen-style formalizations of the more traditional logics are omitted. These systems are discussed in more detail in the next section; see in particular Note 3.2.

Note 2.7. The algebraic counterpart of modal logics was first investigated in connection with Lewis's system S4 in a topological setting by J. C. C. McKinsey and Tarski ([104]). The inference rule of necessitation in $\mathrm{S5}^{\mathrm{G}}$ is considered objectionable by some on the grounds that it appears to embody the counterintuitive logical principle that from the truth of a proposition it can be inferred that it is necessarily true. A case can be made on both philosophical and metalogical grounds for a deductive system like $\mathrm{S5}^{\mathrm{C}}$ whose theorems are exactly those of Lewis's S5 with modus ponens as its only inference rule.

Proofs of the algebraic completeness theorems for the modal systems K and $\mathrm{S5}^{\mathrm{G}}$ (Thms. 2.16(i) and 2.17(i)) can be found in [120] and of the matrix completeness theorem for $\mathrm{S5}^{\mathrm{C}}$ (Thm. 2.19) in [8]. Some further developments in the theory of modal logics in abstract algebraic logic context can be found in [64, 87].

The contrast between $\mathrm{S5}^{\mathrm{G}}$ and $\mathrm{S5}^{\mathrm{C}}$ illustrates the contrast between the inferential and formulaic forms of logic; this reflects two different views of the role of the conditional $\to$. In $\mathrm{S5}^{\mathrm{G}}$ the consequence relation $\vdash_{\mathrm{S5}^{\mathrm{G}}}$ is central and there is little regard for the fact that in this system the conditional $\to$ does not exhibit the usual properties of the material implicative, i.e., the $\to$-deduction theorem fails to hold. In $\mathrm{S5}^{\mathrm{C}}$ the set of theorems is central, and the consequence relation is in effect defined in terms of $\to$ by means of the theorems and the deduction theorem. To make it clear what is at stake here let us denote by S5 the set of theorems of $\mathrm{S5}^{\mathrm{C}}$, or equivalently of $\mathrm{S5}^{\mathrm{G}}$. We define a relation $\vdash^{\to}_{\mathrm{S5}}$ by the condition that $\psi_0, \dots, \psi_{n-1} \vdash^{\to}_{\mathrm{S5}} \varphi$ holds if and only if $\psi_0 \to (\psi_1 \to (\cdots \to (\psi_{n-1} \to \varphi)\cdots))$ is in S5. This relation of course coincides with $\vdash_{\mathrm{S5}^{\mathrm{C}}}$ and hence has the usual properties of the consequence relation of a deductive system, but this depends on the fact that the conditional $\to$ in S5 is essentially the intuitionistic conditional. In the next section we shall see examples among the so-called *substructural logics* of $\mathcal{S}$, which like S5 are characterized by their set of theorems, with conditionals that are weaker that intuitionistic conditional. For such $\mathcal{S}$ the relation $\vdash^{\to}_{\mathcal{S}}$ does not have the usual properties of a consequence relation.

Note 2.8. The predicate calculus, in its standard formalization, is not a deductive system in the sense we have defined. The problem is that, in the standard formalization of first-order predicate logic, the generic-formula symbols (as opposed to individual variables) that occur in the axioms and rules of inference are not allowed to range over all formulas. Consider for example the $\forall$-elimination and $\forall$-introduction rules of Kleene [89]:

$$\forall v\varphi(v) \to \varphi(t) \quad \text{and} \quad \frac{\psi \to \varphi(v)}{\psi \to \forall v\varphi(v)}.$$

In the first, t is a term free for v in $\varphi(v)$ and $\varphi(t)$ is the result of substituting t for all free occurrences of v. In the second, v is not allowed to occur free in ψ. The point is that $\varphi(v)$, $\varphi(t)$, and ψ cannot be replaced by any formulas, as can the sentential variables $p, q, \ldots$ in the axioms and rule schemes of the deductive systems previously considered. Consequently the axioms and rules of predicate calculus cannot be translated into algebraic identities in a direct way as was done before. Before predicate logic can be algebraized, its standard formalization has to be transformed into a deductive system in the present sense. This entails, among other things, a radical transformation of the role of individual variables that will allow us to simulate the metamathematical process of substitution (of terms for variables) at the level of the object language. There are various ways this can be done. One of them is based on the fact that if v does not occur in t, then $\varphi(t)$, the result of substituting t for all free occurrences of v in $\varphi(v)$, is logically equivalent to $\exists v(v = t \wedge \varphi(v))$. This leads to a deductive system PR_ω for predicate logic that we give.

The proof of the completeness theorem for $\mathrm{PR}_{\boldsymbol{\omega}}$ (Thm. 2.21) can be found in [79, Part II]. It is only remotely connected with the Gödel completeness theorem for the first-order predicate logic, but the latter can be given an algebraic form in terms of cylindric algebras as we now describe.

We say that v_n *occurs free in* φ *in the abstract sense* if $\nvdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \mathsf{c}_n\varphi \to \varphi$, i.e., $\nvdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \exists v_n\varphi \to \varphi$. If p is a sentential variable, then $\nvdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \exists v_n p \to p$, since otherwise, because of structurality, we would have $\vdash_{\mathrm{PR}_{\boldsymbol{\omega}}} \exists v_n\varphi \to \varphi$ for every $\varphi \in \mathrm{Fm}$, which is not the case. Thus, for every $n < \omega$, v_n occurs free in p in the abstract sense.

The closest analogue that the sentential variables have in standard predicate logic are the atomic formulas, but they are best viewed as atomic formulas of infinite rank since, as we have seen, every individual variable occurs free in each of them in the abstract sense. To recapture the notion of an atomic formula of finite rank, say k, in this context, we introduce a new constant symbol r into the language together with the infinitely many axioms that say that v_i does not occur free in r in the abstract sense for any i such that $k \leq i < \omega$:

$$\mathsf{c}_i\mathsf{r} \to \mathsf{r}, \quad \text{if } k \leq i < \omega. \tag{3}$$

$\mathrm{PR}_{\boldsymbol{\omega}}$ is a sound and complete formalization of first-order predicate logic in the following sense. Assume for simplicity that the standard predicate logic we are considering contains no function symbols and that each atomic formula is of the form $Rv_0v_1v_2\ldots v_{k-1}$, i.e., the individual variables can occur only in their natural order. It is well known that this does not restrict the expressive power of the logic. Let φ be any sentence of the standard predicate logic. φ can also be considered a sentence of $\mathrm{PR}_{\boldsymbol{\omega}}$ where $Rv_0v_1\ldots v_{k-1}$ is viewed as an alias for the constant r. It can be shown that φ is logically valid iff it is a theorem of the axiomatic extension of $\mathrm{PR}_{\boldsymbol{\omega}}$ by the nonlogical axioms (3) (cf. Monk [106] and also [79, Part II, p.157]).

For more details on the algebraization of predicate logic see [13, Appendix C], [79, Part II] and [3]. For a different approach that leads to a different class of algebras see [77]. For the algebraization of the equational fragment of predicate-logic see Note 3.5 below. Predicate logic can also been formalized within categorical logic; here term-for-variable substitution is simulated functorially. Its algebraization in this context can be found in [136].

Note 2.9. To obtain the axioms and rules of GR we use the fact that GR is equivalent in the sense of Section 4.1 to the applied equational logic of groups, EQ(GR) (Section 3.3.2) via the translations $\boldsymbol{\tau}(p \approx q) = \{p \cdot q^{-1}\}$ and $\rho(p) = \{p \approx e\}$. The axioms and rules of GR were obtained by translating the axioms and rules of EQ(GR), which are the standard equational axioms of groups together with the logical axioms and rules of equality.

Note 2.10. The method of matrices has been a powerful tool in metamathematical research. It provides an alternative to the logistic method for defining deductive systems. Łukasiewicz and independently Post [115] were the first to use matrices to actually define logics, specifically many-valued logics. Two other papers of particular historical interest are [98, 101]. Matrices have been used to define deductive systems that do not have a finite presentation and to establish the independence of various presentations. They have provided a natural way of defining the equivalence of deductive systems over different language types. A general theory of matrices, within the context of the general theory of deductive systems, has been developed by the modern school of propositional logic; for detailed surveys see [32, 138, 140]. In particular, see [138] for proof of Thm. 2.24.

Recently a generalized notion of matrix has proved fruitful in the study of the abstract algebraic logic of Gentzen-style systems. See Note 3.3

3. k-Dimensional Deductive Systems

In this section we define the notion of a *k-dimensional deductive system* and describe its matrix models. We are motivated here by a desire to find a general framework in which both deductive systems, in the sense of Sec. 2, and equational logic can be treated in a uniform way. Our starting point is the observation due to [20] that a deductive system can be viewed as a strict universal Horn theory with a single unary predicate that represents the "assertion" of a formula, i.e., a proposition; this is a Horn theory without equality. Equational logic can be viewed as a strict universal Horn theory with a single binary predicate representing the equality of two formulas (terms). This is a Horn theory *without equality*, that is, there is no primitive binary relation that is interpreted as identity in every model. Strict universal Horn theories without equality and with a single k-ary predicate give rise to the notion of a k-deductive system. After defining these systems and describing their matrix models, we discuss a number of examples of 2-dimensional deductive systems. The one of greatest interest is *free equational logic*. We call the extensions of this system *algebraic*, and they turn out to be one of the key elements in our theory of abstract algebraic logic.

3.1. k-dimensional deductive systems.

Let $\mathcal{L}$ be an arbitrary language type and assume $1 \leq k < \omega$. The basic constituents of a k-deductive system are sequences of formulas of length k, which we call k-formulas. Thus *k-formulas* are the elements of $\mathrm{Fm}_{\mathcal{L}}^k$, where, as usual,

$$\mathrm{Fm}_{\mathcal{L}}^k = \{\langle \varphi_0, \ldots, \varphi_{k-1}\rangle \,:\, \varphi_i \in \mathrm{Fm}_{\mathcal{L}}, i < k\}.$$

A k-formula $\langle \varphi_0, \ldots, \varphi_{k-1}\rangle$ will be denoted by $\boldsymbol{\varphi}$, and in general k-formulas will be denoted by boldface symbols. For 1-formulas we may drop the ordered 1-tuple and boldface notation. By a *k-variable* we will mean a k-formula of the form $\mathbf{p} = \langle p_0, \ldots, p_{k-1}\rangle$, where $p_0, \ldots, p_{k-1}$ are distinct sentential variables.

If $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ is a substitution, and $\boldsymbol{\varphi} = \langle \varphi_0, \dots, \varphi_{k-1} \rangle$ a k-formula, then $\sigma\boldsymbol{\varphi} = \langle \sigma\varphi_0, \dots, \sigma\varphi_{k-1} \rangle$, and for $\Gamma \subseteq \mathrm{Fm}^k$, $\sigma(\Gamma)$ stands for $\{\, \sigma\boldsymbol{\varphi} : \boldsymbol{\varphi} \in \Gamma \,\}$.

Let $\mathcal{L}$ be a sentential language type as in Sec. 2 over the set Va of sentential variables.

Definition 3.1. A *k-dimensional deductive system* $\mathcal{S}$ *(over* $\mathcal{L}$*)* is an ordered pair $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm}^k) \times \mathrm{Fm}^k$ (here $\mathrm{Fm} = \mathrm{Fm}_{\mathcal{L}}$), and the following conditions hold for all $\Gamma, \Delta \subseteq \mathrm{Fm}^k$ and $\varphi \in \mathrm{Fm}^k$.

(i) $\boldsymbol{\varphi} \in \Gamma$ implies $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$;
(ii) $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ and $\Delta \vdash_{\mathcal{S}} \boldsymbol{\psi}$ for every $\boldsymbol{\psi} \in \Gamma$ implies $\Delta \vdash_{\mathcal{S}} \boldsymbol{\varphi}$;
(iii) $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ implies $\Gamma' \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ for some finite $\Gamma' \subseteq \Gamma$;
(iv) $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ implies $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma\boldsymbol{\varphi}$ for every substitution σ.

We sometimes call $\mathcal{S}$ simply a *k-deductive system*. The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation* of $\mathcal{S}$. The condition

(v) $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_{\mathcal{S}} \boldsymbol{\varphi}$

is an easy consequence of conditions (i) and (ii). Observe that a 1-deductive system is just a deductive system as described in Sec. 2.

A *k-inference rule* over $\mathcal{L}$ is a pair $\langle \Gamma, \boldsymbol{\varphi} \rangle$ where Γ is a finite set of k-formulas and $\boldsymbol{\varphi}$ is a single k-formula; if $\Gamma = \emptyset$, then $\boldsymbol{\varphi}$ is called a *k-axiom* over $\mathcal{L}$. A k-formula $\boldsymbol{\psi}$ is *directly derivable* from a set Δ of k-formulas by the k-inference rule $\langle \Gamma, \boldsymbol{\varphi} \rangle$ if there is a substitution $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ such that $\sigma(\Gamma) \subseteq \Delta$ and $\sigma\boldsymbol{\varphi} = \boldsymbol{\psi}$. A set of k-axioms and k-inference rules over $\mathcal{L}$ is a *presentation* of a given k-dimensional deductive system $\mathcal{S}$, if, for all $\Delta \subseteq \mathrm{Fm}^k$ and $\boldsymbol{\psi} \in \mathrm{Fm}^k$, $\Delta \vdash_{\mathcal{S}} \boldsymbol{\psi}$ iff $\boldsymbol{\psi}$ is contained in the smallest set of k-formulas that includes Δ, contains all the substitution instances of the axioms, and is closed under direct derivability with respect to the inference rules.

A set T of k-formulas is called a *theory* of $\mathcal{S}$ (an *$\mathcal{S}$-theory* for short) if $T \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ implies $\boldsymbol{\varphi} \in T$, for each $\boldsymbol{\varphi} \in \mathrm{Fm}^k$. The $\mathcal{S}$-theories are closed under arbitrary intersection and thus form a complete lattice under the partial ordering of set-theoretical inclusion. The set of all $\mathcal{S}$-theories is denoted $\mathrm{Th}\,\mathcal{S}$ and the lattice of $\mathcal{S}$-theories by $\mathbf{Th}\,\mathcal{S} = \langle \mathrm{Th}\,\mathcal{S}, \cap, \vee^{\mathcal{S}} \rangle$. Let $T_i \in \mathrm{Th}\,\mathcal{S}$ for $i \in I$. Their meet in $\mathbf{Th}\,\mathcal{S}$ is $\bigcap_{i \in I} T_i$ and their join is the intersection of all theories that include each T_i, i.e.,

$$\bigvee_{i \in I}^{\mathcal{S}} T_i = \bigcap \{\, S \in \mathrm{Th}\,\mathcal{S} : \ T_i \subseteq S \text{ for all } i \in I \,\}.$$

Given a k-dimensional deductive system $\mathcal{S}$, we define $\mathrm{Cn}_{\mathcal{S}} : \mathcal{P}(\mathrm{Fm}^k) \to \mathcal{P}(\mathrm{Fm}^k)$ by

$$\mathrm{Cn}_{\mathcal{S}}\,\Gamma = \{\, \boldsymbol{\varphi} \in \mathrm{Fm}^k \ : \ \Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi} \,\}.$$

$\mathrm{Cn}_{\mathcal{S}}$ is called the *$\mathcal{S}$-consequence operator*. It is easy to see that $\mathrm{Cn}_{\mathcal{S}}\,\Gamma$ is the smallest $\mathcal{S}$-theory including Γ, i.e., the meet in $\mathbf{Th}\,\mathcal{S}$ of all $\mathcal{S}$-theories that include Γ. An $\mathcal{S}$-theory T is *finitely axiomatized* or *generated* if $T = \mathrm{Cn}_{\mathcal{S}}\,\Gamma$ for some finite $\Gamma \subseteq \mathrm{Fm}^k$. It follows from the assumption that $\mathcal{S}$ is finitary (Def. 3.1(iii)) that the finitely generated theories coincide with the compact elements of $\mathbf{Th}\,\mathcal{S}$. An element a of a complete abstract lattice $\boldsymbol{L}$ is *compact* if $a \leq \bigvee S$ for $S \subseteq L$ implies $a \leq \bigvee S'$ for some finite $S' \subseteq S$. Compactness is an abstract lattice-theoretic property and thus preserved under isomorphism. Hence, if $\mathcal{S}$ and $\mathcal{S}'$ are two deductive systems, then any isomorphism between their respective theory lattices

$\mathbf{Th}\,\mathcal{S}$ and $\mathbf{Th}\,\mathcal{S}'$ must map finitely generated theories into finitely generated theories. We will make essential use of this fact later.

An abstract lattice $\boldsymbol{L}$ is *algebraic* if it is complete and every element is the join of compact elements, where an element a is *compact* if, for every $X \subseteq L$, $a \leq \bigvee X$ implies $a \leq \bigvee X'$ for some finite $X' \subseteq X$. Since the assumption that $\mathcal{S}$ is finitary guarantees that every $\mathcal{S}$-theory is the join of finitely generated theories, a theory is compact in $\mathbf{Th}\,\mathcal{S}$ iff it is finitely generated. Thus $\mathbf{Th}\,\mathcal{S}$ is always algebraic.

For a discussion of algebraic lattices and their connection with consequence relations (more generally, closure operators) see [22] or [103].

Clearly the consequence relation $\vdash_{\mathcal{S}}$ can be recovered from both the set of theories $\mathrm{Th}\,\mathcal{S}$ and the consequence operator $\mathrm{Cn}_{\mathcal{S}}$; hence $\vdash_{\mathcal{S}}$, $\mathrm{Th}\,\mathcal{S}$, and $\mathrm{Cn}_{\mathcal{S}}$ are interderivable in a natural sense, and in fact the deductive system $\mathcal{S}$ is sometimes alternatively defined either as the pair $\langle \mathcal{L}, \mathrm{Th}\,\mathcal{S}\rangle$ or the pair $\langle \mathcal{L}, \mathrm{Cn}_{\mathcal{S}}\rangle$. The following theorem provides a useful characterization of the set of theories of a k-deductive system.

Theorem 3.2. *A set $\mathcal{C}$ of subsets of Fm^k is the set of theories of some k-deductive system iff the following conditions hold.*

(i) *$\mathcal{C}$ is closed under arbitrary intersection, i.e., $\bigcap X \in \mathcal{C}$ for every $X \subseteq \mathcal{C}$;*

(ii) *$\mathcal{C}$ is closed under directed unions, i.e., $\bigcup X \in \mathcal{C}$ for every $X \subseteq \mathcal{C}$ that is upward-directed in the sense that, for every pair $T, T' \in X$, there is a $S \in \mathcal{C}$ such that $T, T' \subseteq S$; and*

(iii) *$\mathcal{C}$ is closed under inverse images of substitutions, i.e., if $T \in \mathcal{C}$, then $\sigma^{-1}(T) \in \mathcal{C}$ for every substitution $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ and every $T \in X$.*

Alternatively, condition (iii) *can be replaced by*

(iii$'$) *$\mathcal{C}$ is closed under inverse images of surjective substitutions.*

Proof. $\Longrightarrow$. Let $\mathcal{S}$ be a k-deductive system. It is clear that $\mathrm{Th}\,\mathcal{S}$ is closed under intersection Let $\{\, T_i : i \in I \,\}$ be an upward-directed subset of $\mathrm{Th}\,\mathcal{S}$, and suppose $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ and $\Gamma \subseteq \bigcup_{i\in I} T_i$. $\Gamma' \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ for some finite $\Gamma' \subseteq \Gamma$. Since the set of T_i is upward-directed, there is a $j \in I$ such that $\Gamma' \subseteq T_j$. Then $\boldsymbol{\varphi} \in T_j \subseteq \bigcup_{i\in I} T_i$. So $\bigcup_{i\in I} T_i$ is in $\mathrm{Th}\,\mathcal{S}$. Thus $\mathrm{Th}\,\mathcal{S}$ is closed under directed unions. Finally, assume $T \in \mathrm{Th}\,\mathcal{S}$ and σ be a substitution. If $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ and $\Gamma \subseteq \sigma^{-1}(T)$, then $\sigma(\Gamma) \vdash_{\mathcal{S}} \sigma\boldsymbol{\varphi}$ and $\sigma(\Gamma) \subseteq T$. So $\sigma\boldsymbol{\varphi} \in T$ and hence $\boldsymbol{\varphi} \in \sigma^{-1}(T)$. Thus $\sigma^{-1}(T) \in \mathrm{Th}\,\mathcal{S}$.

$\Longleftarrow$. Assume conditions (i), (ii), and (iii$'$) hold. Define $\Gamma \vdash_{\mathcal{C}} \boldsymbol{\varphi}$ by the condition that, for every $T \in \mathcal{C}$, $\Gamma \subseteq T$ implies $\boldsymbol{\varphi} \in T$. The first two conditions of Def. 3.1 (with $\vdash_{\mathcal{C}}$ in place of $\vdash_{\mathcal{S}}$) are obvious.

To see that 3.1(iii) holds, assume $\Gamma \vdash_{\mathcal{C}} \boldsymbol{\varphi}$. For each $\Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma$, i.e., for each finite $\Gamma' \subseteq \Gamma$, let $\mathrm{Cn}_{\mathcal{C}}\,\Gamma' = \bigcap\{\, S \in \mathcal{C} : \Gamma' \subseteq S \,\}$. Clearly $\mathrm{Cn}_{\mathcal{C}}\,\Gamma' \in \mathcal{C}$ since $\mathcal{C}$ is closed under intersection. The set $\{\, \mathrm{Cn}_{\mathcal{C}}\,\Gamma' : \Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma \,\}$ is obviously upward-directed. Thus $U = \bigcup\{\, \mathrm{Cn}_{\mathcal{C}}\,\Gamma' : \Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma \,\}$ is a member of $\mathcal{C}$. This implies $\boldsymbol{\varphi} \in U$ since $\Gamma = \bigcup\{\, \Gamma' : \Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma \,\} \subseteq U$. Thus $\boldsymbol{\varphi} \in \mathrm{Cn}_{\mathcal{C}}\,\Gamma'$, i.e., $\Gamma' \vdash_{\mathcal{C}} \boldsymbol{\varphi}$, for some $\Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma$.

Finally, assume $\sigma(\Gamma) \nvdash_{\mathcal{C}} \sigma\boldsymbol{\varphi}$. By what we just proved above $\sigma(\Gamma') \nvdash_{\mathcal{C}} \sigma\boldsymbol{\varphi}$ for every $\Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma$, since $\sigma(\Gamma') \subseteq_{\boldsymbol{\omega}} \sigma(\Gamma)$ for every substitution σ. Consider a fixed but arbitrary $\Gamma' \subseteq_{\boldsymbol{\omega}} \Gamma$. Since there are only finitely many variables in $\Gamma' \cup \{\boldsymbol{\varphi}\}$, there is a surjective substitution σ' such that $\sigma\boldsymbol{\psi} = \sigma'\boldsymbol{\psi}$ for every $\boldsymbol{\psi} \in \Gamma' \cup \{\boldsymbol{\varphi}\}$. By assumption there is a $T \in \mathcal{C}$

such that $\sigma'(\Gamma') \subseteq T$ and $\sigma'\boldsymbol{\varphi} \notin T$. Thus, $\Gamma' \subseteq \sigma'^{-1}\sigma'(\Gamma') \subseteq \sigma'^{-1}(T)$ and $\boldsymbol{\varphi} \notin \sigma'^{-1}(T)$ and $\sigma'^{-1}(T) \in \mathcal{C}$ by hypothesis. Thus $\Gamma' \nvdash_{\mathcal{C}} \boldsymbol{\varphi}$ for every $\Gamma' \subseteq_{\omega} \boldsymbol{\varphi}$, and hence $\Gamma \nvdash \boldsymbol{\varphi}$. This establishes 3.1(iv). So $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{C}} \rangle$ is a k-deductive system.

It remains only to show that $\mathcal{C} = \operatorname{Th} \mathcal{S}$. Suppose $T \in \operatorname{Th} \mathcal{S}$. Then $T \vdash_{\mathcal{C}} \boldsymbol{\varphi}$ implies $\boldsymbol{\varphi} \in T$, i.e., $\operatorname{Cn}_{\mathcal{S}} T \subseteq T$ and hence $\operatorname{Cn}_{\mathcal{S}} T = T$. So $T \in \mathcal{C}$ since $\mathcal{C}$ is closed under intersection. Conversely, assume $T \in \mathcal{C}$, i.e., $\operatorname{Cn}_{\mathcal{C}} T = T$. If $T \vdash_{\mathcal{C}} \boldsymbol{\varphi}$, then $\boldsymbol{\varphi} \in \operatorname{Cn}_{\mathcal{C}} T = T$, so $T \in \operatorname{Th} \mathcal{S}$. □

The connection between k-deductive systems and universal Horn logic is described in Note 3.1. A discussion of how Gentzen-style logical systems can be viewed as generalized deductive systems can be found in Note 3.2.

3.2. **Matrix semantics for k-deductive systems.** Let $\boldsymbol{A}$ be an algebra of type $\mathcal{L}$. A k-tuple $\langle a_0, \dots, a_{k-1} \rangle$ of elements of $\boldsymbol{A}$ will be called a *k-element* of $\boldsymbol{A}$ and denoted by boldface letters, e.g., $\mathbf{a} = \langle a_0, \dots, a_{k-1} \rangle$. A *$k$-matrix* over $\mathcal{L}$ is a pair $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra, and $F \subseteq A^k$. An $\mathcal{L}$-matrix is thus just a 1-matrix over $\mathcal{L}$. The elements of F are called the *designated k-elements* of the matrix. If $\mathfrak{A}$ is a k-matrix, then $\boldsymbol{A}$ denotes its underlying algebra and $F^{\mathfrak{A}}$ its set of designated k-tuples; the superscript "$\mathfrak{A}$" on $F^{\mathfrak{A}}$ is usually omitted.

We define the relation $\vDash_{\mathfrak{A}}$ for a k-matrix $\mathfrak{A}$ as in Sec. 2.3: if $\Gamma \subseteq \operatorname{Fm}^k$ and $\boldsymbol{\varphi} \in \operatorname{Fm}^k$, then we define $\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi}$ to be the relation that holds between Γ and $\boldsymbol{\varphi}$ if, for every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\boldsymbol{\varphi}\}$ in A, we have

$$\boldsymbol{\psi}^{\boldsymbol{A}}(\bar{a}) \in F \text{ for every } \boldsymbol{\psi} \in \Gamma \text{ implies } \boldsymbol{\varphi}^{\boldsymbol{A}}(\bar{a}) \in F.$$

The interpretation $\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar{a})$ of $\boldsymbol{\varphi}$ in $\boldsymbol{A}$ under the assignment $\bar{a}$ is defined in the natural way: Let $\boldsymbol{\varphi} = \langle \varphi_0, \dots, \varphi_{k-1} \rangle$; then we write $\boldsymbol{\varphi} = \boldsymbol{\varphi}(p_0, \dots, p_{n-1})$ to indicate that the variables of φ_i, for $i < k$, are all included in the list $p_0, \dots, p_{n-1}$, and we write $\boldsymbol{\varphi}^{\boldsymbol{A}}(a_0, \dots, a_{n-1})$, or just $\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar{a})$, for $\langle \varphi_0^{\boldsymbol{A}}(a_0, \dots, a_{n-1}), \dots, \varphi_{k-1}^{\boldsymbol{A}}(a_0, \dots, a_{n-1}) \rangle$. Similarly, for $\Gamma \subseteq \operatorname{Fm}^k$, $\Gamma^{\boldsymbol{A}}(\bar{a})$ denotes $\{\boldsymbol{\psi}^{\boldsymbol{A}}(\bar{a}) : \boldsymbol{\psi} \in \Gamma\}$. Finally, for any class M of k-matrices, $\Gamma \vDash_{\mathsf{M}} \boldsymbol{\varphi}$ iff $\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi}$ for all $\mathfrak{A} \in \mathsf{M}$.

The definition of $\vDash_{\mathfrak{A}}$ for k-matrices $\mathfrak{A}$ can be reformulated in more algebraic terms as was done for 1-matrices in Sec. 2.3: If $h : \mathbf{Fm} \to \boldsymbol{A}$ is a homomorphism and $\boldsymbol{\varphi} = \langle \varphi_0, \dots, \varphi_{k-1} \rangle$ is a k-formula, we write $h\boldsymbol{\varphi} = \langle h\varphi_0, \dots, h\varphi_{k-1} \rangle$. Then $\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi}$ iff

$$h\boldsymbol{\psi} \in F \text{ for every } \boldsymbol{\psi} \in \Gamma \quad \text{implies} \quad h\boldsymbol{\varphi} \in F, \qquad \text{for every } h : \mathbf{Fm} \to \boldsymbol{A}.$$

Let $\mathcal{S}$ be a k-dimensional deductive system. A k-matrix $\mathfrak{A}$ is called a *matrix model* of $\mathcal{S}$, or an *$\mathcal{S}$-matrix* for short, if $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ implies $\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi}$, for all $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \operatorname{Fm}^k$. A subset $F \subseteq A^k$ is called an *$\mathcal{S}$-filter*, or simply a *filter* when $\mathcal{S}$ is clear from context, if the matrix $\langle \boldsymbol{A}, F \rangle$ is an $\mathcal{S}$-matrix. Thus the $\mathcal{S}$-matrices are exactly the models (in the first-order sense) of the universal Horn theory $H(\mathcal{S})$ in $\mathcal{L}_D$ associated with $\mathcal{S}$; see Note 3.1.

The set of $\mathcal{S}$-filters on $\boldsymbol{A}$ is denoted by $\operatorname{Fi}_{\mathcal{S}} \boldsymbol{A}$. The $\mathcal{S}$-filters on the formula algebra are exactly the $\mathcal{S}$-theories; this is a consequence of the next lemma. The entire discussion of theories in Sec. 3.1 applies almost without exception to filters on an arbitrary algebra. In particular $\operatorname{Fi}_{\mathcal{S}} \boldsymbol{A}$ is closed under arbitrary intersection and forms an algebraic lattice, which we denote by $\mathbf{Fi}_{\mathcal{S}} \boldsymbol{A}$. For every $X \subseteq A^k$ there is a smallest $\mathcal{S}$-filter including X, called the

$\mathcal{S}$-filter generated by X. It is denoted by $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X$. The finitely generated filters coincide with the compact elements of $\mathbf{Fi}_{\mathcal{S}}\, \boldsymbol{A}$.

Lemma 3.3. *Let $\mathcal{S}$ be a k-dimensional deductive system, $\Gamma \subseteq \mathrm{Fm}^k$, and $\boldsymbol{\varphi} \in \mathrm{Fm}^k$. The following are equivalent.*

(i) $\boldsymbol{\varphi} \in \mathrm{Cn}_{\mathcal{S}}\, \Gamma$, *i.e.*, $\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi}$,
(ii) *For all algebras $\boldsymbol{A}$, and for all $\bar{a} \in A^{\omega}$, $\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar{a}) \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\big(\Gamma^{\boldsymbol{A}}(\bar{a})\big)$,*
(iii) $\boldsymbol{\varphi} \in \mathrm{Fg}^{\mathrm{Fm}}\, \Gamma$.

Proof. The implication (i) $\Rightarrow$ (ii) follows immediately from the definition of $\mathcal{S}$-filter of $\boldsymbol{A}$. The implication (ii) $\Rightarrow$ (iii) is obvious since (iii) is a special case of (ii). It remains to show (iii) implies (i).

So suppose $\Gamma \nvdash_{\mathcal{S}} \boldsymbol{\varphi}$. Let $\mathfrak{A} = \langle \boldsymbol{A}, T\rangle$ with $\boldsymbol{A} = \mathbf{Fm}$ and $T = \{\boldsymbol{\psi} : \Gamma \vdash_{\mathcal{S}} \boldsymbol{\psi}\}$. We claim that T is an $\mathcal{S}$-filter of $\mathbf{Fm}$. In order to show this, we need to verify that $\mathfrak{A}$ is an $\mathcal{S}$-matrix. Let $\Delta \cup \{\boldsymbol{\psi}\} \subseteq \mathrm{Fm}^k$, and assume $\Delta \vdash_{\mathcal{S}} \boldsymbol{\psi}$ and $h : \mathbf{Fm} \to \boldsymbol{A}$ such that $h(\Delta) \subseteq T$. Since $\boldsymbol{A} = \mathbf{Fm}$, h is a substitution. Thus by structurality, $h(\Delta) \vdash_{\mathcal{S}} h\boldsymbol{\psi}$, and hence $h\boldsymbol{\psi} \in T$ as well since T is an $\mathcal{S}$-theory. So T is an $\mathcal{S}$-filter. Thus $\mathrm{Fg}^{\mathbf{Fm}}_{\mathcal{S}}\, \Gamma \subseteq T$ since oviously $\Gamma \subseteq T$. (They are actually equal, but we do not need this fact.) By assumption $\boldsymbol{\varphi} \notin T$, so $\boldsymbol{\varphi} \notin \mathrm{Fg}^{\mathbf{Fm}}_{\mathcal{S}}\, \Gamma$. □

Note that $\mathrm{Cn}_{\mathcal{S}}\, \Gamma = \mathrm{Fg}^{\mathbf{Fm}}\, \Gamma$ for every $\Gamma \subseteq \mathrm{Fm}^k$; thus the $\mathcal{S}$-theories are exactly the $\mathcal{S}$-filters of the formula algebra.

The basic completeness theorem is now immediate.

Theorem 3.4. *Let $\mathcal{S}$ be a k-dimensional deductive system and M the class of all matrix models of $\mathcal{S}$. Then for all $\Gamma \subseteq \mathrm{Fm}^k$ and $\boldsymbol{\varphi} \in \mathrm{Fm}^k$, we have*

$$\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi} \quad \textit{iff} \quad \Gamma \vDash_{\mathsf{M}} \boldsymbol{\varphi}.$$

Proof. The implication from left to right holds by definition of M. For the converse, if $\Gamma \nvdash_{\mathcal{S}} \boldsymbol{\varphi}$, let $\mathfrak{A} = \langle \boldsymbol{A}, T\rangle$ with $\boldsymbol{A} = \mathbf{Fm}$ and $T = \mathrm{Cn}_{\mathcal{S}}\, \Gamma$. By the previous lemma, $\mathfrak{A}$ is a $\mathcal{S}$-matrix. Clearly, $\Gamma \nvDash_{\mathfrak{A}} \boldsymbol{\varphi}$, so $\Gamma \nvDash_{\mathsf{M}} \boldsymbol{\varphi}$. □

A generalized notion of matrix model has recently become an important area of investigation in abstract algebraic logic. The generalized matrix models of a deductive system $\mathcal{S}$ are closely related to the Gentzen systems for $\mathcal{S}$ discussed in Note 3.2. For more details see Note 3.3.

The notion of *completeness* established in this theorem is less significant than it may appear, at least from an algebraic point of view: Every k-deductive system is complete with respect to a class of matrices whose underlying algebra is absolutely free and hence satisfies no identities other than the equational tautologies $\varphi \approx \varphi$. Thus every deductive system has a matrix semantics whose underlying algebras have no meaningful structure and whose semantical significance lies exclusively in the sets of designated k-elements. Contrast this with the algebraic semantics for the various deductive systems we considered in Sec. 2.2. What we need is a more restricted notion of completeness that is applicable to deductive systems of the most general kind and that gives a semantics that is truly algebraic in the sense that the equational logic of the class of underlying algebras is correlated, to the extent this is possible, with the logical properties of the deductive system. It turns out that the key here is the notion of a *reduced matrix*, which we now consider.

Let $\mathfrak{A}$ be a k-matrix, $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$. A congruence Θ of $\boldsymbol{A}$ is *compatible* with F if for all $\mathbf{a}, \mathbf{b} \in A^k$, $a_i \,\Theta\, b_i$ for $i < k$ and $\mathbf{a} \in F$ imply $\mathbf{b} \in F$, i.e., F is a union of Cartesian products of Θ-equivalence classes. We write $\mathbf{a}\, \Theta^k\, \mathbf{b}$ as shorthand for the condition that $a_i \,\Theta\, b_i$ for all $i < k$. Let Φ and Ψ be two congruences of $\boldsymbol{A}$ that are compatible with F. Then the relative product

$$\Phi|\Psi := \{\, \langle a, b\rangle : \exists c \in A\big(\langle a, c\rangle \in \Phi \text{ and } \langle c, b\rangle \in \Psi\big) \,\}$$

is also clearly compatible with F, and consequently, so is the join $\Phi \vee \Psi = \bigcup_{k<\omega} \Phi|^k\Psi$ in the lattice of congruences ($\Phi|^0\Psi = \boldsymbol{\Delta}_A$, the identity congruence, and $\Phi|^{k+1}\Psi = (\Phi|^k\Psi)|(\Phi|\Psi)$). Thus the set of all congruences compatible with F is directed under set-theoretical inclusion and hence the union is again a compatible congruence. So the largest congruence of $\boldsymbol{A}$ compatible with F always exists.

Definition 3.5. Let $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ be a k-matrix. The largest congruence on $\boldsymbol{A}$ compatible with F is called the *Leibniz congruence* of $\boldsymbol{A}$ over F and is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}} F$.

The following technical result expresses a very useful property of Leibniz congruences we will make use of later.

Lemma 3.6. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\mathcal{L}$-algebras and $h : \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ a surjective homomorphism. Then for every $F \subseteq B^k$,*

$$\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big) = h^{-1}(\boldsymbol{\Omega}^{\boldsymbol{B}} F).$$

Proof. $h^{-1}(F) := \{\, \langle a_0, \dots, a_{k-1}\rangle : \langle h(a_0), \dots, h(a_{k-1})\rangle \in F \,\}$. It is easy to check that $h^{-1}(\boldsymbol{\Omega}^{\boldsymbol{B}} F)$ is a congruence on $\boldsymbol{A}$ compatible with $h^{-1}(F)$; so $\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big) \supseteq h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$.

Let $\Theta = h^{-1}(\boldsymbol{\Delta}_B)$, the *relation kernel* of h. Θ is compatible with $h^{-1}(F)$, so $\Theta \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)$. It follows that $h\big(\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)\big)$ is a congruence of $\boldsymbol{B}$ and, because h is surjective, that $h^{-1}h\big(\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)$. So $h\big(\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)\big)$ is compatible with $h\,h^{-1}(F)$, which equals F since h is surjective. Thus $h\big(\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)\big) \subseteq \boldsymbol{\Omega}_{\boldsymbol{B}} F$, and hence

$$\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big) = h^{-1}h\big(\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big)\big) \subseteq h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F).$$

□

Lemma 3.7. *Let $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ be a k-matrix and $a, b \in A$. Then $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ iff, for each $\boldsymbol{\varphi}(x, y_0, \dots, y_{n-1}) \in \mathrm{Fm}^k$ and all $c_0, \dots, c_{n-1} \in A$*

$$\boldsymbol{\varphi}^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1}) \in F \quad \textit{iff} \quad \boldsymbol{\varphi}^{\boldsymbol{A}}(b, c_0, \dots, c_{n-1}) \in F.$$

Proof. Let Θ be the set of all pairs $\langle a, b\rangle$ that satisfy the condition. Then Θ is clearly an equivalence relation. To show it is a congruence relation, consider any $\omega \in \mathcal{L}$ and let m be its rank. Suppose $\langle a_i, b_i\rangle \in \Theta$ for each $i < m$. We must show $\langle \omega^{\boldsymbol{A}}(a_0, \dots, a_{m-1},\ \omega^{\boldsymbol{A}}(b_0, \dots, b_{m-1})\rangle \in \Theta$, and since Θ is transitive, it suffices to show for each $i < m$ that

$$\langle \omega^{\boldsymbol{A}}(\bar{b}_{<i}, a_i, \bar{a}_{>i})\, \omega^{\boldsymbol{A}}(\bar{b}_{<i}, b_i, \bar{a}_{>i})\rangle \in \Theta,$$

where $\bar{b}_{<i} = b_0, \dots, b_{i-1}$ and $\bar{a}_{>i} = a_{i+1}, \dots, a_{m-1}$. I.e., it suffices to show that for each $\boldsymbol{\varphi}(x, y_0, \dots, y_{n-1}) \in \mathrm{Fm}^k$ and all $\bar{c} = c_0, \dots, c_{n-1} \in A^m$,

$$\boldsymbol{\varphi}^{\boldsymbol{A}}\big(\omega^{\boldsymbol{A}}(\bar{b}_{<i}, a_i, \bar{a}_{>i}), \bar{c}\big) \in F \quad \text{iff} \quad \boldsymbol{\varphi}^{\boldsymbol{A}}\big(\omega^{\boldsymbol{A}}(\bar{b}_{<i}, b_i, \bar{a}_{>i}), \bar{c}\big) \in F.$$

Take $\boldsymbol{\psi}(x, z_0, \dots, z_{i-1}, w_{i+1}, \dots, w_{m-1}, y_0, \dots, y_{n-1})$ to be $\boldsymbol{\varphi}(\omega \bar{z}_{<i} x \bar{w}_{>i+1}, \bar{y})$. Then

$$\boldsymbol{\varphi}^{\boldsymbol{A}}\big(\omega^{\boldsymbol{A}}(\bar{b}_{<i}, a_i, \bar{a}_{>i}), \bar{c}\big) = \boldsymbol{\psi}^{\boldsymbol{A}}(a_i, \bar{b}_{<i}, \bar{a}_{>i}, \bar{c}) \in F$$
$$\text{iff} \quad \boldsymbol{\varphi}^{\boldsymbol{A}}\big(\omega^{\boldsymbol{A}}(\bar{b}_{<i}, b_i, \bar{a}_{>i}), \bar{c}\big) = \boldsymbol{\psi}^{\boldsymbol{A}}(b_i, \bar{b}_{<i}, \bar{a}_{>i}, \bar{c}) \in F.$$

Thus Θ is a congruence relation. To see Θ is compatible with F, let $\mathbf{a} \in F$, $\mathbf{b} \in A^k$, such that $\mathbf{a}\, \Theta^k\, \mathbf{b}$. In order to show $\mathbf{b} \in F$ it suffices to show, for each $i < k$, that $\langle \bar{b}_{<i}, a_i, \bar{a}_{>i} \rangle \in F$ implies $\langle \bar{b}_{<i}, b_i, \bar{a}_{>i} \rangle \in F$. To this end, choose $\boldsymbol{\varphi}$ to be the k-variable $\boldsymbol{\varphi}(x, \bar{y}_{<i}, \bar{z}_{>i}) = \langle y_0, \dots, y_{i-1}, x, z_{i+1}, \dots, z_{k-1} \rangle$. Then $\boldsymbol{\varphi}^{\boldsymbol{A}}(a_i, \bar{b}_{<i}, \bar{a}_{>i}) = \langle \bar{b}_{<i}, a_i, \bar{a}_{>i} \rangle \in F$, and hence

$$\boldsymbol{\varphi}^{\boldsymbol{A}}(b_i, \bar{b}_{<i}, \bar{a}_{>i}) = \langle \bar{b}_{<i}, b_i, \bar{a}_{>i} \rangle \in F$$

as well. We have thus shown that Θ is both a congruence of $\boldsymbol{A}$ and compatible with F, and conclude that $\Theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$.

Conversely, if $\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$, then from the fact that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is a congruence relation it can be proved for each component φ_i of $\boldsymbol{\varphi}$ that $\big\langle \varphi_i^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1}), \varphi_i^{\boldsymbol{A}}(b, c_0, \dots, c_{n-1}) \big\rangle \in (\boldsymbol{\Omega}^{\boldsymbol{A}} F)^k$; the proof is a straightforward induction on the length of φ_i. It follows by the compatibility condition that $\boldsymbol{\varphi}^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1}) \in F$ iff $\boldsymbol{\varphi}^{\boldsymbol{A}}(b, c_0, \dots, c_{n-1}) \in F$. Thus $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \Theta$. □

This lemma justifies the term "Leibniz congruence" for $\boldsymbol{\Omega}^{\boldsymbol{A}} F$. It formalizes the well-known Leibniz criterion of equality according to which two objects of a domain are equal iff they have exactly the same properties: $\varphi^{\boldsymbol{A}}(a, c_0, \dots, c_{n-1}) \in F$ expresses the assertion that a has the "property" $\varphi(x, y_0, \dots, y_{n-1})$ relative to the "truth set" F and relative to the fixed but arbitrary system of objects $c_0, \dots, c_{n-1}$. In other words, $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ can be viewed as the abstract relation of logical equivalence (relative to the "truth set" F). This forms the basis of the abstraction of the classical Lindenbaum-Tarski process.

The characterization of the Leibniz congruence in the context of 1-deductive systems was first given in [126]. It has played an important role in the metatheory of sentential logic; see [138]. The term "Leibniz congruence" was first used in [13].

Let $\boldsymbol{A}^* = \boldsymbol{A}/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$ and $\mathfrak{A}^* = \langle \boldsymbol{A}^*, F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F \rangle$. Observe that we must have $\boldsymbol{\Omega}^{\boldsymbol{A}^*}(F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F) = \boldsymbol{\Delta}_{A^*}$, since otherwise the inverse image of $\boldsymbol{\Omega}^{\boldsymbol{A}^*}(F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F)$ under the natural map from $\boldsymbol{A}$ to $\boldsymbol{A}^*$ would be a congruence of $\boldsymbol{A}$ that is compatible with F and strictly larger than $\boldsymbol{\Omega}^{\boldsymbol{A}} F$. A matrix $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$ with the property $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \boldsymbol{\Delta}_A$ is called a *reduced* matrix; thus for any matrix $\mathfrak{A}$ the quotient $\mathfrak{A}^*$ is reduced.

It is easy to verify that for all $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^k$ and any matrix $\mathfrak{A}$ we have

$$\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi} \quad \text{iff} \quad \Gamma \vDash_{\mathfrak{A}^*} \boldsymbol{\varphi}. \tag{4}$$

To see this, suppose $\Gamma \vDash_{\mathfrak{A}} \boldsymbol{\varphi}$. Let $a_0^*, a_1^*, a_2^*, \dots$ be an assignment of elements of $\boldsymbol{A}^*$ to the variables $p_0, p_1, p_2, \dots$ of $\Gamma \cup \{\boldsymbol{\varphi}\}$ such that $\boldsymbol{\psi}^{\boldsymbol{A}^*}(a_0^*, a_1^*, a_2^*, \dots) \in F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$ for all $\boldsymbol{\psi} \in \Gamma$. Choose $a_0, a_1, a_2, \dots \in A$ such that $a_i^* = a_i/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$ for all i. By the basic property of congruences we have that $\boldsymbol{\psi}^{\boldsymbol{A}}(a_0, a_1, a_2, \dots)/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F = \boldsymbol{\psi}^{\boldsymbol{A}^*}(a_0^*, a_1^*, a_2^*, \dots)$ for every $\boldsymbol{\psi} \in \Gamma$. Consequently, since $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is compatible with F, $\boldsymbol{\psi}^{\boldsymbol{A}}(a_0, a_1, a_2, \dots) \in F$ for every $\boldsymbol{\psi} \in \Gamma$, and hence $\boldsymbol{\varphi}^{\boldsymbol{A}}(a_0, a_1, a_2, \dots) \in F$ and thus $\boldsymbol{\varphi}^{\boldsymbol{A}^*}(a_0^*, a_1^*, a_2^*, \dots) \in F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$. This verifies the implication from left to right in (4). The proof of the implication in the opposite direction is verified in a similar manner.

This yields the following sharpening of Thm. 3.4:

Theorem 3.8. *Let $\mathcal{S}$ be a k-dimensional deductive system and M^* the class of all reduced $\mathcal{S}$-matrices. Then for all $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^k$ we have*

$$\Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi} \quad \textit{iff} \quad \Gamma \vDash_{\mathsf{M}^*} \boldsymbol{\varphi}.$$

The algebraic completeness theorems given in Sec. 2.2 for the 1-dimensional deductive systems CPC, IPC, BCK, K, $\mathrm{S5^G}$, PR_ω, and GR can all be viewed as special cases of Theorem 3.8 in a sense we now explain. Consider for example a CPC-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F \rangle$. We observe first of all that, for all $a, b \in A$,

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \quad \text{iff} \quad a \leftrightarrow^{\boldsymbol{A}} b \in F.$$

To see this let $\equiv_F \,=\, \{\, \langle a, b \rangle : a \leftrightarrow^{\boldsymbol{A}} b \in F \,\}$. Then $\equiv_F$ is a congruence relation on $\boldsymbol{A}$. This is established by essentially the same argument we outlined in Sec. 2.2.1 to show that $\equiv_T$ is a congruence relation on $\mathbf{Fm}$ for an arbitrary CPC-theory T. Note also that for the same reason the quotient $\boldsymbol{A}/{\equiv_F}$ is a Boolean algebra. By modus ponens, $a \leftrightarrow^{\boldsymbol{A}} b, a \in F$ implies $b \in F$. So $\equiv_F$ is compatible with F and hence $\equiv_F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Conversely, assume $\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$. Then $\langle a \leftrightarrow^{\boldsymbol{A}} a,\ a \leftrightarrow^{\boldsymbol{A}} b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ since $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is a congruence relation. But $a \leftrightarrow^{\boldsymbol{A}} a \in F$, and hence $a \leftrightarrow^{\boldsymbol{A}} b \in F$ by the compatibility of $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ with F. So $a \equiv_F b$. This shows that $\boldsymbol{\Omega}^{\boldsymbol{A}} F = {\equiv_F}$.

Since $p \leftrightarrow (p \leftrightarrow \top)$ is a tautology, $a \leftrightarrow^{\boldsymbol{A}} \top^{\boldsymbol{A}} \in F$ for every $a \in F$, and thus $\langle a, \top^{\boldsymbol{A}} \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ for every $a \in F$, i.e, $\top^{\boldsymbol{A}}/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F = F$. Hence the reduced CPC-matrices are all of the form $\langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\} \rangle$, with $\boldsymbol{A}$ a Boolean algebra. This shows that, for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, the semantic entailment relation $\Gamma \approx \top \vDash_{\boldsymbol{A}} \varphi \approx \top$ defined in Sec. 2.2 is the same as the $\Gamma \vDash_{\mathfrak{A}} \varphi$ defined above with $\mathfrak{A} = \langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\} \rangle$. It also shows that every CPC-filter F on any algebra $\boldsymbol{A}$, not necessarily a Boolean algebra, is the inverse image of a Boolean filter on the Boolean algebra $\boldsymbol{A}^*$, in fact of the trivial filter $\{\top^{\boldsymbol{A}^*}\}$.

Similar reasoning applies to the examples IPC, BCK, K, $\mathrm{S5^G}$, and PR_ω. If $\langle \boldsymbol{A}, F \rangle$ is a GR-matrix, then $F = e^{\boldsymbol{A}}/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$, and hence the reduced GR-matrices are of the form $\langle \boldsymbol{A}, \{e^{\boldsymbol{A}}\} \rangle$, with $\boldsymbol{A}$ a group.

The completeness theorem for $\mathrm{S5^C}$ (Thm. 2.19) can also be viewed as a special case of Thm. 3.8, but of a different kind because the filter of a reduced $\mathrm{S5^C}$-matrix need not be a singleton. Indeed, the reduced $\mathrm{S5^C}$-matrices are precisely the reduced filtered monadic algebras $\langle \boldsymbol{A}, F \rangle$ defined in the paragraph before Thm. 2.19. To see this recall that F is a filter in the sense of Sec. 2.2 iff F is an $\mathrm{S5^C}$-filter, and it is an open filter in the sense of Sec. 2.2 iff it is an $\mathrm{S5^G}$-filter. Thus $\langle \boldsymbol{A}, F \rangle$ is a filtered monadic algebra iff it is an $\mathrm{S5^C}$-matrix whose underlying algebra is monadic, and it is reduced as a filtered monadic algebra iff F includes no proper $\mathrm{S5^G}$-filter.

Suppose $\langle \boldsymbol{A}, F \rangle$ is reduced as a filtered monadic algebra, and let $G = \top^{\boldsymbol{A}}/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$. G is clearly an $\mathrm{S5^C}$-filter and since $\Box^{\boldsymbol{A}} \top^{\boldsymbol{A}} = \top^{\boldsymbol{A}}$ it is also an $\mathrm{S5^G}$-filter. Consequently, $G = \{\top^{\boldsymbol{A}}\}$. $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is trivially compatible with G, so $\boldsymbol{\Omega}^{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G = \boldsymbol{\Delta}_A$, and hence $\langle \boldsymbol{A}, F \rangle$ is reduced as an $\mathrm{S5^C}$-matrix.

Suppose conversely that $\langle \boldsymbol{A}, F\rangle$ is not reduced as a filtered monadic algebra. Then F includes a proper $\mathrm{S5}^{\mathrm{G}}$-filter G. Then $\boldsymbol{\Omega}^{\boldsymbol{A}} G \neq \boldsymbol{\Delta}_A$. But $\boldsymbol{\Omega}^{\boldsymbol{A}} G$ is also compatible with F since, if $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} G$ with $a \in F$, then $\top^{\boldsymbol{A}} = a \to^{\boldsymbol{A}} a$ and $\langle a \to^{\boldsymbol{A}} a,\, a \to^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} G$, and hence $a \to^{\boldsymbol{A}} b \in G \subseteq F$. We can now conclude by modus ponens that $b \in F$. Thus $\boldsymbol{\Omega}^{\boldsymbol{A}} F \supseteq \boldsymbol{\Omega}^{\boldsymbol{A}} G$ and hence $\langle \boldsymbol{A}, F\rangle$ fails to be reduced as an $\mathrm{S5}^{\mathrm{C}}$-matrix.

3.3. **Examples.** A number of different logics can either be viewed as 2-deductive systems or can be naturally reformulated as such. The most important example of a 2-deductive system is equational. A 2-formula $\langle \varphi, \psi\rangle$ is to be interpreted in this context as the equation $\varphi \approx \psi$.

3.3.1. *Free equational logic* ($\mathrm{EQ}_{\mathcal{L}}$). Let $\mathcal{L} = \{\omega_i : i \in I\}$ be any language. The 2-dimensional deductive system $\mathrm{EQ}_{\mathcal{L}}$ has for its axioms and rules:

(E1) $\langle p, p\rangle$;

(ER1) $$\frac{\langle p, q\rangle}{\langle q, p\rangle};$$

(ER2) $$\frac{\langle p, q\rangle, \langle q, r\rangle}{\langle p, r\rangle}.$$

($\mathrm{ER3}_\omega$) $$\frac{\langle p_0, q_0\rangle, \dots, \langle p_{n-1}, q_{n-1}\rangle}{\langle \omega p_0, \dots, p_{n-1},\, \omega q_0, \dots, q_{n-1}\rangle}, \quad \text{for each } \omega \in \mathcal{L},\ n \text{ the rank of } \omega.$$

The subscript "$\mathcal{L}$" on $\mathrm{EQ}_{\mathcal{L}}$ will normally be omitted if clear from context.

A EQ-matrix is a 2-matrix $\mathfrak{A} = \langle \boldsymbol{A}, \Theta\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and Θ is a congruence relation on $\boldsymbol{A}$. The set of congruences on $\boldsymbol{A}$ is denoted by $\mathrm{Co}\,\boldsymbol{A}$; thus $\mathrm{Co}\,\boldsymbol{A} = \mathrm{Fi}_{\mathrm{EQ}}\,\boldsymbol{A}$. Let Θ and Φ be congruences of $\boldsymbol{A}$. Then Φ is compatible with Θ (viewed as a set of designated pairs of elements of $\boldsymbol{A}$) iff $\Phi \subseteq \Theta$. To see this assume first of all that $\Phi \subseteq \Theta$. If $\langle a_1, b_1\rangle \in \Theta$ and $a_1 \,\Phi\, a_2$ and $b_1 \,\Phi\, b_2$, then $\langle a_2, b_2\rangle \in \Theta$ by the transitivity of Θ. So Φ is compatible with $\boldsymbol{A}$. Assume conversely that Φ is compatible with $\boldsymbol{A}$. Let $\langle a, b\rangle \in \Phi$. $\langle a, a\rangle \in \Theta$ since Θ is reflexive. Thus from $\langle a, b\rangle, \langle a, a\rangle \in \Phi$ and compatibility we have $< a, b\rangle \in \Theta$. So $\Phi \subseteq \Theta$.

Thus $\boldsymbol{\Omega}^{\boldsymbol{A}} \Theta = \Theta$ for every congruence Θ, and hence the reduced EQ-matrices are all of the form $\langle \boldsymbol{A}, \Delta_A\rangle$. For a 2-formula $\boldsymbol{\varphi} = \langle \varphi_0, \varphi_1\rangle$ and a reduced EQ-matrix $\mathfrak{A} = \langle \boldsymbol{A}, \Delta_A\rangle$, we have $\vDash_{\mathfrak{A}} \boldsymbol{\varphi}$ iff $\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar a) = \langle \varphi_0^{\boldsymbol{A}}(\bar a), \varphi_1^{\boldsymbol{A}}(\bar a)\rangle \in \Delta_A$ for all $\bar a \in A^\omega$ iff $\vDash_{\boldsymbol{A}} \varphi_0 \approx \varphi_1$. Satisfaction of a 2-formula in a reduced EQ-matrix $\mathfrak{A}$ thus amounts to satisfaction of an equation in the underlying algebra $\boldsymbol{A}$ of $\mathfrak{A}$.

3.3.2. *Applied equational logic* ($\mathrm{EQ}(\mathsf{K})$). See Note 3.4. We associate with any quasivariety K of type $\mathcal{L}$ a 2-dimensional deductive system $\mathrm{EQ}(\mathsf{K})$. The reduced $\mathrm{EQ}(\mathsf{K})$-matrices are of the form $\langle \boldsymbol{A}, \Delta_A\rangle$, for $\boldsymbol{A} \in \mathsf{K}$.

$\mathrm{EQ}(\mathsf{K})$ is an extension of EQ, so it has (E1), (ER1), (ER2), and ($\mathrm{ER3}_\omega$), for $\omega \in \mathcal{L}$, among its axioms and rules of inference. Let K be axiomatized by a set Id of identities and a set Qd of quasi-identities. Then adjoin to the axioms of EQ:

$$\langle \varphi, \psi\rangle, \quad \text{for every identity } \forall \bar p(\varphi \approx \psi) \in \mathrm{Id},$$

and to the rules of inference of EQ:

$$\frac{\langle \xi_0, \eta_0\rangle, \dots, \langle \xi_{n-1}, \eta_{n-1}\rangle}{\langle \varphi, \psi\rangle},$$

for every quasi-identity $\forall\bar{p}\big(\xi_0 \approx \eta_0 \,\&\, \cdots \,\&\, \xi_{n-1} \approx \eta_{n-1} \Longrightarrow \varphi \approx \psi\big) \in \mathrm{Qd}$. Since $\mathrm{EQ}(\mathsf{K})$ extends EQ, any $\mathrm{EQ}(\mathsf{K})$-matrix $\mathfrak{A}$ is of the form $\langle \boldsymbol{A}, \Theta\rangle$ with Θ a congruence of $\boldsymbol{A}$. The added axioms and rules stipulate that $\langle \boldsymbol{A}, \Theta\rangle$ is an $\mathrm{EQ}(\mathsf{K})$-matrix iff $\boldsymbol{A}/\Theta \in \mathsf{K}$. A congruence with this property is said to be *relative to* K, or a K-*congruence* for short. The set of all K-congruences on $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$; thus $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A} = \mathrm{Fi}_{\mathrm{EQ}(\mathsf{K})}\,\boldsymbol{A}$.

The reduced $\mathrm{EQ}(\mathsf{K})$-matrices are precisely the matrices $\langle \boldsymbol{A}, \Delta_A\rangle$ such that $\boldsymbol{A} \in \mathsf{K}$. Note that it follows that $\mathrm{EQ}(\mathsf{K})$ does not depend on the particular axiomatization of K.

Conversely, if $\mathcal{S}$ is a 2-deductive system extending EQ, i.e., $\vdash_{\mathcal{S}} \supseteq \vdash_{\mathrm{EQ}}$, then the class of reduced $\mathcal{S}$-matrices consists of all matrices $\langle \boldsymbol{A}, \Delta_A\rangle$, where $\boldsymbol{A} \in \mathsf{K}$ for some quasivariety K; in fact, K is the quasivariety defined by the identities and quasi-identities associated with the axioms and rules of $\mathcal{S}$. Furthermore, $\mathcal{S} = \mathrm{EQ}(\mathsf{K})$. The quasivarieties over $\mathcal{L}$ are thus in one-one correspondence with the 2-deductive systems extending EQ.

The logic of partially ordered algebras can also be formalized as a 2-deductive system with the 2-formulas representing the partial ordering, i.e., the 2-formula $\langle \varphi, \psi\rangle$ is now to be interpreted as the inequality $\varphi \preccurlyeq \psi$.

3.3.3. *Free partially ordered logic* ($\mathrm{PO}_{\mathcal{L}}$). Let $\mathcal{L} = \{\, \omega_i : i \in I \,\}$ be any language type. The axioms and rules of inference are the same as for the free equational logic except that the symmetry rule (ER1) is omitted. A 2-matrix $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ over $\mathcal{L}$ is a $\mathrm{PO}_{\mathcal{L}}$-matrix if A is an $\mathcal{L}$-algebra and $F \subseteq A^2$ is a reflexive, transitive relation, i.e., a quasiorder, with the property (guaranteed by the rules ($\mathrm{ER3}_\omega$)) that each fundamental operation is monotone in each argument. Note that $\boldsymbol{\Omega}^{\boldsymbol{A}} F = \{\, \langle a, b\rangle : \langle a, b\rangle, \langle b, a\rangle \in F \,\}$, and thus $F/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is a partial order on $A/\,\boldsymbol{\Omega}^{\boldsymbol{A}} F$. In fact, the reduced $\mathrm{PO}_{\mathcal{L}}$-matrices are precisely the partially ordered algebras of type $\mathcal{L}$.

Applied partially ordered logics are obtained by adjoining extralogical axioms and rules to those of $\mathrm{PO}_{\mathcal{L}}$. We consider two examples: semilattices, as ordered algebras, and a certain fragment of intuitionistic linear logic.

3.3.4. *Semilattices* (SL). $\mathcal{L} = \{\wedge\}$ with $\wedge$ binary.

(E1) $\langle p, p\rangle$;

(SL1) $\langle p,\, p \wedge p\rangle$;

(SL2) $\langle p \wedge q,\, p\rangle$;

(SL3) $\langle p \wedge q,\, q\rangle$;

(ER2) $\dfrac{\langle p, q\rangle,\ \langle q, r\rangle}{\langle p, r\rangle}$.

($\mathrm{ER3}_\wedge$) $\dfrac{\langle p_0, q_0\rangle,\ \langle p_1, q_1\rangle}{\langle p_0 \wedge p_1,\ q_0 \wedge q_1\rangle}$.

An alternative presentation of SL that corresponds more closely to the standard definition of semilattices is obtained by omitting (SL1) and replacing ($\mathrm{ER3}_\wedge$) by

(SLR1) $\dfrac{\langle r, p\rangle, \langle r, q\rangle}{\langle r, p \wedge q\rangle}$.

The proof of equivalence of these two presentations illustrates the different view of semilattices we get by thinking of them as a partially ordered system rather than as an equational system. As an example we give a proof of ($\mathrm{ER3}_\wedge$) in the alternative presentation.

(1) $\langle p_0, q_0\rangle$, assumption
(2) $\langle p_1, q_1\rangle$, assumption
(3) $\langle p_0 \wedge p_1,\, p_0\rangle$, (SL2)
(4) $\langle p_0 \wedge p_1,\, q_0\rangle$, (1), (3), and (ER2)
(5) $\langle p_0 \wedge p_1,\, q_1\rangle$, similar to proof of (4)
(6) $\langle p_0 \wedge p_1,\, q_0 \wedge q_1\rangle$, (4), (5), and (SLR1).

The SL-matrices are the 2-matrices $\big\langle\langle A, \wedge\rangle, \leq\big\rangle$, where $\leq$ is a quasiordering of A such that $a \wedge b$ is a maximal (but not necessarily unique) lower bound of a and b. $\big\langle\langle A, \wedge\rangle,\, \leq\big\rangle$ is reduced precisely when $\langle A, \wedge\rangle$ is a semilattice and $\leq$ is the associated partial ordering.

The 2-deductive systems LA of *lattices* and DL of *distributive lattices*, both with $\mathcal{L} = \{\wedge, \vee\}$, are extensions of SL by the obvious axioms and rules.

Sequent calculi form another natural class of multi-dimensional deductive system, although they are not finite dimensional; see Note 3.3. A sequent calculus, with a fusion connective that allows the set of formulas on the left-hand side of a sequent to be combined in a single formula, can be reformulated as a 2-deductive system. Here $\langle \varphi, \psi\rangle$ stands for the sequent $\varphi \Rightarrow \psi$ with a single formula on the left-hand side. Roughly speaking, a sequent calculus with this property constitutes a generalization of partially ordered logic in which the fundamental operations may be anti-monotone with respect to the ordering in some arguments. Linear logic (with the tensor connective $*$ as the fusion connective) is of this kind.

3.3.5. $(*, \rightarrow)$*-fragment of linear logic* ($\mathrm{LL}^{*,\rightarrow}$). $\mathcal{L} = \{*, \rightarrow\}$, both binary operations.

(E1) $\langle p, p\rangle$;

(ER2) $\dfrac{\langle p, q\rangle,\ \langle q, r\rangle}{\langle p, r\rangle}$;

(ER3$_*$) $\dfrac{\langle p_0, q_0\rangle,\ \langle p_1, q_1\rangle}{\langle p_0 * p_1,\ q_0 * q_1\rangle}$;

(LL1) $\langle p * q,\ q * p\rangle$;

(LL2) $\langle p * (q * r),\ (p * q) * r\rangle$;

(LL3) $\langle (p * q) * r,\ p * (q * r)\rangle$;

(LLR1) $\dfrac{\langle p_0, q_0\rangle, \langle p_1, q_1\rangle}{\langle q_0 \rightarrow p_1,\ p_0 \rightarrow q_1\rangle}$;

(LLR2) $\dfrac{\langle p * q,\ r\rangle}{\langle p,\ q \rightarrow r\rangle}$;

(LLR3) $\dfrac{\langle p,\ q \rightarrow r\rangle}{\langle p * q,\ r\rangle}$.

The reduced $\mathrm{LL}^{*,\rightarrow}$-matrices $\big\langle\langle A, *, \rightarrow\rangle,\, \leq\big\rangle$ are known as *partially ordered commutative residuated semigroups.*

3.3.6. $(*, \rightarrow, \wedge)$*-fragment of linear logic* ($\mathrm{LL}^{*,\rightarrow,\wedge}$). $\mathcal{L} = \{*, \rightarrow, \wedge\}$. The axioms and rules of inference for $\mathrm{LL}^{*,\rightarrow,\wedge}$ are those of SL and $\mathrm{LL}^{*,\rightarrow}$ taken together.

BCK, $\mathrm{LL}^{*,\rightarrow}$ and $\mathrm{LL}^{*,\rightarrow,\wedge}$ are examples of what are called *substructural logics.* See Note 3.2.

A universal equational deductive system is discussed in Note 3.5.

3.4. **Algebraic deductive systems.** The deductive systems EQ(K) will play a crucial role in the definition of algebraizable deductive systems we consider below in Sec. 4. For any algebra $\boldsymbol{A}$ and set $\Gamma \cup \{\boldsymbol{\varphi}\}$ of 2-formulas we write

$$\Gamma \vDash_{\langle \boldsymbol{A}, \boldsymbol{\Delta}_A \rangle} \boldsymbol{\varphi}$$

more suggestively as

$$\{\, \psi_0 \approx \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\boldsymbol{A}} \varphi_0 \approx \varphi_1.$$

Similarly, for a class K of algebras and $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^2$ we may write

$$\Gamma \vDash_{\{\, \langle \boldsymbol{A}, \boldsymbol{\Delta}_A \rangle : \boldsymbol{A} \in \mathsf{K} \,\}} \boldsymbol{\varphi} \quad \text{as} \quad \{\, \psi_0 \approx \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{K}} \varphi_0 \approx \varphi_1.$$

Observe that the relations $\vDash_{\boldsymbol{A}}$ and $\vDash_{\mathsf{K}}$ defined here are precisely the ones given in Sec. 2.1.2.

For the 2-dimensional deductive systems EQ(K), where K a quasivariety, the completeness theorem for reduced matrix semantics, Thm. 3.8, thus assumes the following form.

Theorem 3.9. *Let* K *be a quasivariety and* $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^2$. *Then*

$$\Gamma \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\varphi} \quad \textit{iff} \quad \{\, \psi_0 \approx \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{K}} \varphi_0 \approx \varphi_1.$$

For 2-deductive systems of the form EQ(K), where K is a quasivariety, the completeness theorem for reduced matrix semantics thus yields a completeness theorem with respect to the class of *algebras* K in a natural way, and K is called an *algebraic semantics* for EQ(K). Conversely, any 2-dimensional deductive system $\mathcal{S}$ such that $\Gamma \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\varphi}$ iff $\{\, \psi_0 \approx \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{K}} \varphi_0 \approx \varphi_1$ holds for some quasivariety K must be of the form EQ(K). These observations justify the following:

Definition 3.10. A 2-deductive system $\mathcal{S}$ is called *algebraic* if it is an extension of EQ, i.e., $\mathcal{S}$ is of the form EQ(K) for some quasivariety K.

In Sec. 3.3.4 semilattices were viewed as the reduced matrix models of the applied partially ordered 2-deductive system SL, and hence as partially ordered sets endowed with a meet operation. They can also be viewed as the reduced matrix models of an applied equational system EQ(SL) where $\mathcal{L} = \{\wedge\}$ and SL is the variety of semilattices. Here are the axioms and rules of inference for EQ(SL). Notice that they are obtained by adjoining the usual equational axioms for semilattices to the axioms and rules of EQ.

(E1) $\langle p, p \rangle$;
(SL1) $\langle p,\, p \wedge p \rangle$;
(SL4) $\langle p \wedge q,\, q \wedge p \rangle$;
(SL5) $\langle p \wedge (q \wedge r),\, (p \wedge q) \wedge r \rangle$;
(ER1) $\dfrac{\langle p, q \rangle}{\langle q, p \rangle}$;
(ER2) $\dfrac{\langle p, q \rangle,\ \langle q, r \rangle}{\langle p, r \rangle}$;
(ER3$_\wedge$) $\dfrac{\langle p_0, q_0 \rangle,\ \langle p_1, q_1 \rangle}{\langle p_0 \wedge p_1,\, q_0 \wedge q_1 \rangle}$.

In a similar way we obtain presentations of the equational systems $\mathrm{EQ}_{\{\wedge,\vee\}}(\mathsf{LA})$ and $\mathrm{EQ}_{\{\wedge,\vee\}}(\mathsf{DL})$ for lattices and distributive lattices, respectively.

We take the algebraic 2-deductive systems as the paradigm for an algebraizable deductive system in our theory of abstract algebraic logic. An arbitrary k-deductive system is defined to be *algebraizable* if it is equivalent to some algebraic system. The precise sense in which they are equivalent is discussed in Sec. 4.2 below.

3.5. **The deduction-detachment theorem in k-deductive systems.** We introduce an abstract version of the deduction theorem that makes sense for any k-deductive system. In Sec. 2 we saw several examples of the deduction theorem. In these examples the deduction theorem was always formulated as an equivalence, one implication of which was the deduction theorem proper. In those cases where the rule of detachment is the only rule of inference the implication in the other direction follows by a single application of detachment, but if other rules are present they are also involved in obtaining the opposite implication. For example, in the system $\mathrm{S5^G}$ of modal logic the deduction theorem 2.15 assumed the form, for $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$,

$$\Gamma \cup \{\varphi\} \vdash_{\mathrm{S5^G}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{S5^G}} \Box\,\varphi \to \psi.$$

The implication from left to right is the deduction theorem for $\mathrm{S5^G}$ proper. The implication from right to left is easily demonstrated as follows. Given $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$, suppose $\Gamma \vdash_{\mathrm{S5^G}} \Box\,\varphi \to \psi$. From $\Gamma \cup \varphi$ we infer $\Box\,\varphi$ by the rule of necessitation, and then ψ by applying the rule of modus ponens to the formulas $\Box\,\varphi$ and $\Box\,\varphi \to \psi$; thus $\Gamma \cup \{\varphi\} \vdash_{\mathrm{S5^G}} \psi$.

The implication from right to left can be viewed as a generalized detachment rule. To emphasize the fact that we are incorporating both implications in the abstract version of the deduction theorem we will speak of the *deduction-detachment theorem* rather than just the deduction theorem. We say a 1-dimensional deductive system $\mathcal{S}$ has the *deduction-detachment theorem* (DDT) if there exists a finite set $E(p,q) = \{\eta_0(p,q), \dots, \eta_{m-1}(p,q)\}$ of formulas in two variables such that, for all $\Gamma \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}$, we have

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} E(\varphi, \psi).$$

Here $\Gamma \vdash_{\mathcal{S}} E(\varphi, \psi)$ is an abbreviation for the conjunction of the assertions $\Gamma \vdash_{\mathcal{S}} \eta_i(\varphi, \psi)$, $i < m$. If $\mathcal{S}$ has the DDT with respect to $E(p,q)$, we call $E(p,q)$ a *deduction-detachment set* for $\mathcal{S}$. In the examples of Sec. 2.2, $E(p,q) = \{p \to q\}$ is a deduction-detachment set for each of CPC, IPC, and $\mathrm{S5^C}$, while for $\mathrm{S5^G}$ we can take $E(p,q) = \{\Box\, p \to q\}$.

The generalization of the notion of a deduction-detachment set to k-dimensional deductive systems is straightforward. Let $\mathcal{S}$ be a k-deductive system and let

$$\mathbf{p} = \langle p_0, \dots, p_{k-1} \rangle, \qquad \mathbf{q} = \langle q_0, \dots, q_{k-1} \rangle$$

be k-variables.

Definition 3.11. Let $\mathcal{S}$ be a k-deductive system. A finite set of k-formulas in $2k$ variables $E(\mathbf{p}, \mathbf{q}) = \{\boldsymbol{\eta}_0(\mathbf{p}, \mathbf{q}), \dots, \boldsymbol{\eta}_{m-1}(\mathbf{p}, \mathbf{q})\}$ is called a *deduction-detachment* set for $\mathcal{S}$ if, for all $\Gamma \subseteq \mathrm{Fm}^k$, and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathrm{Fm}^k$, we have

$$\Gamma, \boldsymbol{\varphi} \vdash_{\mathcal{S}} \boldsymbol{\psi} \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} E(\boldsymbol{\varphi}, \boldsymbol{\psi}).$$

$\mathcal{S}$ has the *deduction-detachment theorem* (DDT, for short) if it possesses some deduction-detachment set.

For 1-deductive systems this definition reduces to the one given above.

In Sec. 5 we examine the DDT in the 2-deductive systems EQ(K) of applied equational logic defined in Sec. 3.3.2. For example the equational system EQ(DL) of distributive lattices has the deduction-detachment theorem with the deduction-detachment system $\{\boldsymbol{\eta}_0(\mathbf{p},\mathbf{q}),\boldsymbol{\eta}_1(\mathbf{p},\mathbf{q})\}$ with $\boldsymbol{\eta}_0(\mathbf{p},\mathbf{q}) = (p_0 \wedge p_1) \wedge q_0 \approx (p_0 \wedge p_1) \wedge q_1$ and $\boldsymbol{\eta}_0(\mathbf{p},\mathbf{q}) = (p_0 \vee p_1) \vee q_0 \approx (p_0 \vee p_1) \vee q_1$ (see Sec. 5.2.4). In Sec. 5 we prove that, for any quasivariety K, EQ(K) has the deduction-detachment theorem in the sense of Def. 3.11 if and only if the quasivariety K has equationally definable principal relative congruences (Thm. 5.4).

A local form of the deduction-detachment theorem for multi-dimensional deductive systems can also be formulated, but we will not need it in this paper; see Note 2.7. For more information on the DDT and its generalizations see Note 3.6.

3.6. Notes.

Note 3.1. A k-dimensional deductive system can be viewed as a universal Horn theory with a single k-ary predicate. Let $\mathcal{L}$ be the underlying sentential language of $\mathcal{S}$. Let $\mathcal{L}_D$ be the first-order language *without equality* whose extra-logical constants are the primitive connectives of $\mathcal{L}$, now thought of as operation symbols of the appropriate rank, together with a single predicate symbol D of rank k. With each axiom $\boldsymbol{\varphi}$ and each inference rule $\langle\{\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_{n-1}\},\,\boldsymbol{\varphi}\rangle$ of $\mathcal{S}$ we associate respectively the universally quantified atomic sentence $\forall\bar{p}(D\boldsymbol{\varphi})$ and the universal Horn sentence

$$\forall\bar{p}(D\boldsymbol{\psi}_0 \wedge \cdots \wedge D\boldsymbol{\psi}_{n-1} \to D\boldsymbol{\varphi}),$$

where $\bar{p}$ is a list of all variables occurring in $\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_{n-1}$, or $\boldsymbol{\varphi}$. Let $H(\mathcal{S})$ be the elementary (first-order, strict) universal Horn theory over $\mathcal{L}_D$ axiomatized by these sentences. Then it can be shown that, for all $\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_{n-1},\boldsymbol{\varphi} \in {\mathrm{Fm}_{\mathcal{L}}}^k$,

$$\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_{n-1} \vdash_{\mathcal{S}} \boldsymbol{\varphi} \quad \text{iff} \quad \vdash_{H(\mathcal{S})} \forall\bar{p}(D\boldsymbol{\psi}_0 \wedge \cdots \wedge D\boldsymbol{\psi}_{n-1} \to D\boldsymbol{\varphi}),$$

where $\vdash_{H(\mathcal{S})}$ denotes consequence relative to $H(\mathcal{S})$ in the usual first-order sense. Conversely, with every (strict) universal Horn theory H in the first-order language $\mathcal{L}_D$ we can associate a deductive system $\mathcal{S}(H)$ in the obvious way, that is, for every universally quantified atomic sentence $\forall\bar{p}(D\boldsymbol{\varphi})$ among the axioms of H we take $\boldsymbol{\varphi}$ as an axiom of $\mathcal{S}(H)$, and for every universal Horn sentence

$$\forall\bar{p}(D\boldsymbol{\psi}_0 \wedge \cdots \wedge D\boldsymbol{\psi}_{n-1} \to D\boldsymbol{\varphi})$$

among the axioms of H we take $\langle\{\boldsymbol{\psi}_0,\dots,\boldsymbol{\psi}_{n-1}\},\boldsymbol{\varphi}\rangle$ to be a rule of $\mathcal{S}(H)$. For every universal Horn theory H we have $H\big(\mathcal{S}(H)\big) = H$, and for every deductive system $\mathcal{S}$, $\mathcal{S}\big(H(\mathcal{S})\big) = \mathcal{S}$.

We have restricted ourselves here to strict universal Horn logic with a single relation, but it should be clear that there is no essential difficulty in recasting any strict universal Horn logic as a generalized k-deductive system. For related results in this broader context see [23, 36, 42, 43, 44, 50, 51, 52, 119].

Note 3.2. Gentzen-style formalizations of logic can be viewed as a natural extension of the notion of k-deductive system. By an *$\boldsymbol{\omega}$-formula* (of type $\mathcal{L}$) we mean any finite nonempty sequence of formulas. In the context of Gentzen-style formalisms $\boldsymbol{\omega}$-formulas are called *sequents* and often written $\varphi_0,\dots,\varphi_{k-1} \Rightarrow \varphi_k$ or in some similar form. The set of all sequents is denoted by $\mathrm{Fm}^{(\boldsymbol{\omega})}$. $\boldsymbol{\omega}$-deductive systems are defined in the same way as k-deductive systems but with sequents as the basic syntactic unit. More precisely, a $\boldsymbol{\omega}$-*deductive system* or a *sequent calculus* is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ where $\vdash_{\mathcal{G}} \subseteq \mathcal{P}(\mathrm{Fm}^{(\boldsymbol{\omega})}) \times \mathrm{Fm}^{(\boldsymbol{\omega})}$, satisfying the conditions (i)–(iv) of Def. 3.1 mutatis mutandis. The following rules are often assumed to hold for $\mathcal{G}$.

$$\frac{\bar{\xi}, \psi, \vartheta, \bar{\xi}' \Rightarrow \varphi}{\bar{\xi}, \vartheta, \psi, \bar{\xi}' \Rightarrow \varphi}\ (\textit{exchange}); \quad \frac{\bar{\xi}, \bar{\xi}' \Rightarrow \varphi}{\bar{\xi}, \psi, \bar{\xi}' \Rightarrow \vartheta}\ (\textit{weakening}); \quad \frac{\bar{\xi}, \psi, \psi, \bar{\xi}' \Rightarrow \varphi}{\bar{\xi}, \psi, \bar{\xi}' \Rightarrow \vartheta}\ (\textit{contraction}),$$

$$\frac{\psi^0_0, \dots, \psi^0_{k_0-1} \Rightarrow \varphi_0\, ; \dots ; \psi^{n-1}_0, \dots, \psi^{n-1}_{k_{n-1}-1} \Rightarrow \varphi_{n-1}\, ; \varphi_0, \dots, \varphi_{n-1} \Rightarrow \vartheta}{\psi^0_0, \dots, \psi^0_{k_0-1}, \dots, \psi^{n-1}_0, \dots, \psi^{n-1}_{k_{n-1}-1} \Rightarrow \vartheta} \quad (\textit{cut}).$$

Here ξ and ξ' represent arbitrary finite sequences of formulas and $\bar{\xi}, \psi, \vartheta, \bar{\xi}'$ denotes the concatenation of the sequences $\bar{\xi}$, $\langle \psi, \vartheta \rangle$, and $\bar{\xi}'$. The rules of exchange, weakening, and contraction together are called the *structural rules* (not to be confused with the *structure* axiom of k-deductive systems, Def. 3.1(iv)). $\mathcal{G}$ is said to be a *Gentzen system* for a 1-deductive system $\mathcal{S}$ if $\psi_0, \dots, \psi_{k-1} \vdash_{\mathcal{S}} \varphi$ iff $\psi_0, \dots, \psi_{k-1} \Rightarrow \varphi$ is a theorem of $\mathcal{G}$. Clearly in this case the structural and cut rules must hold in $\mathcal{G}$.

Historically sequent calculi have been used to characterize logics that are defined by a set of theorems, rather than by a consequence relation, and for which there is a connective $\to$ with respect to which it is closed under modus ponens; recall that the modal logic S5 considered in Note 2.2 is an example of a logic of this kind. A sequent calculus $\mathcal{G}$ is said to be a *Gentzen system* for a logic $\mathcal{S}$ of this kind if $\psi_0 \to (\varphi_1 \to (\cdots \to (\varphi_{k-1} \to \varphi_k) \cdots))$ is theorem of $\mathcal{S}$ iff $\psi_0, \dots, \psi_{k-1} \Rightarrow \varphi$ is a theorem of $\mathcal{G}$. In particular φ is a theorem of $\mathcal{S}$ iff $\Rightarrow \varphi$ (i.e., the one-term sequent $\langle \varphi \rangle$) is a theorem of $\mathcal{G}$. A logic (that is characterized by its theorems and that has a connective $\to$ such that theorems are closed under modus ponens) is said to be *substructural* if it has a Gentzen system that fails to satisfy all the structural rules. Important examples of nonstructural logics are relevance and entailment logics where weakening fails ([1, 2]), B-C-K (its set of theorems) and related logics where contraction fails ([112]), linear logic (the set of theorems of the 1-dimensional version) where both weakening and contraction fail ([71, 135]), and logics where all three of weakening, contraction, and exchange fail ([91]). For a collection of survey papers on substructural logics see [47]; for their algebraic semantics see [109, 110, 111, 123, 135].

Recently sequent calculii have begun to play a prominent role in abstract algebraic logic, both in their own right and as Gentzen systems that arise naturally from certain deductive systems. In the latter case their influence is felt mainly through the notion of generalized matrices; see the following note. The monograph by Font and Jansana [59] is a systematic exposition of pioneering work in this area. See also [70, 121, 122].

Note 3.3. By a *generalized* (*matrix*) *model* of a deductive system $\mathcal{S}$ we mean a pair $\langle \boldsymbol{A}, \mathcal{C} \rangle$ where $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ is an algebraic closed-set system of $\mathcal{S}$-filters of $\boldsymbol{A}$, i.e., $\mathcal{C} \subseteq \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ and $\mathcal{C}$ is closed under arbitrary intersection and unions of upward-directed chains. Of particular interest are the generalized models of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{C}}\, \boldsymbol{A} \rangle$ and certain generalized models derived from them in a natural way; these are called the *full generalized models* of $\mathcal{S}$. The full generalized models are useful because in some cases the collection of all filters on an algebra can provide much more information about the algebraic properties of $\mathcal{S}$ than can be obtained from the filters considered in isolation. This method was used to great effect in [65] to study the algebraic properties of the disjunction-conjunction fragment of CPC. A general theory of generalized models in abstract algebraic logic can be found in [59].

Generalized matrices are the natural models of sequent calculi that satisfy the structural and cut rules (see the preceding note). Deductive systems $\mathcal{S}$ with the property that the class of all full generalized models of $\mathcal{S}$ coincide with the class of all models of a Gentzen system for $\mathcal{S}$ are studied in some detail in [59]. Those $\mathcal{S}$ with a *fully adequate Gentzen system* in this sense turn out to be an interesting class of deductive systems; see [60, 61].

Note 3.4. Equational logic the sense of Sec. 3.3.2 differs from the equational logic of identities as it is commonly understood in universal algebra. In the latter attention is normally restricted to the case K is a variety, and hence the set Qd of proper quasi-identities is empty. In addition, one is normally interested only in the identities that can be derived, i.e., the theorems of $\mathrm{EQ}(\mathsf{K})$, as opposed to the quasi-identities (the rules of $\mathrm{EQ}(\mathsf{K})$).

Note 3.5. All the 2-deductive systems considered in this way can be viewed as variants of equational logic. There is a more radical variant that in a certain sense encompasses in a single formalism all the applied equational logics of Sec. 3.3.2 over all language types $\mathcal{L}$. We call it hyperequational logic and denote it by $\mathrm{HEQ}_{\boldsymbol{\omega}}$. It itself is an applied equational logic of the form $\mathrm{EQ}(\mathsf{V})$ where V is the variety of abstract algebras of clones. The language type $\mathcal{H}$ of $\mathrm{HEQ}_{\boldsymbol{\omega}}$ contains an infinite sequence $\mathsf{S}^0, \mathsf{S}^1, \mathsf{S}^2, \dots$ of binary operation symbols and an infinite sequence $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \dots$ of 0-ary operation symbols.

Hyperequational logic ($(\mathrm{HEQ}_{\boldsymbol{\omega}})$ is the applied equational logic defined by the following axioms for all $n, m, l < \boldsymbol{\omega}$. To emphasize the equational character of $\mathrm{HEQ}_{\boldsymbol{\omega}}$ we write $\varphi \approx \psi$ for the 2-formula $\langle \varphi, \psi \rangle$.

(HE1) $\mathsf{S}^n_{\mathsf{v}_n}(x) \approx x$;
(HE2) $\mathsf{S}^n_x(\mathsf{v}_n) \approx x$;
(HE3) $\mathsf{S}^n_x(\mathsf{v}_m) \approx \mathsf{v}_m, \quad \text{if } n \neq m$;
(HE4) $\mathsf{S}^n_x \mathsf{S}^n_y(z) \approx \mathsf{S}^n_{\mathsf{S}^n_x(y)}(z)$;
(HE5) $\mathsf{S}^n_{x(m/l)} \mathsf{S}^m_y(z) \approx \mathsf{S}^m_{\mathsf{S}^n_{x(m/l)}(y)} \mathsf{S}^n_{x(m/l)}(z)$, where $x(m/l) = \mathsf{S}^m_{\mathsf{v}_l}(x)$ and m, n, and l are all distinct.

In the following we explain the sense in which hyperequational logic constitutes an adequate formalization of all of that part of applied equational logic that deals only with axiomatic extensions of free equational logic (no nonlogical inference rules). We also discuss how it is used to reduce the study of all such applied equational logics to the study of the theories of a single applied equational logic.

Let $\mathbf{Cl}(\boldsymbol{A})$ be the algebra of clones of an algebra $\boldsymbol{A}$ over an arbitrary language type $\mathcal{L}$. The elements of $\mathbf{Cl}(\boldsymbol{A})$ are term (i.e., derived) operations of $\boldsymbol{A}$ and the fundamental operations are the operations of substitution in the clone algebra (i.e., the semantical version of term-for-variable substitution at the syntax level), together with the 0-ary projection operations. Because term operations are naturally structured by their arity, the logic of clone algebras is often formulated as a multisorted system ([134]), or as a categorical logic

([94]). But it can also be given the form of a 2-deductive system in the sense of Def 3.1; when formalized as a singlesorted system it is convenient to view the elements of $\mathbf{Cl}(\boldsymbol{A})$ formally as operations of infinite arity. We call the system hyperequational logic because it is now common to refer to the identities of $\mathbf{Cl}(\boldsymbol{A})$ as the *hyperidentities* of $\boldsymbol{A}$. The version we present here, $\mathrm{HEQ}_{\boldsymbol{\omega}}$, is essentially due to Feldman [54] with a simplification by Cirulis [28]; for another version see [108]. Semantically, the denotations of formulas of $\mathcal{H}$ in $\mathbf{Cl}(\boldsymbol{A})$ are functions $f, g : A^{\boldsymbol{\omega}} \to A$. For all $\bar{a} = \langle a_0, a_1, a_2, \dots \rangle$ in $A^{\boldsymbol{\omega}}$ we have $\mathsf{S}^n_g(f)(\bar{a}) = f(a_0, \dots, a_{n-1}, g(\bar{a}), a_{n+1}, \dots)$ (we write $\mathsf{S}^n_g(f)$ for $\mathsf{S}^n(f, g)$) and a_n for $\mathsf{v}_n(\bar{a})$).

A formula φ is said to be *independent of* m (*in the abstract sense*) if $\vdash_{\mathrm{HEQ}_\omega} \mathsf{S}^m_{\mathsf{v}_l}(\varphi) \approx \varphi$ for some $l \neq m$; it is shown in [54] that this does not depend on which $l \neq m$ is chosen. It follows that the last axiom can be equivalently formulated as the conditional identity

$$\mathsf{S}^n_x \mathsf{S}^m_y(z) \approx \mathsf{S}^m_{\mathsf{S}^n_x(y)} \mathsf{S}^n_x(z), \quad \text{whenever } x \text{ is independent of } m.$$

The algebras defined by the above axioms when the relation symbol "$\approx$" is interpreted as identity are called *substitution algebras*. They form a variety SA and $\mathrm{HEQ}_{\boldsymbol{\omega}}$ coincides with $\mathrm{EQ}(\mathsf{SA})$, the applied equational logic over $\mathcal{H}$ determined by SA. Conversely, every applied equational logic $\mathcal{S}_{\mathsf{V}}$, where V is a variety of language type $\mathcal{L}$, can be viewed as a theory T_{V} of $\mathrm{HEQ}_{\boldsymbol{\omega}}$, i.e., a congruence on the formula algebra $\mathbf{Fm}_{\mathcal{H}}(\{\, x_\omega : \omega \in \mathcal{L} \,\})$. Let V be an arbitrary variety over the language type $\mathcal{L}$. Let X be a set of variables in one-one correspondence with $\mathcal{L}$. Let x_ω be the variable associated with $\omega \in \mathcal{L}$. T_{V} is the theory of $\mathrm{HEQ}_{\boldsymbol{\omega}}$ generated by the following 2-formulas. First of all, for each $\omega \in \mathcal{L}$ we include the infinite set of generators

$$\langle \mathsf{S}^i_{\mathsf{v}_{i+1}}(x_\omega),\, x_\omega \rangle, \quad \text{all } i \text{ such that } n \le i < \boldsymbol{\omega},$$

where n is the rank of ω. Finally, we transform each identity of some equational base for V into a 2-formula of $\mathcal{H}$. For example, consider the variety SG of semigroups. $\mathcal{L} = \{\cdot\}$ where the rank of $\cdot$ is 2. We write $\mathsf{v}_0, \mathsf{v}_1, \mathsf{v}_2, \dots$ for the variables of $\mathcal{L}$-formulas, both to distinguish them from the variables of $\mathcal{H}$-formulas and to emphasize the connection with the projection operations on clones. The multiplication operation $\cdot$ is represented by $x.$ in the $\mathcal{H}$-formula algebra and by $\mathsf{v}_0 \cdot \mathsf{v}_1$ in the $\mathcal{L}$-formula algebra. The representation $\mathsf{v}_0 \cdot \mathsf{v}_1$ indicates that the operation depends on at most 0 and 1. $(\mathsf{v}_0 \cdot \mathsf{v}_1) \cdot \mathsf{v}_2$ is obtained by simultaneously substituting $(\mathsf{v}_0 \cdot \mathsf{v}_1)$ for v_0 and v_2 for v_1 in $\mathsf{v}_0 \cdot \mathsf{v}_1$. Thus $(\mathsf{v}_0 \cdot \mathsf{v}_1) \cdot \mathsf{v}_2$ is represented by $\mathsf{S}^0_{x.} \mathsf{S}^1_{\mathsf{v}_2}(x.)$. Similarly, $\mathsf{v}_0 \cdot (\mathsf{v}_1 \cdot \mathsf{v}_2)$ is represented by $\mathsf{S}^1_{\mathsf{S}^0_{\mathsf{v}_1} \mathsf{S}^1_{\mathsf{v}_2}(x.)}(x.)$. So the associative law $(\mathsf{v}_0 \cdot \mathsf{v}_1) \cdot \mathsf{v}_2 \approx \mathsf{v}_0 \cdot (\mathsf{v}_1 \cdot \mathsf{v}_2)$ is transformed into a generator

$$\big\langle \mathsf{S}^0_{x.} \mathsf{S}^1_{\mathsf{v}_2}(x.),\, \mathsf{S}^1_{\mathsf{S}^0_{\mathsf{v}_1} \mathsf{S}^1_{\mathsf{v}_2}(x.)}(x.) \big\rangle$$

of the congruence T_{SG} on $\mathbf{Fm}_{\mathcal{H}}(\{x.\})$; the only other generators are the ones specifying the arity of $\cdot$. Any $\mathcal{L}$-formula φ can be translated into a $\mathcal{H}$-formula φ', in the way indicated above, that contains no variable other than $x.$. Moreover, it follows from the representation theory for locally finite substitution algebras proved in [28] that $\varphi \approx \psi$ is a theorem of $\mathcal{S}_{\mathsf{SG}}$, i.e., a semigroup identity, iff $\langle \varphi', \psi' \rangle \in T_{\mathsf{SG}}$, and an analogous result holds for every variety, no matter what the language. It is in this sense that hyperequational logic is an adequate formalization of all of the part of applied equational logic that deals with only axiomatic extensions of free equational logic (no nonlogical inference rules). Moreover, it allows us to reduce the metatheory of the whole of applied equational logic to the study of theories of a single applied equational logic.

Note 3.6. A local form of the deduction theorem was given in Thm. 2.11 and discussed in Note 2.7; more details about the local deduction theorem can be found in Note 5.1 below. The deduction theorem in Gentzen-style systems is investigated in [122]. In [61] it is shown that another variant of the deduction theorem serves to characterize those protoalgebraic deductive systems (see Sec. 4.4.1 below) that have a fully adequate Genzten system (see Note 3.3); see also [60]. For additional work on the deduction theorem and its variants in abstract algebraic logic see [15, 34, 35].

As mentioned in Note 2.6 the deduction theorem seems to be intimately involved in the problem as to when the equivalent quasivariety of an algebraizable deductive system (see Def. 4.4 below) is actually a variety; see also Note 4.3. This problem is addressed in [39, 40, 59].

4. Algebraizable Deductive Systems

In this section we introduce a notion of equivalence among deductive systems of arbitrary finite dimension, over a given language. It will allow us to give a precise meaning to the notions of the *equivalent algebraic semantics* of a deductive system and of an *algebraizable logic*. Many metalogical properties are preserved under our notion of equivalence; we will see in the next section that the property of possessing a deduction-detachment theorem is one of them.

4.1. **Equivalence of deductive systems.** Let $\mathcal{S}_1$ be a k-dimensional deductive system and $\mathcal{S}_2$ an l-dimensional deductive system, where $1 \leq k, l < \omega$. $\mathcal{S}_1$ and $\mathcal{S}_2$ are assumed to be over the same language $\mathcal{L}$. By a (k, l)-*translation* we mean a finite set $\boldsymbol{\tau}$ of l-formulas in a single k-variable, i.e., for some n

$$\boldsymbol{\tau}(\langle p_0, \dots, p_{k-1}\rangle) = \{\, \langle \tau_0^i(p_0, \dots, p_{k-1}), \dots, \tau_{l-1}^i(p_0, \dots, p_{k-1})\rangle : i < n \,\}, \quad \text{or}$$
$$\boldsymbol{\tau}(\mathbf{p}) = \{\, \boldsymbol{\tau}^i(\mathbf{p}) : i < n \,\} \quad \text{for short.}$$

If $\boldsymbol{\varphi}$ is a k-formula, $\boldsymbol{\varphi} = \langle \varphi_0, \dots, \varphi_{k-1}\rangle$, then $\boldsymbol{\tau}(\boldsymbol{\varphi})$ will denote the set of l-formulas

$$\{\, \boldsymbol{\tau}^i(\boldsymbol{\varphi}) : i < n \,\} = \{\, \langle \tau_0^i(\varphi_0, \dots, \varphi_{k-1}), \dots, \tau_{l-1}^i(\varphi_0, \dots, \varphi_{k-1})\rangle : i < n \,\}.$$

For $\Gamma \subseteq \mathrm{Fm}^k$, $\boldsymbol{\tau}(\varGamma)$ will denote the set $\bigcup\{\boldsymbol{\tau}(\boldsymbol{\varphi}) : \boldsymbol{\varphi} \in \varGamma\}$.

Definition 4.1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be k- and l-deductive systems, respectively, with $1 \leq k, l < \omega$.

(i) A (k, l)-translation $\boldsymbol{\tau}$ is an *interpretation* of $\mathcal{S}_1$ in $\mathcal{S}_2$ if, for all $\varGamma \subseteq \mathrm{Fm}^k$ and $\boldsymbol{\varphi} \in \mathrm{Fm}^k$, we have

$$\varGamma \vdash_{\mathcal{S}_1} \boldsymbol{\varphi} \qquad \text{iff} \qquad \boldsymbol{\tau}(\varGamma) \vdash_{\mathcal{S}_2} \boldsymbol{\tau}(\boldsymbol{\varphi}). \tag{5}$$

(ii) $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equivalent* if there is an interpretation $\boldsymbol{\tau}$ of $\mathcal{S}_1$ in $\mathcal{S}_2$ and an interpretation $\boldsymbol{\rho}$ of $\mathcal{S}_2$ in $\mathcal{S}_1$ such that $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are *inverses* of one another in the following sense:

$$\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}_1} \boldsymbol{\rho}\big(\boldsymbol{\tau}(\boldsymbol{\varphi})\big), \quad \text{for} \quad \boldsymbol{\varphi} \in \mathrm{Fm}^k, \tag{6}$$

$$\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\boldsymbol{\varphi})\big), \quad \text{for} \quad \boldsymbol{\varphi} \in \mathrm{Fm}^l. \tag{7}$$

Here $\dashv\vdash$ is the relation of *interderivability*; so (6) says $\{\boldsymbol{\varphi}\} \vdash_{\mathcal{S}_1} \boldsymbol{\psi}$, for all $\boldsymbol{\psi} \in \boldsymbol{\rho}(\boldsymbol{\tau}(\boldsymbol{\varphi}))$, and $\boldsymbol{\rho}(\boldsymbol{\tau}(\boldsymbol{\varphi})) \vdash_{\mathcal{S}_1} \boldsymbol{\varphi}$. (6) and (7) together are called the *invertibility* conditions. $\boldsymbol{\rho}$ is a *right inverse* of $\boldsymbol{\tau}$ if (7) holds, and $\boldsymbol{\tau}$ is a *right inverse* of $\boldsymbol{\rho}$ if (6) holds. If both conditions hold $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are said to be simply *inverses* of one another.

Note that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent iff there is a (k,l)-translation $\boldsymbol{\tau}$ and an (l,k)-translation $\boldsymbol{\rho}$ such that the four conditions (5)–(7) and

$$\Gamma \vdash_{\mathcal{S}_2} \boldsymbol{\varphi} \qquad \text{iff} \qquad \boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}_1} \boldsymbol{\rho}(\boldsymbol{\varphi}) \tag{8}$$

hold. The following theorem shows that the two conditions (5) and (7) are sufficient. By symmetry, (6) and (8) are also sufficient.

Theorem 4.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be k-dimensional and l-dimensional deductive systems, respectively, with $1 \le k,\ l < \omega$. Then $\mathcal{S}_1$ is equivalent to $\mathcal{S}_2$ if there is an interpretation of $\mathcal{S}_1$ in $\mathcal{S}_2$ that has a right inverse, i.e., there is a (k,l)-translation $\boldsymbol{\tau}$ and an (l,k)-translation $\boldsymbol{\rho}$ such that $\boldsymbol{\tau}$ is an interpretation of $\mathcal{S}_1$ in $\mathcal{S}_2$ and, for all $\boldsymbol{\varphi} \in \mathrm{Fm}^l$, $\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\boldsymbol{\varphi})\big)$.*

Proof. We first need to verify that, under the assumptions, ρ is an interpretation. Let $\Gamma \subseteq \mathrm{Fm}^l$, $\boldsymbol{\varphi} \in \mathrm{Fm}^l$. Then $\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\boldsymbol{\varphi})\big)$, and similarly, $\Gamma \dashv\vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\Gamma)\big)$. Thus

$$\Gamma \vdash_{\mathcal{S}_2} \boldsymbol{\varphi} \quad \text{iff} \quad \boldsymbol{\tau}\big(\boldsymbol{\rho}(\Gamma)\big) \vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\boldsymbol{\varphi})\big) \quad \text{iff} \quad \boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}_1} \boldsymbol{\rho}(\boldsymbol{\varphi}),$$

since $\boldsymbol{\tau}$ is an interpretation. This shows $\boldsymbol{\rho}$ is an interpretation. To show that $\boldsymbol{\tau}$ is a right inverse of $\boldsymbol{\rho}$ note that, for $\boldsymbol{\varphi} \in \mathrm{Fm}^k$,

$$\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}_1} \boldsymbol{\rho}\big(\boldsymbol{\tau}(\boldsymbol{\varphi})\big) \quad \text{iff} \quad \boldsymbol{\tau}(\boldsymbol{\varphi}) \dashv\vdash_{\mathcal{S}_2} \boldsymbol{\tau}\big(\boldsymbol{\rho}\big(\boldsymbol{\tau}(\boldsymbol{\varphi})\big)\big),$$

and the right-hand side of the equivalence holds since by assumption $\boldsymbol{\rho}$ is a right inverse of $\boldsymbol{\tau}$. □

The notions of equivalence, translation, and interpretation defined above could be called *deductive* to distinguish them from the related notions of *definitional* equivalence, etc. that are the more familiar ones in the literature. The connection between the two notions of equivalence and interpretation is discussed in more detail in the Note 4.1.

4.1.1. *Example.* The 1-dimensional deductive system CPC of classical propositional logic (see Sec. 2.2.1) and the 2-dimensional deductive system EQ(BA) (Sec. 3.3.2)—both over the language $\mathcal{L} = \{\to, \wedge, \vee, \neg, \bot, \top\}$—are equivalent. Indeed, let $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ respectively be the $(1,2)$- and $(2,1)$-translations

$$\boldsymbol{\tau}(p) = \{\langle p, \top\rangle\}, \qquad \boldsymbol{\rho}(\langle p, q\rangle) = \{p \leftrightarrow q\},$$

where again $p \leftrightarrow q$ stands for $(p \to q) \wedge (q \to p)$. Let M be the class of all matrix-models of CPC. Then $\mathsf{M} = \{\, \langle \boldsymbol{A}, F\rangle : \boldsymbol{A}$ an algebra, F a CPC-filter$\,\}$, and

$$\mathsf{M}^* = \{\, \langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle : \boldsymbol{A} \in \mathsf{BA}\,\}$$

is the class of all reduced matrix-models of CPC. Similarly, let L be the class of all matrix-models of the algebraic 2-deductive system EQ(BA) correlated with the variety of Boolean algebras. Then $\mathsf{L} = \{ \langle \boldsymbol{A}, \Theta \rangle : \boldsymbol{A} \text{ an algebra}, \Theta \in \mathrm{Co}_{\mathsf{BA}}\, \boldsymbol{A} \}$, and

$$\mathsf{L}* = \{ \langle \boldsymbol{A}, \boldsymbol{\Delta}_A \rangle : \boldsymbol{A} \in \mathsf{BA} \}.$$

For all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ we have

$$\begin{aligned}
\Gamma \vdash_{\mathrm{CPC}} \varphi \quad &\text{iff} \quad \Gamma \vDash_{\mathsf{Mo}^*\mathrm{CPC}} \varphi, && \text{by reduced matrix completeness theorem for CPC (3.8)}\\
&\text{iff} \quad \Gamma \approx \top \vDash_{\mathsf{BA}} \varphi \approx \top, && \text{by trivial properties of equality}\\
&\text{iff} \quad \{ \langle \psi, \top \rangle : \psi \in \Gamma \} \vdash_{\mathrm{EQ}(\mathsf{BA})} \langle \varphi, \top \rangle, && \text{by Thm. 3.9}\\
&\text{iff} \quad \boldsymbol{\tau}(\Gamma) \vdash_{\mathrm{EQ}(\mathsf{BA})} \boldsymbol{\tau}(\boldsymbol{\varphi}), && \text{by definition of } \boldsymbol{\tau}.
\end{aligned}$$

Thus $\boldsymbol{\tau}$ is an interpretation of CPC in EQ(BA).

To show CPC and EQ(BA) are equivalent, by Thm. 4.2 it suffices to show for $\langle \varphi_0, \varphi_1 \rangle \in \mathrm{Fm}^2$,

$$\langle \varphi_0, \varphi_1 \rangle \dashv\vdash_{\mathrm{EQ}(\mathsf{BA})} \langle \varphi_0 \leftrightarrow \varphi_1, \top \rangle.$$

By Thm. 3.9, this is equivalent to

$$\varphi_0 \approx \varphi_1 \mathrel{=\!\!\mid\!\vDash}_{\mathsf{BA}} \varphi_0 \leftrightarrow \varphi_1 \approx \top,$$

i.e., for $\boldsymbol{A} \in \mathsf{BA}$, $\bar{a} \in A^\omega$,

$$\varphi_0^{\boldsymbol{A}}(\bar{a}) = \varphi_1^{\boldsymbol{A}}(\bar{a}) \quad \text{iff} \quad (\varphi_0 \leftrightarrow \varphi_1)^{\boldsymbol{A}} = \top^{\boldsymbol{A}}.$$

But this is evidently true; so indeed, CPC is equivalent to EQ(BA).

From this equivalence we get that $\boldsymbol{\rho}$ must be an interpretation of EQ(BA) in CPC, that is, for all $\{ \langle \psi_0^i, \psi_1^i \rangle : i \in I \} \cup \{\langle \psi_0, \psi_1 \rangle\} \subseteq \mathrm{Fm}^2$,

$$\{ \langle \psi_0^i, \psi_1^i \rangle : i \in I \} \vdash_{\mathrm{EQ}(\mathsf{BA})} \langle \psi_0, \psi_1 \rangle \quad \text{iff} \quad \boldsymbol{\rho}\big(\{ \langle \psi_0^i, \psi_1^i \rangle : i \in I \}\big) \vdash_{\mathrm{CPC}} \boldsymbol{\rho}\big(\langle \psi_0, \psi_1 \rangle\big).$$

In view of the definitions of EQ(BA) and $\boldsymbol{\rho}$ this is equivalent to

$$\{ \psi_0^i \approx \psi_1^i : i \in I \} \vDash_{\mathsf{BA}} \varphi_0 \approx \varphi_1 \quad \text{iff} \quad \{ \psi_0^i \leftrightarrow \psi_1^i : i \in I \} \vdash_{\mathrm{CPC}} \varphi_0 \leftrightarrow \varphi_1,$$

which is just the inverse completeness theorem for CPC (I.2.1.4).

4.1.2. *Example.* Equivalent but nonidentical systems can be found even within the exclusive domain of 1-dimensional deductive systems. For example, a 3-valued paraconsistent logic first considered by D'Ottaviano and Da Costa [48] turns out to be equivalent to the ($\to$, $\neg$)-fragment of Łukasiewicz's 3-valued logic. To see this, let $\boldsymbol{A} = \langle \{0, \frac{1}{2}, 1\}, \to, \neg \rangle$ be the 3-element Wajsberg algebra (a reduct of the 3-element MV-algebra) whose operations are given in Table 1.

Let $\mathcal{L} = \{\to, \neg\}$, $\mathfrak{J}_3 = \langle \boldsymbol{A}, \{1, 2\} \rangle$, and $\mathfrak{L}_3 = \langle \boldsymbol{A}, \{1\} \rangle$. Then $\mathrm{J}_3 = \langle \mathcal{L}, \vDash_{\mathfrak{J}_3} \rangle$ is a paraconsistent logic studied in [48] (see also [53]), and $\mathrm{Ł}_3 = \langle \mathcal{L}, \vDash_{\mathfrak{L}_3} \rangle$ is the familiar 3-valued logic of Łukasiewicz (see [121] for a discussion and references). It is easy to check from the defining matrices that $\vdash_{\mathrm{J}_3} (\neg p \to p) \to p$ but $\nvdash_{\mathrm{Ł}_3} (\neg p \to p) \to p$. So the two deductive systems are different. Let $\Diamond x = \neg x \to x$ and $\Box x = \neg \Diamond \neg x$ (see Table 1), and let the

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

	$\Diamond$
0	0
$\frac{1}{2}$	1
1	1

	$\Box$
0	0
$\frac{1}{2}$	0
1	1

TABLE 1. J_3 and $\mathrm{Ł}_3$

(1-1)-translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ be given by $\boldsymbol{\tau}(p) = \{\Diamond p\}$ and $\boldsymbol{\rho}(p) = \{\Box p\}$. It is easy to see that, for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vDash_{\mathfrak{J}_3} \varphi \quad \text{iff} \quad \{\Diamond \psi : \psi \in \Gamma\} \vDash_{\mathfrak{L}_3} \Diamond \varphi;$$

hence $\boldsymbol{\tau}$ is an interpretation of J_3 in $\mathrm{Ł}_3$. Furthermore, for $\varphi \in \mathrm{Fm}$, we have $\varphi \dashv\vdash_{\mathrm{Ł}_3} \Diamond \Box \varphi$; hence $\varphi \dashv\vdash_{\mathrm{Ł}_3} \Diamond \Box \varphi$. Thus by Thm. 4.2 J_3 and $\mathrm{Ł}_3$ are equivalent.

The systems J_3 and $\mathrm{Ł}_3$ are algebraizable with the same equivalent algebraic semantics, viz., the variety of 3-valued Wajsburg algebras, see Sec. 4.3.1 below. So it is possible that different deductive systems can have the same equivalent algebraic semantics via different translations.

One of our main concerns in the paper is how various properties transfer between equivalent deductive systems. For example, the property of having the deduction-detachment theorem transfers by Thm. 5.7 below. In the following theorem we see how each presentation of a deductive system $\mathcal{S}$ can be automatically transformed into a presentation of any system equivalent to $\mathcal{S}$. In the statement of the theorem, for any finite sets Γ and Φ of k-formulas we use the abbreviation $\langle \Gamma, \Phi \rangle$ for the set of rules of the form $\langle \Gamma, \boldsymbol{\varphi} \rangle$ where $\boldsymbol{\varphi}$ ranges over all the k-formulas Φ, and for any set Ir of k-rules and any (k,l)-translation $\boldsymbol{\tau}$, $\boldsymbol{\tau}(\mathrm{Ir})$ denotes the set of $\boldsymbol{\tau}$-translates of the rules in Ir, i.e., the set of l-rules $\langle \boldsymbol{\tau}(\Gamma), \boldsymbol{\tau}(\varphi) \rangle$ for all $\langle \Gamma, \varphi \rangle \in \mathrm{Ir}$.

Theorem 4.3. *Assume $\mathcal{S}$ and $\mathcal{S}'$ are k- and l-deductive systems that are equivalent under the interpretations $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ and $\boldsymbol{\rho} : \mathcal{S}' \to \mathcal{S}$. Consider any presentation of $\mathcal{S}$ by a set* Ax *of axioms and a set* Ir *of inference rules. Then $\mathcal{S}'$ is presented by the axioms $\boldsymbol{\tau}(\mathrm{Ax}) = \{ \boldsymbol{\tau}\boldsymbol{\varphi} : \boldsymbol{\varphi} \in \mathrm{Ax} \}$ and a set of inference rules that consist of the rules in $\boldsymbol{\tau}(\mathrm{Ir}) = \{ \langle \boldsymbol{\tau}(\Gamma), \boldsymbol{\tau}(\varphi) \rangle : \langle \Gamma, \boldsymbol{\varphi} \rangle \in \mathrm{Ir} \}$ together with the two rules*

$$\langle \boldsymbol{\tau}\boldsymbol{\rho}(\mathbf{p}), \mathbf{p} \rangle \quad \text{and} \quad \langle \{\mathbf{p}\}, \boldsymbol{\tau}\boldsymbol{\rho}(\mathbf{p}) \rangle \tag{9}$$

Proof. Clearly, each l-formula in $\boldsymbol{\tau}(\mathrm{Ax})$ is a theorem of $\mathcal{S}'$ and each rule in $\boldsymbol{\tau}(\mathrm{Ir})$ and in (9) is a derived rule of $\mathcal{S}'$. It remains only to show that, if $\Gamma \cup \{\boldsymbol{\varphi}\}$ is any set of l-formulas such that $\Gamma \vdash_{\mathcal{S}'} \boldsymbol{\varphi}$, then $\boldsymbol{\varphi}$ is provable from Γ by means of the axioms $\boldsymbol{\tau}(\mathrm{Ax})$ and the inference rules $\boldsymbol{\tau}(\mathrm{Ir})$ together with the rules (9). From $\Gamma \vdash_{\mathcal{S}'} \boldsymbol{\varphi}$ we have $\boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\boldsymbol{\varphi})$ since $\boldsymbol{\rho}$ is an interpretation. Let $\boldsymbol{\psi} \in \boldsymbol{\rho}(\boldsymbol{\varphi})$ and let $\boldsymbol{\vartheta}_0, \dots, \boldsymbol{\vartheta}_n = \boldsymbol{\psi}$ be a proof of $\boldsymbol{\psi}$ from $\boldsymbol{\rho}(\Gamma)$ by the axioms Ax and inference rules Ir. Clearly $\boldsymbol{\tau}(\boldsymbol{\vartheta}_0), \dots, \boldsymbol{\tau}(\boldsymbol{\vartheta}_n) = \boldsymbol{\tau}(\boldsymbol{\psi})$ is proof of $\boldsymbol{\tau}(\boldsymbol{\psi})$ from $\boldsymbol{\tau}(\Gamma)$ by the axioms $\boldsymbol{\tau}(\mathrm{Ax})$ and inference rules $\boldsymbol{\tau}(\mathrm{Ir})$. So each l-formula of $\boldsymbol{\tau}\boldsymbol{\rho}(\boldsymbol{\varphi})$ is provable from $\boldsymbol{\tau}\boldsymbol{\rho}(\Gamma)$ by $\boldsymbol{\tau}(\mathrm{Ax})$ and $\boldsymbol{\tau}(\mathrm{Ir})$. Since each l-formula in $\boldsymbol{\tau}\boldsymbol{\rho}(\Gamma)$ is directly derivable from Γ

by the rules $\langle\{\mathbf{p}\}, \boldsymbol{\tau}\boldsymbol{\rho}(\mathbf{p})\rangle$, and $\boldsymbol{\varphi}$ is directly derivable from $\boldsymbol{\tau}\boldsymbol{\rho}(\boldsymbol{\varphi})$ by the rule $\langle\boldsymbol{\tau}\boldsymbol{\rho}(\mathbf{p}), \mathbf{p}\rangle$, we get the desired result. □

4.2. **Algebraizable deductive systems.** In Def. 3.10 the so-called algebraic deductive systems were defined to be the 2-deductive systems EQ(K), where K can be any quasivariety (see Sec. 3.3.2). We discussed how they can be viewed as the paradigm for algebraic logic and that a logic is algebraizable if it is equivalent to EQ(K) for some quasivariety K. Now that we have a precise notion of equivalence in place we can make the definition formal.

Definition 4.4. Let $\mathcal{S}$ be a k-dimensional deductive system. $\mathcal{S}$ is *algebraizable* if $\mathcal{S}$ is equivalent to EQ(K) for some quasivariety K, called the *equivalent* quasivariety of $\mathcal{S}$.

In the example of Sec. 4.1.1 we showed that CPC is equivalent to EQ(BA), with equivalent quasivariety the variety of Boolean algebras.

The defining conditions of algebraizability are usually stated in the following somewhat different form. Suppose $\mathcal{S}$ is a 1-dimensional deductive system that is equivalent to EQ(K), where K is some quasivariety. Let $\boldsymbol{\tau}$ be an interpretation of $\mathcal{S}$ in EQ(K) and ρ an interpretation of EQ(K) in $\mathcal{S}$ such that, for all $\varphi \in \mathrm{Fm}$, $\varphi \dashv\vdash_{\mathcal{S}} \rho(\boldsymbol{\tau}(\varphi))$, and for all $\boldsymbol{\varphi} \in \mathrm{Fm}^2$, $\boldsymbol{\varphi} \dashv\vdash_{\mathrm{EQ(K)}} \boldsymbol{\tau}(\rho(\boldsymbol{\varphi}))$. Now $\boldsymbol{\tau}(p) = \{\, \boldsymbol{\tau}^i(p) : i < m \,\}$, for some 2-formulas $\boldsymbol{\tau}^i(p)$, $i < m$, in a single variable p, and we may write $\boldsymbol{\tau}^i(p) = \langle \delta_i(p), \varepsilon_i(p)\rangle$, $i < m$. Also, we have $\rho(\langle p_0, p_1\rangle) = \{\, \rho^j(p_0, p_1) : j < n \,\}$ for certain 1-formulas $\rho^j(p_0, p_1)$, which we will denote also by $\rho^j(p_0, p_1) = \Delta_j(p_0, p_1)$. We thus have, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \tag{10}$$

$$\text{iff} \quad \{\, \langle \delta_i(\psi), \varepsilon_i(\psi)\rangle : i < m, \psi \in \Gamma \,\} \vdash_{\mathcal{S}_{\mathsf{K}}} \{\, \langle \delta_i(\varphi), \varepsilon_i(\varphi)\rangle : i < m \,\} \tag{11}$$

$$\text{iff} \quad \{\, \delta_i(\psi) \approx \varepsilon_i(\psi) : i < m, \psi \in \Gamma \,\} \models_{\mathsf{K}} \{\, \delta_i(\varphi) \approx \varepsilon_i(\varphi) : i < m \,\}, \tag{12}$$

where (11) holds because $\boldsymbol{\tau}$ is an interpretation and (12) follows from Thm. 3.9.

On similar grounds, the fact that ρ is an interpretation translates into the following: for $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^2$,

$$\begin{aligned} &\{\, \psi_0 \approx \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \models_{\mathsf{K}} \varphi_0 \approx \varphi_1 \\ &\qquad\qquad \text{iff} \quad \{\Delta_j(\psi_0, \psi_1) : j < n\} \vdash_{\mathcal{S}} \{\Delta_j(\varphi_0, \varphi_1) : j < n\}. \end{aligned} \tag{13}$$

The invertibility conditions translate into: for all $\varphi \in \mathrm{Fm}$,

$$\varphi \quad \dashv\vdash_{\mathcal{S}} \quad \{\, \Delta_j\big(\delta_i(\varphi), \varepsilon_i(\varphi)\big) : i < m, j < n \,\}, \tag{14}$$

and for all $\boldsymbol{\psi} \in \mathrm{Fm}^2$,

$$\psi_0 \approx \psi_1 \quad \rightleftharpoons\models_{\mathsf{K}} \quad \{\, \delta_i\big(\Delta_j(\psi_0, \psi_1)\big) \approx \varepsilon_i\big(\Delta_j(\psi_0, \psi_1)\big) : i < m, j < n \,\}. \tag{15}$$

These were precisely the conditions for algebraizability stipulated in [13]. As observed in Thm. 4.2, conclusions (13) and (14) suffice to guarantee the translations establish an equivalence.

4.2.1. *Examples of algebraizable 1-deductive systems.* In Sec. 4.1.1 we saw that CPC is algebraizable, with equivalent quasivariety the variety of Boolean algebras. All the examples in Sec. 2.2 except $\mathrm{S5^C}$ are algebraizable: IPC has equivalent variety HA, BCK has equivalent quasivariety BCK, and K and $\mathrm{S5^G}$ have equivalent varieties MA and MO, respectively. All of these except BCK and BCK* use the same translations used for the equivalence of CPC and BA; for BCK and BCK* we take $\boldsymbol{\tau} = \{\langle p, p \to p\rangle\}$ and $\rho = \{p \to q,\, q \to p\}$.

First-order predicate logic with $\alpha \leq \omega$ variables, when formalized as a 1-dimensional deductive system in the sense of Sec. 2.2.6, is algebraizable with the class of *cylindric algebras of dimension* α as its equivalent variety. For some other examples of algebraizable 1-dimensional deductive systems see [13, 63, 95, 118].

These examples show that the abstract notion of algebraizability formalized in Def. 4.4 agrees with the usual one when specialized to the familiar logics and produces the expected algebraic semantics. But a number of alternative definitions of algebraizability have been proposed, some of which can be viewed as a modification of the one given in Def. 4.4. The first attempt to provide a systematic and abstract treatment of algebraic logic can be found in Rasiowa [120]. Rasiowa's theory is quite close to the one presented here but differs in some important aspects. These are discussed in Note 4.2 along with alternative notions of algebraizability.

4.2.2. *Examples of algebraizable 2-deductive systems.* One of the interesting features of a precise notion of abstract algebraizability is that it can be applied to reveal instances of the Lindenbaum-Tarski process where one would not expect to find them. A large number of apparently disparate phenomena can be unified at least conceptually this way. A good example of this kind is the dual character of a semilattice as both an algebra and a partially ordered structure. This duality can be naturally formulated as the equivalence of two 2-dimensional deductive systems. The concept of a semilattice as an algebra is captured by the algebraic 2-deductive system EQ(SL), an extension of free equational logic, where SL is the variety of semilattices. On the other hand, the concept of a semilattice as a partially ordered algebra is captured by the 2-deductive system SL, an extension of free partially ordered logic, presented in Sec. 3.3.4. Note that SL is not "algebraic" in the classical sense in which EQ(SL) is algebraic; its reduced matrices are of the form $\langle \boldsymbol{A}, \leq\rangle$, with $\leq$ a partial ordering relation, while the reduced matrices of EQ(SL) are of the form $\langle \boldsymbol{A}, \boldsymbol{\Delta}_A\rangle$. That these two deductive systems are equivalent in the sense of Def. 4.1(ii) can be viewed as the formal expression of the well known fact that a semilattice can be thought of as either a partially ordered set or an algebra. Alternatively, the algebraic character of semilattices as partially ordered sets can be expressed in the fact that SL is algebraizable with SL as its equivalent algebraic semantics.

For simplicity, we will use SL to denote all three of the following: the variety of semilattices over the language $\mathcal{L} = \{\wedge\}$, the class of reduced matrix models of EQ(SL), i.e., the class of all matrices $\langle \boldsymbol{A}, \boldsymbol{\Delta}_A\rangle$ where $\boldsymbol{A}$ is a semilattice, and the class of reduced matrix models of SL, i.e., the class of all matrices $\langle \boldsymbol{A}, \leq\rangle$ where $\boldsymbol{A}$ is a semilattice and $\leq$ is the partial ordering of $\boldsymbol{A}$.

Consider the $(2, 2)$-translations

$$\boldsymbol{\tau}(\langle p_0, p_1\rangle) = \{\langle p_0,\, p_0 \wedge p_1\rangle\} \quad \text{and} \quad \boldsymbol{\rho}(\langle p_0, p_1\rangle) = \{\langle p_0, p_1\rangle, \langle p_1, p_0\rangle\}.$$

In SL the 2-formula $\langle p_0, p_1\rangle$ represents the inequality $p_0 \leq p_1$ and in EQ(SL) it represents the equality $p_0 \approx p_1$. So $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ can be more suggestively written in the form

$$\boldsymbol{\tau}(p_0 \leq p_1) = \{p_0 \approx p_0 \wedge p_1\} \quad \text{and} \quad \boldsymbol{\rho}(p_0 \approx p_1) = \{p_0 \leq p_1,\, p_1 \leq p_0\}.$$

The fact that these turn out to be invertible interpretations between SL and EQ(SL) reflects the fact that in a semilattice $x \leq y$ iff $x = x \wedge y$ and $x = y$ iff $x \leq y$ and $y \leq x$.

For all $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^2$ we have

$$\begin{aligned}\Gamma \vdash_{\mathrm{SL}} \boldsymbol{\varphi} \quad &\text{iff} \quad \{\, \psi_0 \leq \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{SL}} \varphi_0 \leq \varphi_1 \\ &\text{iff} \quad \{\, \psi_0 \approx \psi_0 \wedge \psi_1 : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{SL}} \varphi_0 \approx \varphi_0 \wedge \varphi_1 \\ &\text{iff} \quad \{\, \boldsymbol{\tau}(\boldsymbol{\psi}) : \boldsymbol{\psi} \in \Gamma \,\} \vDash_{\mathsf{SL}} \boldsymbol{\tau}(\boldsymbol{\varphi}) \\ &\text{iff} \quad \boldsymbol{\tau}(\Gamma) \vdash_{\mathrm{EQ}(\mathsf{SL})} \boldsymbol{\tau}(\boldsymbol{\varphi}).\end{aligned}$$

Thus $\boldsymbol{\tau}$ is an interpretation of SL in EQ(SL).

$$\begin{aligned}\boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\varphi})) &= \boldsymbol{\tau}(\boldsymbol{\rho}(\varphi_0 \approx \varphi_1)) \\ &= \boldsymbol{\tau}\big(\{\varphi_0 \leq \varphi_1,\, \varphi_1 \leq \varphi_0\}\big) \\ &= \{\varphi_0 \approx \varphi_0 \wedge \varphi_1,\, \varphi_1 \approx \varphi_1 \wedge \varphi_0\}.\end{aligned}$$

Since $\varphi_0 \approx \varphi_1 \mathrel{=\!\!|\!\!\vDash_{\mathsf{SL}}} \boldsymbol{\tau}(\boldsymbol{\rho}(\varphi_0 \approx \varphi_1))$, we have $\boldsymbol{\varphi} \dashv\vdash_{\mathrm{EQ}(\mathsf{SL})} \boldsymbol{\tau}(\boldsymbol{\rho}(\boldsymbol{\varphi}))$. Thus SL and EQ($\mathsf{SL}$) are equivalent by Thm. 4.2.

We illustrate the application of the algorithm given in Thm. 4.3 by using the above interpretations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ to translate the alternative presentation of SL given in Sec. 3.3.3 into a presentation of EQ(SL).

$$\begin{array}{ll} p \approx p \wedge p, & \boldsymbol{\tau}((\mathrm{E1})) \\ p \wedge q \approx (p \wedge q) \wedge p, & \boldsymbol{\tau}((\mathrm{Sl2})) \\ p \wedge q \approx (p \wedge q) \wedge q, & \boldsymbol{\tau}((\mathrm{Sl3})) \\ \dfrac{p \approx p \wedge q,\ q \approx q \wedge r}{p \approx p \wedge r}, & \boldsymbol{\tau}((\mathrm{ER2})) \\ \dfrac{r \approx r \wedge p,\ r \approx r \wedge q}{r \approx r \wedge (p \wedge q)}, & \boldsymbol{\tau}((\mathrm{SLR1})) \\ \dfrac{p_0 \approx p_0 \wedge p_1,\ p_1 \approx p_1 \wedge p_0}{p_0 \approx p_1}, & \langle \boldsymbol{\tau\rho}(\mathbf{p}),\, \mathbf{p} \rangle \\ \dfrac{p_0 \approx p_1}{p_0 \approx p_0 \wedge p_1,\ p_1 \approx p_1 \wedge p_0}, & \langle \{\mathbf{p}\},\, \boldsymbol{\tau\rho}(\mathbf{p}) \rangle \end{array}$$

These two identities and four quasi-identities constitute an alternative axiom system for the variety of semilattices. It is much different and more complicated than the standard one, but notice that it is a complete axiomatization in that it does not assume the Birkhoff axioms and rules of free equational logic (Sec. 3.3.1) as does the standard one. Note that, in the presence of the Birkhoff axioms, the rule $\langle \{\mathbf{p}\},\, \boldsymbol{\tau\rho}(\mathbf{p}) \rangle$ is a consequence of axiom $\boldsymbol{\tau}((\mathrm{E1}))$.

The same translations give the equivalence of the partially ordered and equational deductive systems of lattices, LA and EQ(LA), and of distributive lattices, DL and EQ(DL). LA and DL denote the varieties of lattices and distributive lattices, respectively.

In general, for any algebraizable k-deductive system $\mathcal{S}$ with equivalent quasivariety K, a defining set of identities and quasi-identities for K, that is, a presentation of EQ(K), can be obtained from any given presentation of $\mathcal{S}$ by means of Thm. 4.3. The identities are the $\boldsymbol{\tau}$-transforms of the axioms of $\mathcal{S}$ and the quasi-identities are the $\boldsymbol{\tau}$-transforms of the inference rules of $\mathcal{S}$ together with the following quasi-identities corresponding to the rules of (9).

$$\&_{i<n,j<m}\, \varepsilon_i(\boldsymbol{\rho}_j(p,q)) \approx \delta_i(\boldsymbol{\rho}_j(p,q)) \implies p \approx q \quad \text{and} \tag{16}$$

$$p \approx q \Longrightarrow \varepsilon_i(\boldsymbol{\rho}_j(p,q)) \approx \delta_i(\boldsymbol{\rho}_j(p,q) \text{ for all } i < n, j < m, \tag{17}$$

where

$$\boldsymbol{\tau}(\mathbf{p}) = \{\varepsilon_0(\mathbf{p}) \approx \delta_0(\mathbf{p}), \dots, \varepsilon_{n-1}(\mathbf{p}) \approx \delta_{n-1}(\mathbf{p})\}, \quad \boldsymbol{\rho}(\langle p,q\rangle) = \{\boldsymbol{\rho}_0(p,q), \dots, \boldsymbol{\rho}_{m-1}(p,q)\}.$$

Note that in the presence of the Birkhoff axiom and rules the quasi-identities (17) are equivalent to the identities $\varepsilon_i(\boldsymbol{\rho}_j(p,p)) \approx \delta_i(\boldsymbol{\rho}_j(p,p))$, $i < n, j < m$.

4.3. **Intrinsic characterizations of algebraizability.** Def. 4.4 on its face provides no means of testing the algebraizability of a deductive system if the equivalent quasivariety and the two inverse interpretations are not known and cannot be surmised. An intrinsic characterization of algebraizability is needed for this purpose. There are several different ones. For the first one we need the to know the interpretations, but not the equivalent quasivariety.

Theorem 4.5. *A k-deductive system $\mathcal{S}$ is algebraizable iff there is a $(k,2)$-translation $\boldsymbol{\tau}(\langle p_0,\dots,p_{k-1}\rangle)$ and a $(2,k)$-translation $\boldsymbol{\rho}(\langle p_0,p_1\rangle)$ such that the following hold.*

(i) $\vdash_{\mathcal{S}} \boldsymbol{\rho}(\langle p,p\rangle)$;
(ii) $\boldsymbol{\rho}(\langle p,q\rangle) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\langle q,p\rangle)$;
(iii) $\boldsymbol{\rho}(\langle p,q\rangle), \boldsymbol{\rho}(\langle q,r\rangle) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\langle p,r\rangle)$;
(iv) $\boldsymbol{\rho}(\langle p_0,q_0\rangle),\dots,\boldsymbol{\rho}(\langle p_{n-1},q_{n-1}\rangle) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\langle \omega p_0,\dots,p_{k-1}, \omega q_0,\dots,q_{k-1}\rangle)$, *for each* $\omega \in \mathcal{L}$, *with* n *the rank of* $\boldsymbol{\omega}$,
(v) $\mathbf{p} \dashv\vdash_{\mathcal{S}} \boldsymbol{\rho\tau}(\mathbf{p})$, *where* $\mathbf{p} = \langle p_0,\dots,p_{k-1}\rangle$.

Proof. Define the 2-deductive system $\mathcal{S}'$ by the condition that, for all $\Gamma \cup \{\boldsymbol{\varphi}\} \subseteq \mathrm{Fm}^2$,

$$\Gamma \vdash_{\mathcal{S}'} \boldsymbol{\vartheta} \quad \text{iff} \quad \boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\boldsymbol{\varphi}). \tag{18}$$

It is easy to verify that $\vdash_{\mathcal{S}'}$ satisfies the five conditions of Definition 3.1, and hence $\langle \mathcal{L}, \vdash_{\mathcal{S}'}\rangle$ is a 2-deductive system. For example, suppose $\Gamma \vdash_{\mathcal{S}'} \boldsymbol{\varphi}$ and $\sigma : \mathbf{Fm} \to \mathbf{Fm}$ is a substitution. Then $\sigma\boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}} \sigma\boldsymbol{\rho}(\boldsymbol{\varphi})$. But $\sigma\boldsymbol{\rho}(\Gamma) = \boldsymbol{\rho}(\sigma\Gamma)$ and $\sigma\boldsymbol{\rho}(\boldsymbol{\varphi}) = \boldsymbol{\rho}(\sigma\boldsymbol{\varphi})$. Hence $\boldsymbol{\rho}(\sigma\Gamma) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\sigma\boldsymbol{\varphi})$ and thus $\sigma\Gamma \vdash_{\mathcal{S}'} \sigma\boldsymbol{\varphi}$. So $\mathcal{S}'$ is substitution invariant.

The conditions (i)–(iv) guarantee that the axioms and inference rules (ER1)–(ER3)$_\omega$, $\omega \in \mathcal{L}$, of free equational logic (Sec. 3.3.1) are valid in $\mathcal{S}'$. Thus $\mathcal{S}'$ is an extension of EQ and hence of the form EQ(K) for some quasivariety K. $\boldsymbol{\rho}$ is an interpretation of EQ(K) in $\mathcal{S}$ by (18) and $\boldsymbol{\tau}$ is its right inverse by (v). So $\mathcal{S}$ is algebraizable by Thm. 4.2, and K is its equivalent quasivariety.

Conversely, assume $\mathcal{S}$ is algebraizable with equivalent quasivariety K and inverse interpretations $\boldsymbol{\tau} : \mathcal{S} \to \mathrm{EQ(K)}$ and $\boldsymbol{\tau} : \mathrm{EQ(K)} \to \mathcal{S}$. For all $\langle \psi_0^0, \psi_1^0\rangle, \dots, \langle \psi_0^{n-1}, \psi_1^{n-1}\rangle$, $\langle \varphi_0, \varphi_1\rangle \in \mathrm{Fm}^2$ we have

$$\psi_0^0 \approx \psi_1^0, \dots, \psi_0^{n-1} \approx \psi_1^{n-1} \vDash_{\mathsf{K}} \varphi_0 \approx \varphi_1 \quad \text{iff} \quad \boldsymbol{\rho}(\langle \psi_0^0, \psi_1^0\rangle), \dots, \boldsymbol{\rho}(\langle \psi_0^{n-1}, \psi_1^{n-1}\rangle) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\langle \varphi_0, \varphi_1\rangle).$$

Conditions (i)–(iv) are immediate consequences of this equivalence, and (v) holds because $\boldsymbol{\tau}$ is a right inverse of $\boldsymbol{\rho}$. □

4.3.1. *Example.* This theorem can be applied to show the algebraizability of the 3-valued logic of Łukasiewicz Ł$_3$ (see Sec. 4.1.2). Let $\boldsymbol{\tau}$ be the $(1,2)$-translation $\boldsymbol{\tau}(p) = \{\langle p, p \to p\rangle\}$ and ρ the $(2,1)$-translation $\rho(\langle p_0, p_1\rangle) = \{p_0 \to p_1, p_1 \to p_0\}$. The verification of the entailments (i)–(v) of Thm. 4.5 is straightforward.

The equivalent algebraic semantics for Ł$_3$ is the variety of Wajsburg algebas ([62]), which is the class of $(\to, \neg)$-subreducts of MV algebras of exponent 3 ([24]). For a comprehensive treatment of the algebras of many-valued logics see [27].

Since equivalence between deductive systems is clearly transitive, it follows that the 3-valued paraconsistent logic J$_3$ is also algebraizable (Sec. 4.1.2) By Cor. 4.15

4.3.2. *Equivalential deductive systems.* Let $\mathcal{S}$ be an algebraizable k-deductive system with equivalent quasivariety K and inverse interpretations $\boldsymbol{\tau} : \mathcal{S} \to \mathrm{EQ}(\mathsf{K})$ and $\boldsymbol{\rho} : \mathrm{EQ}(\mathsf{K}) \to \mathcal{S}$. Let $\boldsymbol{\varphi} = \langle \varphi_0, \dots, \varphi_{k-1}\rangle$ and $\boldsymbol{\psi} = \langle \psi_0, \dots, \psi_{k-1}\rangle$ be arbitrary k-formulas. By basic properties of equality we have

$$\boldsymbol{\tau}(\langle \varphi_0, \dots, \varphi_{k-1}\rangle),\ \varphi_0 \approx \psi_0, \dots, \varphi_{k-1} \approx \psi_{k-1} \vDash_{\mathsf{K}} \boldsymbol{\tau}(\langle \psi_0, \dots, \psi_{k-1}\rangle).$$

Thus

$$\boldsymbol{\rho}\boldsymbol{\tau}(\boldsymbol{\varphi}),\ \boldsymbol{\rho}(\langle \varphi_0, \psi_0\rangle), \dots, \boldsymbol{\rho}(\langle \varphi_{k-1}, \psi_{k-1}\rangle) \vdash_{\mathcal{S}} \boldsymbol{\rho}\boldsymbol{\tau}(\boldsymbol{\psi}).$$

Abbreviating the set $\boldsymbol{\rho}(\langle \varphi_0, \psi_0\rangle) \cup \dots \cup \boldsymbol{\rho}(\langle \varphi_{k-1}, \psi_{k-1}\rangle)$ by $\boldsymbol{\rho}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ we get

(19) $$\boldsymbol{\varphi},\ \boldsymbol{\rho}(\boldsymbol{\varphi}, \boldsymbol{\psi}) \vdash_{\mathcal{S}} \boldsymbol{\psi} \quad (\boldsymbol{\rho}\text{-}detachment)$$

A general (possibly infinite) $(2,k)$-translation $\boldsymbol{\rho}$ such that the four rules, (i)–(iv), of the hypothesis of Theorem 4.5 together with $\boldsymbol{\rho}$-detachment are derivable in $\mathcal{S}$ is called an *equivalence system* for $\mathcal{S}$. A deductive system that has a (finite) equivalence system is called (*finitely*) *equivalential*. From the above argument we see that every algebraizable deductive system is finitely equivalential, but the class of finitely equivalential systems is much more extensive and the class of equivalential systems even wider yet. Equivalential deductive systems were introduced in [117] and have been extensively studied in [29].

The importance of equivalence systems for abstract algebraic logic derives from the fact that they define the Leibniz congruence in the following natural sense and are in fact characterized by the property. Let $\boldsymbol{\rho}(\langle p_0, p_1\rangle) = \{\, \boldsymbol{\rho}_i(\langle p_0, p_1\rangle) : i \in I \,\}$ be a general $(2,k)$-translation. For any algebra $\boldsymbol{A}$ and all $a, b \in A$ let $\boldsymbol{\rho}^{\boldsymbol{A}}(a,b) = \{\, \boldsymbol{\rho}^{\boldsymbol{A}}_i(a,b) : i \in I \,\} \subseteq A^k$ $\boldsymbol{\rho}$ is said to *define Leibniz congruences* in a k-deductive system $\mathcal{S}$ if, for every algebra $\boldsymbol{A}$ and $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$,

$$\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\,F = \{\, \langle a, b\rangle \in A^2 : \boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F \,\}.$$

Theorem 4.6. *A general $(2,k)$-translation is an equivalence system for a k-deductive system $\mathcal{S}$ iff it defines Leibniz congruences in $\mathcal{S}$.*

Proof. Suppose $\boldsymbol{\rho}$ is an equivalence system for a k-deductive system $\mathcal{S}$. Fix an arbitrary algebra $\boldsymbol{A}$ and $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and define

$$\varPhi := \{\, \langle a, b\rangle \in A^2 : \boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F \,\}.$$

The entailments (i)–(v) of Thm. 4.5 guarantee that $\varPhi$ is a congruence relation, and $\boldsymbol{\rho}$-detachment, (19), guarantees that $\varPhi$ is compatible with F. So $\varPhi \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\,F$. To show the inclusion in the opposite direction assume $a \equiv b \pmod{\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\,F}$. Then, for each $i \in I$, $\boldsymbol{\rho}^{\boldsymbol{A}}_i(a,a) \equiv \boldsymbol{\rho}^{\boldsymbol{A}}_i(a,b) \pmod{\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\,F}$. But $\boldsymbol{\rho}^{\boldsymbol{A}}_i(a,a) \in F$ by reflexivity, Thm. 4.5(i). Thus

$\boldsymbol{\rho}_i^{\boldsymbol{A}}(a,b) \in F$ since $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F$ is compatible with F. This shows $\Phi = \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F$. Hence $\boldsymbol{\rho}$ defines Leibniz congruences in $\mathcal{S}$.

Suppose now that $\boldsymbol{\rho}$ defines Leibniz congruences in $\mathcal{S}$. Since $p \equiv p \pmod{\boldsymbol{\Omega} T}$ for every $T \in \mathrm{Th}\,\mathcal{S}$, we have that $\boldsymbol{\rho}(\langle p,p\rangle) \subseteq T$ for every $T \in \mathrm{Th}\,\mathcal{S}$, i.e., reflexivity is a theorem (in fact a possibly infinite set of theorems) of $\mathcal{S}$. Suppose $\boldsymbol{\rho}(\langle \varphi,\psi\rangle) \subseteq T$. Then $\varphi \equiv \psi$ $(\mathrm{mod}\ \boldsymbol{\Omega} T)$. Hence $\psi \equiv \varphi \pmod{\boldsymbol{\Omega} T}$, and thus $\boldsymbol{\rho}(\langle \psi,\varphi\rangle) \subseteq T$ by assumption. Since this holds for every $T \in \mathrm{Th}\,\mathcal{S}$ and all $\varphi,\psi \in \mathrm{Fm}$, symmetry, Thm. 4.5(ii), is a derived infinite rule of $\mathcal{S}$. In a similar way we get that Thm. 4.5(iii), (iv) are derived rules of $\mathcal{S}$. Finally, assume $\{\boldsymbol{\varphi}\} \cup \boldsymbol{\rho}(\boldsymbol{\varphi},\boldsymbol{\psi}) \subseteq T$. Then $\boldsymbol{\varphi} \equiv \boldsymbol{\psi} \pmod{(\boldsymbol{\Omega} T)^k}$ and hence $\boldsymbol{\psi} \in T$ since $\boldsymbol{\Omega} T$ is compatible with T. So $\boldsymbol{\rho}$-detachment is a rule of $\mathcal{S}$, and hence $\boldsymbol{\rho}$ is an equivalence system for $\mathcal{S}$. □

The key to the other intrinsic characterizations of algebraizability is the abstraction of the correspondence between filters and congruences that is characteristic of the familiar algebraizable logics. Let $\boldsymbol{A}$ be a Boolean algebra and F a Boolean filter of $\boldsymbol{A}$. Let $\equiv_F = \{\, \langle a,b\rangle : a \leftrightarrow^{\boldsymbol{A}} b \in F \,\}$. It is well known that the mapping $F \mapsto \equiv_F$ is a one-one correspondence between Boolean filters and congruences of $\boldsymbol{A}$, in fact an isomorphism between the lattices of Boolean filters and congruences. The Boolean filters of $\boldsymbol{A}$ are just the CPC-filters of $\boldsymbol{A}$ and every congruence is a BA-congruence since BA is a variety. Moreover, as we saw in the remarks following Thm. 3.8, $\equiv_F = \boldsymbol{\Omega}^{\boldsymbol{A}} F$, the Leibniz congruence defined by F. So, in the case that $\mathcal{S}$ is CPC and K is BA, we have that for every $\boldsymbol{A} \in \mathsf{K}$, the mapping $F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the lattice of K-congruences of $\boldsymbol{A}$. (Actually, it is easy to see that this holds for any $\mathcal{L}$-algebra $\boldsymbol{A}$, not just Boolean algebras.) The same is true of all the familiar algebraizable logics. It turns out that the existence of such an isomorphism can be used to characterize algebraizable deductive systems and their equivalent quasivarieties. Recall that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the largest congruence of $\boldsymbol{A}$ compatible with F (see Def. 3.5) and hence does not depend on properties of the deductive system or its equivalent quasivariety. We shall also see how this can be used to give an intrinsic characterization of algebraizability that does not depend on prior knowledge of the equivalent quasivariety.

The next theorem is the key to the intrinsic characterizations of algebraizability. According to it, two deductive systems are equivalent (in the sense of Def. 4.1(ii)) if and only if there exists an isomorphism between the respective theory lattices that, like the formation of the Leibniz congruence, commutes with substitutions in a sense explained shortly. Recall that the $\mathcal{S}$-theories form an algebraic lattice $\mathbf{Th}\ \mathcal{S}$, and that $\mathrm{Cn}_{\mathcal{S}}\,\Gamma = \{\, \varphi : \Gamma \vdash_{\mathcal{S}} \varphi \,\}$ is the theory generated by $\Gamma \subseteq \mathrm{Fm}^k$ (Sec. 3.1). $\mathrm{Cn}_{\mathcal{S}}$ is called the *$\mathcal{S}$-consequence operator.* More generally, for any $\mathcal{L}$-algebra $\boldsymbol{A}$, the $\mathcal{S}$-filters on $\boldsymbol{A}$ form an algebraic lattice $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A} = \langle \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}, \cap, \vee^{\mathcal{S}}\rangle$; for arbitrary joins in $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ we use the symbol $\bigvee^{\mathcal{S}}$. $\mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}\,X$ is the $\mathcal{S}$-filter generated by $X \subseteq A^k$.

Let $\mathcal{S}$ be a k-deductive system and $\boldsymbol{A}$ any $\mathcal{L}$-algebra. For any homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$ we define $h_{\mathcal{S}}^{\boldsymbol{B}} : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}$ by setting $h_{\mathcal{S}}^{\boldsymbol{B}}(F) = \mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{B}}\,h(F)$ for each $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$.

Lemma 4.7. *Let $\boldsymbol{A}$ be any $\mathcal{L}$-algebra. For every endomorphism $h : \boldsymbol{A} \to \boldsymbol{A}$ and $X \subseteq A^k$, $h_{\mathcal{S}}^{\boldsymbol{A}}\,\mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}\,X = \mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}\,h(X)$.*

Proof. $X \subseteq h^{-1}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(X)\big) \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$. So $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X \subseteq h^{-1}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(X)\big)$, i.e., $h(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(X)$. Hence $h^{\boldsymbol{A}}_{\mathcal{S}}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(X) = h^{\boldsymbol{A}}_{\mathcal{S}}(X)$. Conversely, $h(X) \subseteq h(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X) \subseteq h^{\boldsymbol{A}}_{\mathcal{S}}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X$. Hence $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(X) \subseteq h^{\boldsymbol{A}}_{\mathcal{S}}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X$. □

As an easy corollary we get that $h^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigvee^{\mathcal{S}}_{i\in I} F_i\big) = \bigvee^{\mathcal{S}}_{i\in I} h^{\boldsymbol{A}}_{\mathcal{S}} F_i$ for any set $\{\, F_i : i \in I \,\}$ of $\mathcal{S}$-filters of $\boldsymbol{A}$. In fact, clearly $h^{\boldsymbol{A}}_{\mathcal{S}} F_i \subseteq h^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigvee^{\mathcal{S}}_{i\in I} F_i\big)$ for each $i \in I$. This gives the inclusion from right to left. For the opposite inclusion we have $h^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigvee^{\mathcal{S}}_{i\in I} F_i\big) = h^{\boldsymbol{A}}_{\mathcal{S}}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigcup_{i\in I} F_i\big) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h\big(\bigcup_{i\in I} F_i\big) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigcup_{i\in I} h(F_i)\big) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigcup_{i\in I} h^{\boldsymbol{A}}_{\mathcal{S}}(F_i)\big) = \bigvee^{\mathcal{S}}_{i\in I} h^{\boldsymbol{A}}_{\mathcal{S}}(F_i)$.

Lem. 4.7 can be paraphrased by saying that endomorphisms commute with filter generation on $\boldsymbol{A}$. Each interpretation induces a transformation between the filters of the source and target deductive systems that also commutes with filter generation as we now show. Let $\boldsymbol{\tau}$ be a (k, l)-translation. Recall that by definition $\boldsymbol{\tau}$ is of the form

$$\boldsymbol{\tau}(\mathbf{p}) = \{\, \boldsymbol{\tau}^i(\mathbf{p}) : i < n \,\},$$

where $\mathbf{p} = \langle p_0, \dots, p_{k-1}\rangle$ (a k-variable) and

$$\boldsymbol{\tau}^i(p_0, \dots, p_{k-1}) = \big\langle \tau^i_0(p_0, \dots, p_{k-1}), \dots, \tau^i_{l-1}(p_0, \dots, p_{k-1})\big\rangle$$

for each $i < n$. Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. For each k-sequence $\mathbf{a} = \langle a_0, \dots, a_{k-1}\rangle \in \boldsymbol{A}^k$, let

$$\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}) = \{\, \boldsymbol{\tau}^{i\boldsymbol{A}}(\mathbf{a}) : i < n \,\},$$

where $\boldsymbol{\tau}^{i\boldsymbol{A}}(\mathbf{a}) = \big\langle \tau^{i\boldsymbol{A}}_0(a_0, \dots, a_{k-1}), \dots, \tau^{i\boldsymbol{A}}_{l-1}(a_0, \dots, a_{k-1})\big\rangle$ for each $i < n$. Finally, $\boldsymbol{\tau}^{\boldsymbol{A}}(X) = \bigcup\{\, \boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}) : \mathbf{a} \in X \,\}$ for each $X \subseteq A^k$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be k- and l-deductive systems, respectively. Recall that a (k, l)-translation is an interpretation of $\mathcal{S}$ in $\mathcal{S}'$ if $\mathit{\Gamma} \vdash_{\mathcal{S}} \boldsymbol{\varphi}$ iff $\boldsymbol{\tau}(\mathit{\Gamma}) \vdash_{\mathcal{S}'} \boldsymbol{\tau}(\boldsymbol{\varphi})$. For each interpretation $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ and each $\mathcal{L}$-algebra $\boldsymbol{A}$ we define $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'} : \mathcal{P}(A^k) \to \mathrm{Fi}_{\mathcal{S}'}\, \boldsymbol{A}$ by setting $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(X) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X)$ for every $X \subseteq A$.

Lemma 4.8. *Let $\boldsymbol{A}$ be any $\mathcal{L}$-algebra. For every interpretation $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ and every $X \subseteq A^k$, $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}} X$.*

Proof. The proof is similar to the proof of Lem. 4.7. Let

$$F = (\boldsymbol{\tau}^{\boldsymbol{A}})^{-1}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X)\big) = \{\, \mathbf{a} \in A^k : \boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}) \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X) \,\}.$$

We claim that $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$. Suppose $\boldsymbol{\psi}_0(\bar p), \dots, \boldsymbol{\psi}_{n-1}(\bar p) \vdash_{\mathcal{S}} \boldsymbol{\varphi}(\bar p)$, and $\bar a \in A^m$ such that $\boldsymbol{\psi}^{\boldsymbol{A}}_0(\bar a), \dots, \boldsymbol{\psi}^{\boldsymbol{A}}_{n-1}(\bar a) \in F$. Since $\boldsymbol{\tau}$ is an interpretation,

$$\boldsymbol{\tau}\big(\boldsymbol{\psi}_0(\bar p)\big), \dots, \boldsymbol{\tau}\big(\boldsymbol{\psi}_{n-1}(\bar p)\big) \vdash_{\mathcal{S}'} \boldsymbol{\tau}\big(\boldsymbol{\varphi}(\bar p)\big).$$

But $\boldsymbol{\tau}^{\boldsymbol{A}}\big(\boldsymbol{\psi}^{\boldsymbol{A}}_0(\bar a)\big), \dots, \boldsymbol{\tau}^{\boldsymbol{A}}\big(\boldsymbol{\psi}^{\boldsymbol{A}}_{n-1}(\bar a)\big) \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(S)$. So $\boldsymbol{\tau}^{\boldsymbol{A}}\big(\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar a)\big) \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X)$, and hence $\boldsymbol{\varphi}^{\boldsymbol{A}}(\bar a) \in F$. This proves the claim.

Since it is clear that $X \subseteq F$, it follows from the claim that $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X \subseteq F$, and hence $\boldsymbol{\tau}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X\big) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X)$. Thus $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X)$. Conversely, $\boldsymbol{\tau}^{\boldsymbol{A}}(X) \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, x\big) \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X$. Hence $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(X) \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X$. □

As an easy corollary of the lemma we have that $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}$ is a complete join semilattice homomorphism, i.e., $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\big(\bigvee^{\mathcal{S}}_{i\in I} F_i\big) = \bigvee^{\mathcal{S}'}_{i\in I} \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F_i)$ for any subset $\{\, F_i : i \in I \,\}$ of $\mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$.

$$\begin{array}{ccc}
\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} & \xrightarrow{\alpha} & \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A} \\
{\scriptstyle h_{\mathcal{S}}}\downarrow & & \downarrow{\scriptstyle h_{\mathcal{S}'}} \\
\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} & \xrightarrow{\alpha} & \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A}
\end{array}$$

FIGURE 2

Let $\mathcal{S}$ and $\mathcal{S}'$ be k- and l deductive systems and let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A mapping $\alpha : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A}$ is said to *commute with endomorphisms* if, for every endomorphism $h : \boldsymbol{A} \to \boldsymbol{A}$, $h^{\boldsymbol{A}}_{\mathcal{S}'} \circ \alpha = \alpha \circ h^{\boldsymbol{A}}_{\mathcal{S}}$, i.e., the diagram in Figure 2 commutes.

Lemma 4.9. *Let $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ be an interpretation between the deductive systems $\mathcal{S}$ and $\mathcal{S}'$. Then for every algebra $\boldsymbol{A}$ the mapping $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'} : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A}$ commutes with endomorphisms.*

Proof. We first show $\boldsymbol{\tau}^{\boldsymbol{A}} h(\mathbf{a}) = h\,\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a})$ for every $\mathbf{a} \in A^k$.

$$\begin{aligned}
\boldsymbol{\tau}^{\boldsymbol{A}} h(\mathbf{a}) &= \boldsymbol{\tau}^{\boldsymbol{A}} h\big(\langle a_0, \dots, a_{k-1}\rangle\big) \\
&= \boldsymbol{\tau}^{\boldsymbol{A}}\big(\langle h a_0, \dots, h a_{k-1}\rangle\big) \\
&= \big\{\, \big\langle \tau_0^{i\boldsymbol{A}}(h a_0, \dots, h a_{k-1}), \dots, \tau_{l-1}^{i\boldsymbol{A}}(h a_0, \dots, h a_{k-1}) \big\rangle : i < n \,\big\} \\
&= \big\{\, \big\langle h\big(\tau_0^{i\boldsymbol{A}}(a_0, \dots, a_{k-1})\big), \dots, h\big(\tau_{l-1}^{i\boldsymbol{A}}(a_0, \dots, a_{k-1})\big) \big\rangle : i < n \,\big\} \\
&= h\,\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}).
\end{aligned}$$

Let $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$.

$$\begin{aligned}
\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'} h_{\mathcal{S}}(F) &= \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\,\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, h(F) \\
&= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\big(\boldsymbol{\tau}^{\boldsymbol{A}} h(F)\big), && \text{by Lem. 4.8} \\
&= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\big(h \boldsymbol{\tau}^{\boldsymbol{A}}(F)\big) \\
&= h_{\mathcal{S}'}\,\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(F)\big), && \text{by Lem. 4.7} \\
&= h_{\mathcal{S}'} \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F).
\end{aligned}$$

□

Lemma 4.10. *Assume $\mathcal{S}$ and $\mathcal{S}'$ are equivalent under the interpretations $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ and $\boldsymbol{\rho} : \mathcal{S}' \to \mathcal{S}$. Then for any $\mathcal{L}$-algebra $\boldsymbol{A}$,*

$$\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'} : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A} \quad \textit{and} \quad \boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}} : \mathrm{Fi}_{\mathcal{S}'}\,\boldsymbol{A} \to \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$$

are lattice isomorphisms and are inverses of one another.

Proof. Recall that by the definition of equivalence we have for every $\boldsymbol{\varphi} \in \mathrm{Fm}^k$ and every $\boldsymbol{\psi} \in \mathrm{Fm}^l$,

$$\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}} \boldsymbol{\rho\tau}(\boldsymbol{\varphi}) \quad \text{and} \quad \boldsymbol{\psi} \dashv\vdash_{\mathcal{S}'} \boldsymbol{\tau\rho}(\boldsymbol{\psi}). \tag{20}$$

From the first equivalence of (20) we get for each $\mathbf{a} \in A^k$

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\{\mathbf{a}\} = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}),$$

and from the second equivalence we get for each $\mathbf{b} \in A^l$

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\{\mathbf{b}\} = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}\boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{b}). \tag{21}$$

More generally, for every $X \subset A^k$ we have

$$\begin{aligned}\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X &= \bigvee\nolimits^{\mathcal{S}}\{\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\{\mathbf{a}\} : \mathbf{a} \in X \,\}\\ &= \bigvee\nolimits^{\mathcal{S}}\{\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}) : \mathbf{a} \in X \,\}\\ &= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}} \bigcup\{\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(\mathbf{a}) : \mathbf{a} \in X \,\}\\ &= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(X).\end{aligned}$$

Thus

$$\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(X) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, X \tag{22}$$

Similarly, for $X \subseteq A^l$, $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}\boldsymbol{\rho}^{\boldsymbol{A}}(X) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, X$.

We have already seen that $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}$ and $\boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}}$ are join-semilattice homomorphisms. Thus it suffices to show that they are inverses of one another. For any $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ we have

$$\begin{aligned}\boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}}\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F) &= \boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}}\, \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}(F)\\ &= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}(F), && \text{by Lem. 4.8}\\ &= \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\, F, && \text{by (22)}\\ &= F.\end{aligned}$$

Similarly, using Lemma 4.8 we get $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}\boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}}(F) = F$ for each $F \in \mathrm{Fi}_{\mathcal{S}'}\, \boldsymbol{A}$. □

We are now prepared for the main theorem on which all the various intrinsic characterizations of algebraizability are based.

Theorem 4.11. *Assume $\mathcal{S}$ and $\mathcal{S}'$ are k- and l-deductive systems. The following are equivalent.*

(i) *$\mathcal{S}$ and $\mathcal{S}'$ are equivalent.*
(ii) *There exists an isomorphism from $\mathbf{Th}\ \mathcal{S}$ to $\mathbf{Th}\ \mathcal{S}'$ that commutes with substitutions, i.e., endomorphisms of the formula algebra.*
(iii) *For each $\mathcal{L}$-algebra $\boldsymbol{A}$ there exists an isomorphism from $\mathbf{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ to $\mathbf{Fi}_{\mathcal{S}'}\, \boldsymbol{A}$ that commutes with endomorphisms.*

Proof. The implication from (i) to (iii) follows from Lems. 4.9 and 4.10, and (ii) is just a special case of (iii). So it remains only to establish the implication from (ii) to (i).

Let $f : \mathbf{Th}\ \mathcal{S} \to \mathbf{Th}\ \mathcal{S}'$ be a lattice isomorphism that commutes with substitutions, i.e., $f\sigma_{\mathcal{S}}(T) = \sigma_{\mathcal{S}'}f(T)$ for each $T \in \mathrm{Th}\ \mathcal{S}$. Consider $f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}) \in \mathrm{Th}\ \mathcal{S}'$, where $\mathbf{p} = \langle p_0, \dots, p_{k-1}\rangle$ (with the p_i distinct) is a fixed k-variable. $\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}$ is finitely generated and hence compact in $\mathbf{Th}\ \mathcal{S}$. Thus $f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\})$ is compact in $\mathbf{Th}\ \mathcal{S}'$ and hence finitely generated (see the remarks in Sec. 3.2). So

$$f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}) = \mathrm{Cn}_{\mathcal{S}'}\{\, \boldsymbol{\eta}^i(p_0, \dots, p_{k-1}, q_0, \dots, q_{m-1}) : i < n \,\},$$

where $\boldsymbol{\eta}^i(\mathbf{p}, \bar{q}) = \boldsymbol{\eta}^i(p_0, \dots, p_{k-1}, q_0, \dots, q_{m-1})$ is an l-formula for each $i < n$ and $q_0, \dots, q_{m-1}$ is a list of all the variables distinct from the p_j that occur in at least one of the $\boldsymbol{\eta}^i$. Let σ be any substitution such that $\sigma p_i = p_i$ and $\sigma q_j = p_0$ for all $i < k$ and $j < m$. Then

$$\begin{aligned} f\big(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}\big) &= f\big(\mathrm{Cn}_{\mathcal{S}}\{\sigma\mathbf{p}\}\big), && \text{since } \sigma\mathbf{p} = \mathbf{p} \\ &= f\big(\sigma_{\mathcal{S}}\,\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}\big), && \text{by Lem. 4.7} \\ &= \sigma_{\mathcal{S}'}\big(f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\})\big), && \text{since } f \text{ commutes with } \sigma \\ &= \sigma_{\mathcal{S}'}\big(\mathrm{Cn}_{\mathcal{S}'}\{\,\boldsymbol{\eta}^i(\mathbf{p}, q_0, \dots, q_{m-1}) : i < n\,\}\big) \\ &= \mathrm{Cn}_{\mathcal{S}'}\big(\sigma\{\,\boldsymbol{\eta}^i(\mathbf{p}, q_0, \dots, q_{m-1}) : i < n\,\}\big), && \text{by Lem. 4.7} \\ &= \mathrm{Cn}_{\mathcal{S}'}\{\,\boldsymbol{\eta}^i(\mathbf{p}, p_0, \dots, p_0) : i < n\,\}. \end{aligned}$$

Let $\boldsymbol{\tau}^i(\mathbf{p}) = \boldsymbol{\eta}^i(\mathbf{p}, p_0, \dots, p_0)$ for $i < n$. Define

$$\boldsymbol{\tau}(\mathbf{p}) = \{\,\boldsymbol{\tau}^i(\mathbf{p}) : i < n\,\}.$$

Let $\boldsymbol{\varphi} \in \mathrm{Fm}^k$ and let σ' be any substitution such that $\sigma'\mathbf{p} = \boldsymbol{\varphi}$.

$$\begin{aligned} f\big(\mathrm{Cn}_{\mathcal{S}}\{\boldsymbol{\varphi}\}\big) &= f\big(\mathrm{Cn}_{\mathcal{S}}\{\sigma'\mathbf{p}\}\big), && \text{since } \sigma'\mathbf{p} = \boldsymbol{\varphi} \\ &= f\big(\sigma'_{\mathcal{S}}\,\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\}\big), && \text{by Lem. 4.7} \\ &= \sigma'_{\mathcal{S}'}\big(f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\})\big), && \text{since } f \text{ commutes with substitution} \\ &= \sigma'_{\mathcal{S}'}\big(\mathrm{Cn}_{\mathcal{S}'}\{\,\boldsymbol{\tau}^i(\mathbf{p}) : i < n\,\}\big) \\ &= \mathrm{Cn}_{\mathcal{S}'}\big(\sigma'\{\,\boldsymbol{\tau}^i(\mathbf{p}) : i < n\,\}\big), && \text{by Lem. 4.7} \\ &= \mathrm{Cn}_{\mathcal{S}'}\{\,\boldsymbol{\tau}^i(\boldsymbol{\varphi}) : i < n\,\} \\ &= \mathrm{Cn}_{\mathcal{S}'}\,\boldsymbol{\tau}(\boldsymbol{\varphi}). \end{aligned}$$

For $\Gamma \subseteq \mathrm{Fm}^k$,

$$f(\mathrm{Cn}_{\mathcal{S}}\,\Gamma) = f\Big(\bigvee_{\boldsymbol{\varphi}\in\Gamma}^{\mathcal{S}} \mathrm{Cn}_{\mathcal{S}}\{\boldsymbol{\varphi}\}\Big) = \bigvee_{\boldsymbol{\varphi}\in\Gamma}^{\mathcal{S}'} f(\mathrm{Cn}_{\mathcal{S}}\{\boldsymbol{\varphi}\}) = \bigvee_{\boldsymbol{\varphi}\in\Gamma}^{\mathcal{S}'} \mathrm{Cn}_{\mathcal{S}'}\{\boldsymbol{\tau}(\boldsymbol{\varphi})\} = \mathrm{Cn}_{\mathcal{S}'}\,\boldsymbol{\tau}(\Gamma).$$

Thus, for $\Gamma \cup \{\boldsymbol{\varphi}\}) \subseteq \mathrm{Fm}^k$,

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}} \boldsymbol{\varphi} \quad &\text{iff} \quad \mathrm{Cn}_{\mathcal{S}}\{\boldsymbol{\varphi}\} \subseteq \mathrm{Cn}_{\mathcal{S}}\,\Gamma \\ &\text{iff} \quad f(\mathrm{Cn}_{\mathcal{S}}\{\boldsymbol{\varphi}\}) \subseteq f(\mathrm{Cn}_{\mathcal{S}}\,\Gamma) \\ &\text{iff} \quad \mathrm{Cn}_{\mathcal{S}'}\,\boldsymbol{\tau}(\boldsymbol{\varphi}) \subseteq \mathrm{Cn}_{\mathcal{S}'}\,\boldsymbol{\tau}(\Gamma) \\ &\text{iff} \quad \boldsymbol{\tau}(\Gamma) \vdash_{\mathcal{S}'} \boldsymbol{\tau}\{\varphi\}. \end{aligned}$$

Thus $\boldsymbol{\tau}$ is an interpretation of $\mathcal{S}$ in $\mathcal{S}'$ and $\boldsymbol{\tau}_{\mathcal{S}'} = f$. Similarly, there is an interpretation $\boldsymbol{\rho}$ of $\mathcal{S}'$ in $\mathcal{S}$ such that $\boldsymbol{\rho}_{\mathcal{S}} = f^{-1}$.

Let $\boldsymbol{\varphi} \in \mathrm{Fm}^l$. Then $\mathrm{Cn}_{\mathcal{S}'}(\boldsymbol{\tau}\boldsymbol{\rho}(\boldsymbol{\varphi})) = f\big(\mathrm{Cn}_{\mathcal{S}}\,\boldsymbol{\rho}(\boldsymbol{\varphi})\big) = ff^{-1}\big(\mathrm{Cn}_{\mathcal{S}'}\{\boldsymbol{\varphi}\}\big) = \mathrm{Cn}_{\mathcal{S}'}\{\boldsymbol{\varphi}\}$, i.e., $\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}'} \boldsymbol{\tau}\boldsymbol{\rho}(\boldsymbol{\varphi})$. Similarly, for $\boldsymbol{\varphi} \in \mathrm{Fm}^k$, $\boldsymbol{\varphi} \dashv\vdash_{\mathcal{S}} \boldsymbol{\rho}\boldsymbol{\tau}(\boldsymbol{\varphi})$. Thus $\mathcal{S}$ and $\mathcal{S}'$ are equivalent under the interpretations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$. □

The following observation about the theorem will be useful in the sequel. Note that in the proof the substitutions σ and σ' can be taken to be surjective. Thus the conclusion of the theorem remains valid when "substitution" and "endomorphism" are replaced respectively by "surjective substitution" and "surjective endomorphism" in parts (i) and (ii).

We note also that the construction of the interpretation $\boldsymbol{\tau}$ is not effective, depending as it does on the compactness of $f(\mathrm{Cn}_{\mathcal{S}}\{\mathbf{p}\})$ in the lattice $\mathbf{Th}\,\mathcal{S}'$; similarly for $\boldsymbol{\rho}$.

4.4. **The Leibniz operator.** Theorem 4.11 has several important corollaries. It can be used to obtain a characterization of algebraizability in terms of the Leibniz operator that in turn can be used to show that, if a k-deductive system $\mathcal{S}$ is algebraizable, then it can be algebraized in essentially only one way.

For any algebra $\boldsymbol{A}$ the formation of the Leibniz congruence defines a natural map

$$\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{P}(A^k) \to \mathrm{Co}\ \boldsymbol{A}, \quad (F \mapsto \boldsymbol{\Omega}^{\boldsymbol{A}} F)$$

from the set of all k-subsets of A to the set of congruences of $\boldsymbol{A}$. Recall that $\boldsymbol{\Omega}^{\boldsymbol{A}} F$ is the largest congruence of $\boldsymbol{A}$ compatible with F (see Sec. 3.2). $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is called the *Leibniz operator* on $\boldsymbol{A}$. Let $\mathcal{S}$ be a k-dimensional deductive system over $\mathcal{L}$, and let $\boldsymbol{A}$ be an algebra of type $\mathcal{L}$. The restriction of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ to the set of $\mathcal{S}$-filters is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ and called the $\mathcal{S}$-*Leibniz operator.* Thus

$$\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Co}\ \boldsymbol{A}.$$

4.4.1. *Protoalgebraic deductive systems.* The Leibniz operator has proved to be a very useful tool in studying the algebraic properties of deductive systems. In particular it has been used to classify deductive systems by virtue of how closely they approximate algebraic systems in some sense. The broadest class of deductive systems that have been identified in this way are the protoalgebraic systems. A k-deductive system is *protoalgebraic* if $\boldsymbol{\Omega}_{\mathcal{S}}$ is monotonic in the sense that, for every algebra $\boldsymbol{A}$ and all $F, F' \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$, $F \subseteq F'$ implies $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} F \subseteq \boldsymbol{\Omega}^{\mathcal{S}} F'$. Protoalgebraic systems share with the algebraic systems $\mathrm{EQ}(\mathsf{K})$ many of the basic properties that determine the latter's characteristic algebraic nature. As a consequence the matrix model classes of protoalgebraic systems have many of the characteristic properties of quasivarieties. One such property is the following *correspondence theorem.* The theory of protoalgebraic deductive systems forms an integral part of abstract algebraic logic and there is an extensive literature on the subject; see [12, 16, 32].

Lemma 4.12 (Correspondence Theorem)**.** *Let $\mathcal{S}$ be a protoalgebraic deductive system, $\boldsymbol{A}, \boldsymbol{B}$ algebras, and $h : \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ a surjective homomorphism. For every $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and $G \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{B}$,*

$$h^{-1}\big(h^{\boldsymbol{B}}_{\mathcal{S}}(F) \vee^{\mathcal{S}} G\big) = F \vee^{\mathcal{S}} h^{-1}(G). \tag{23}$$

Proof. Clearly, F and $h^{-1}(G)$ are both included in $h^{-1}\big(h^{\boldsymbol{B}}_{\mathcal{S}}(F)\vee^{\mathcal{S}}G\big)$, so $h^{-1}\big(h^{\boldsymbol{B}}_{\mathcal{S}}(F)\vee^{\mathcal{S}}G\big) \supseteq F \vee^{\mathcal{S}} h^{-1}(G)$. To get the inclusion in the opposite direction consider any $H \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ such that $h^{-1}(G) \subseteq H$. The relation kernel of h, $h^{-1}(\boldsymbol{\Delta}_B)$, is compatible with $h^{-1}(G)$. Thus by protoalgebraicity $h^{-1}(\boldsymbol{\Delta}_B) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}(G) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, H$. Hence $h^{-1}(\boldsymbol{\Delta}_B)$ is compatible with H, i.e., $h^{-1}h(H) = H$. It follows from this that $h^{-1}h^{\boldsymbol{B}}_{\mathcal{S}}(H) = H$, because in general we have

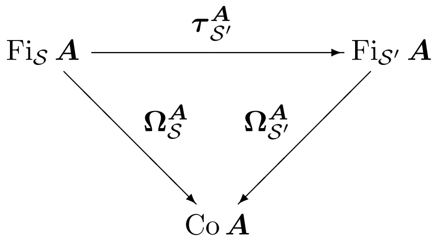

FIGURE 3

$h^{-1}h^{\boldsymbol{B}}_{\mathcal{S}}(H) = \bigcap\{\, K \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A} : H \subseteq K \text{ and } h^{-1}h(K) = K\,\}$. Thus $h^{\boldsymbol{B}}_{\mathcal{S}}(H) = hh^{-1}h^{\boldsymbol{B}}_{\mathcal{S}}(H) = h(H)$. Taking $H = F \vee^{\mathcal{S}} h^{-1}(G)$ we get

$$h^{-1}\big(h^{\boldsymbol{B}}_{\mathcal{S}}(F) \vee^{\mathcal{S}} G\big) \subseteq h^{-1}\big(h^{\boldsymbol{B}}_{\mathcal{S}}(H) \vee^{\mathcal{S}} G\big) = h^{-1}h^{\boldsymbol{B}}_{\mathcal{S}}(H) = h^{-1}h(H) = H.$$

The last of these equalities holds since H is compatible with $h^{-1}(\boldsymbol{\Delta}_B)$. □

Taking the filter G in (23) to be the smallest $\mathcal{S}$-filter on $\boldsymbol{B}$ we get that $h^{-1}h^{\boldsymbol{B}}_{\mathcal{S}}(F) = F$ for every $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ that includes $h^{-1}(G)$. This establishes a one-one inclusion-preserving correspondence between these $\mathcal{S}$-filters and the set of all $\mathcal{S}$-filters of $\boldsymbol{B}$. This is the other more familiar formulation of the correspondence property. It is equivalent to the one in Thm. 4.12. It turns out that the converse of the correspondence theorem holds; this is easy to show. So the correspondence property is equivalent to protoalgebraicity. The following lemma gives another characteristic property of protoalgebraic systems that will be used in the sequel.

Lemma 4.13. *If $\mathcal{S}$ is protoalgebraic, then $\boldsymbol{\Omega}_{\mathcal{S}}$ preserves intersections, i.e., for every set $\{\, F_i : i \in I\,\}$ of $\mathcal{S}$-filters on an algebra $\boldsymbol{A}$ we have $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} \bigcap_{i\in I} F_i = \bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} F_i$.*

Proof. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ is monotonic, $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} \bigcap_{i\in I} F_i \subseteq \bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} F_i$. $\bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} F_i$ is compatible with each F_i and hence with their intersection. Thus by definition of the Leibniz congruence, $\bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} F_i \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} \bigcap_{i\in I} F_i$. □

As in the case of the correspondence theorem, it is easy to show that $\mathcal{S}$ is protoalgebraic iff $\boldsymbol{\Omega}_{\mathcal{S}}$ preserves intersections.

Lemma 4.14. *Assume $\mathcal{S}$ and $\mathcal{S}'$ are k- and l-deductive systems that are equivalent under the interpretations $\boldsymbol{\tau} : \mathcal{S} \to \mathcal{S}'$ and $\boldsymbol{\rho} : \mathcal{S}' \to \mathcal{S}$. Then for any $\mathcal{L}$-algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} = \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'} \circ \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}'}$, i.e., the diagram in Figure 3 commutes.*

Proof. We first show that for every $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ and $\mathbf{a} \in A^l$,

$$\mathbf{a} \in \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F) \quad \text{iff} \quad \boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) \subseteq F. \tag{24}$$

If $\mathbf{a} \in \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$, then $\boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) \subseteq \boldsymbol{\rho}^{\boldsymbol{A}}\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F) \subseteq \boldsymbol{\rho}^{\boldsymbol{A}}_{\mathcal{S}}\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F) = F$. Conversely, if $\boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) \subseteq F$, then $\boldsymbol{\tau}^{\boldsymbol{A}}\boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$. But $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\, \boldsymbol{\tau}^{\boldsymbol{A}}\boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}'}\{\mathbf{a}\}$ by (21). So (24) holds.

To prove the lemma it suffices to prove that, for every $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$ and every $\Theta \in \mathrm{Co}\, \boldsymbol{A}$, Θ is compatible with F iff Θ is compatible with $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$. Furthermore, by symmetry, it

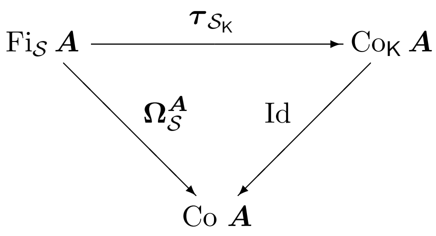

FIGURE 4

suffices to prove only one of the two implications. So assume $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and $\Theta \in \mathrm{Co}\,\boldsymbol{A}$ is compatible with F.

Let $\boldsymbol{\rho}(\mathbf{p}) = \{\, \boldsymbol{\rho}^i(\mathbf{p}) : i < n \,\}$, where $\mathbf{p} = \langle p_0, \dots, p_{l-1} \rangle$ and

$$\boldsymbol{\rho}^i(p_0, \dots, p_{l-1}) = \big\langle \rho^i_0(p_0, \dots, p_{l-1}), \dots, \rho^i_{k-1}(p_0, \dots, p_{l-1}) \big\rangle \quad \text{for each } i < n.$$

Let $\mathbf{a} = \langle a_0, \dots, a_{l-1} \rangle \in \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$. Assume $\mathbf{b} = \langle b_0, \dots, b_{l-1} \rangle \in A^l$ such that $\mathbf{a}\,\Theta^l\,\mathbf{b}$, i.e., $a_i \,\Theta\, b_i$ for all $i < l$. Then

$$\begin{aligned} \mathbf{a} \in \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F) \quad &\text{iff} \quad \boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{a}) \subseteq F \\ &\text{iff} \quad \boldsymbol{\rho}^{i\boldsymbol{A}}(a_0, \dots, a_{l-1}) \in F \text{ for all } i < n, \end{aligned}$$

which, because $\rho^{i\boldsymbol{A}}_j(a_0, \dots, a_{l-1}) \,\Theta\, \rho^{i\boldsymbol{A}}_j(b_0, \dots, b_{l-1})$ for all $j < k$ and Θ is compatible with F,

$$\begin{aligned} &\text{implies} \quad \boldsymbol{\rho}^{i\boldsymbol{A}}(b_0, \dots, b_{l-1}) \in F \text{ for all } i < n, \\ &\text{iff} \quad \boldsymbol{\rho}^{\boldsymbol{A}}(\mathbf{b}) \subseteq F \\ &\text{iff} \quad \mathbf{b} \in \boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F). \end{aligned}$$

So Θ is compatible with $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$. □

Theorem 4.15. *Let $\mathcal{S}$ be a k-deductive system and let K be a quasivariety.*

(i) *Assume $\mathcal{S}$ is an algebraizable k-deductive system and K is its equivalent quasivariety. Let $\boldsymbol{\tau}$ be any invertible interpretation of $\mathcal{S}$ in* $\mathrm{EQ}(\mathsf{K})$. *Then for every algebra $\boldsymbol{A}$ of type $\mathcal{L}$,*

$$\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})} = \boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} : \mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A} \cong \mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A}.$$

(ii) *Conversely, if for every algebra $\boldsymbol{A}$*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} : \mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A} \cong \mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A},$$

then $\mathcal{S}$ is algebraizable with equivalent quasivariety K.

Proof. (i). Recall that $\mathbf{Fi}_{\mathrm{EQ}(\mathsf{K})}\,\boldsymbol{A} = \mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A}$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}\,\Theta = \Theta$ for every $\Theta \in \mathrm{Co}\,\boldsymbol{A}$; so $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}$ is the identity function. Thus the commutativity of the diagram in Figure 3 reduces to the commutativity of the diagram in Figure 4.

(ii). Assume $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ is an isomorphism between $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and $\mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A} = \mathbf{Fi}_{\mathrm{EQ}(\mathsf{K})}\,\boldsymbol{A}$ for every algebra $\boldsymbol{A}$. To show $\mathcal{S}$ is algebraizable with equivalent quasivariety K it suffices by

Thm. 4.11 and the remarks immediately following it to show that $\mathbf{\Omega}^{\boldsymbol{A}}$ commutes with surjective endomorphisms. Let $h : \boldsymbol{A} \to \boldsymbol{A}$ and $F \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$. We must show that

$$h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F = \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{\boldsymbol{A}}_{\mathcal{S}}(F). \tag{25}$$

Since the $\mathcal{S}$-Leibniz operator is an isomorphism it is in particular monotonic, i.e., $G \subseteq G'$ implies $\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, G \subseteq \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, G'$ for all $G, G' \in \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A}$. So $\mathcal{S}$ is protoalgebraic and by the correspondence theorem (Lem. 4.12) we have $h^{-1}h^{\boldsymbol{A}}_{\mathcal{S}}(F) = F \vee^{\mathcal{S}} h^{-1}(F_0)$, where F_0 is the smallest $\mathcal{S}$-filter on $\boldsymbol{A}$. Since $\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ is an isomorphism,

$$\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}h^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F \vee^{\mathrm{EQ}(\mathsf{K})} \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}(F_0).$$

$$\begin{aligned}
h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}h^{\boldsymbol{A}}_{\mathcal{S}}(F) &= h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}h^{-1}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{\boldsymbol{A}}_{\mathcal{S}}(F), && \text{by Lem. 3.6}\\
&= h\, h^{-1}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{\boldsymbol{A}}_{\mathcal{S}}(F), && \text{since } h^{-1}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h(F) \text{ is a K-congruence that includes the relation kernel of } h\\
&= \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{\boldsymbol{A}}_{\mathcal{S}}(F), && \text{since } h \text{ is surjective.}
\end{aligned}$$

Similarly, $h_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}(F_0) = \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F_0$ which is the smallest K-congruence of $\boldsymbol{A}$ since $\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ is an isomorphism. Thus we have

$$\begin{aligned}
\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{\boldsymbol{A}}_{\mathcal{S}}(F) &= h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\big(\mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F\ \vee^{\mathrm{EQ}(\mathsf{K})}\ \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}(F_0)\big)\\
&= h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F\ \vee^{\mathrm{EQ}(\mathsf{K})}\ h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, h^{-1}(F_0), && \text{since } h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})} \text{ is a join-semilattice homomorphism}\\
&= h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F\ \vee^{\mathrm{EQ}(\mathsf{K})}\ \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F_0\\
&= h^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F, && \text{since } \mathbf{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}\, F_0 \text{ is the smallest K-congruence.}
\end{aligned}$$

This establishes (25). □

Note that in order to establish that $\mathcal{S}$ is algebraizable with equivalent quasivariety K it is sufficient to show that $\mathbf{\Omega}_{\mathcal{S}}$ is an isomorphism between the lattices of $\mathcal{S}$-theories and of K-congruences on the formula algebra, i.e.,

$$\mathbf{\Omega}_{\mathcal{S}} :\ \mathbf{Th}_{\mathcal{S}} \cong \mathbf{Co}_{\mathsf{K}}\, \mathbf{Fm}.$$

This is an immediate consequence of Thm. 4.11. The class of underlying algebras of class M of matrices is denoted by $\mathrm{Alg}\,\mathsf{M}$, formally, $\mathrm{Alg}\,\mathsf{M} = \{\, \boldsymbol{A} : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\,\}$.

Corollary 4.16. *Every algebraizable deductive system has a unique equivalent quasivariety, i.e., if $\mathcal{S}$ is equivalent to $\mathrm{EQ}(\mathsf{K})$ and $\mathrm{EQ}(\mathsf{K}')$ for quasivarieties K and K', then $\mathsf{K} = \mathsf{K}'$. Moreover, the unique equivalent quasivariety is $\mathrm{Alg}\,\mathsf{M}^*$, where M is the class of all matrix-models of $\mathcal{S}$.*

Proof. If $\mathcal{S}$ is equivalent to both $\mathrm{EQ}(\mathsf{K})$ and $\mathrm{EQ}(\mathsf{K}')$ then $\mathrm{EQ}(\mathsf{K})$ and $\mathrm{EQ}(\mathsf{K}')$ are equivalent. Let $\boldsymbol{\tau}$ be an invertible interpretation of $\mathrm{EQ}(\mathsf{K})$ in $\mathrm{EQ}(\mathsf{K}')$. Then by Thm. 4.15, $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K}')} = \mathbf{\Omega}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}$ for every algebra $\boldsymbol{A}$. Thus $\boldsymbol{\tau}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K}')}$ must be the identity function since its domain $\mathrm{Fi}_{\mathrm{EQ}(\mathsf{K})}\, \boldsymbol{A}$ coincides with $\mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}$. So $\mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A} = \mathrm{Co}_{\mathsf{K}'}\, \boldsymbol{A}$ for every algebra $\boldsymbol{A}$, and hence $\mathsf{K} = \mathsf{K}'$.

For the last part of the corollary observe that from the isomorphism $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} : \mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A} \cong \mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A}$ we have $\boldsymbol{A} \in \mathsf{M}^*$ iff there exists an $F \in \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}} = \boldsymbol{\Delta}_A$ iff $\boldsymbol{\Delta}_A \in \mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$ iff $\boldsymbol{A} \in \mathsf{K}$. □

4.4.2. *Example.* The 3-valued Łukasiewicz logic $Ł_3$ is algebraizable and its equivalent semantics is the variety of Wajsburg algebras (Sec. 4.3.1). The 3-valued paraconsistent logic J_3 has the same algebraic semantics since it is equivalent to $Ł_3$ (Sec. 4.1.2).

Corollary 4.17. *Let $\mathcal{S}$ be an algebraizable k-deductive system. Then for each algebra $\boldsymbol{A}$ the $\mathcal{S}$-Leibniz operator $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} : \mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Co}\,\boldsymbol{A}$ is injective and continuous, i.e., for any directed set $\{\, F_i : i \in I \,\}$ of $\mathcal{S}$-filters of $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} \bigcup_{i\in I} F_i = \bigcup_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i$.*

Proof. $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} : \mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A} \cong \mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A}$ is a lattice isomorphism, and hence injective. Since $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ is algebraic and the collection $\{\, F_i : i \in I \,\}$ is directed, $\bigcup_{i\in I} F_i = \bigvee^{\mathcal{S}}_{i\in I} F_i$. Similarly, since $\mathbf{Co}_{\mathsf{K}}\,\boldsymbol{A}$ is algebraic, and $\{\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i : i \in I\}$ is also a directed set, we also have $\bigcup_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i = \bigvee^{\mathrm{EQ}(\mathsf{K})}_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i$. Thus $\boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}}\big(\bigcup_{i\in I} F_i\big) = \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}}\big(\bigvee^{\mathcal{S}}_{i\in I} F_i\big) = \bigvee^{\mathrm{EQ}(\mathsf{K})}_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i = \bigcup_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}^{\boldsymbol{A}} F_i$. □

It turns out that the following converse of Cor. 4.17 holds, and this together with Cor. 4.17 gives another characterization of algebraizability that, in contrast to Thms. 4.5 and 4.15 does not require a priori knowledge of either the equivalent quasivariety or the inverse interpretations.

Theorem 4.18. *Let $\mathcal{S}$ be a k-deductive system. If the $\mathcal{S}$-Leibniz operator $\boldsymbol{\Omega}_{\mathcal{S}} : \mathrm{Th}_{\mathcal{S}}\,\boldsymbol{A} \to \mathrm{Co}\,\mathbf{Fm}$ is injective and continuous, then $\mathcal{S}$ is algebraizable.*

Proof. Since $\boldsymbol{\Omega}_{\mathcal{S}}$ is injective and monotonic (monotonicity follows from continutity), its image $\boldsymbol{\Omega}_{\mathcal{S}}(\mathrm{Th}\,\mathcal{S})$ forms a complete lattice under set-theoretical inclusion that is isomorphic to $\mathbf{Th}\,\mathcal{S}$. Thus by Thm. 4.15(ii) and the remark immediately following it we see that in order to establish the theorem it suffices to show that $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big) = \mathrm{Co}_{\mathsf{K}}\,\mathbf{Fm}$ $(= \mathrm{Th}\,\mathrm{EQ}(\mathsf{K}))$ for some quasivariety K. Moreover, in view of Thm. 3.2, it suffices to show that $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ is closed under intersection, directed unions, and inverse surjective substitutions.

For each $\Theta \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ let $T_\Theta = \bigcap\{\, S \in \mathrm{Th}\,\mathcal{S} : \boldsymbol{\Omega}_{\mathcal{S}}\,S = \Theta \,\}$. $\mathcal{S}$ is protoalgebraic (since $\boldsymbol{\Omega}_{\mathcal{S}}$ is monotonic). Thus, by Thm. 4.13, $\boldsymbol{\Omega}_{\mathcal{S}}\,T_\Theta = \bigcap\{\, \boldsymbol{\Omega}_{\mathcal{S}}\,S : \boldsymbol{\Omega}_{\mathcal{S}}\,S = \Theta \,\} = \Theta$. Thus T_Θ is the smallest $\mathcal{S}$-theory whose Leibniz congruence is Θ. Note that, if $\Theta, \Theta' \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ and $\Theta \subseteq \Theta'$, then $T_\Theta \subseteq T_{\Theta'}$ since $\boldsymbol{\Omega}_{\mathcal{S}}(T_\Theta \cap T_{\Theta'}) = \boldsymbol{\Omega}_{\mathcal{S}}\,T_\Theta \cap \boldsymbol{\Omega}_{\mathcal{S}}\,T_{\Theta'} = \Theta \cap \Theta' = \Theta$, and thus $T_\Theta \subseteq T_\Theta \cap T_{\Theta'} \subseteq \Theta'$. Consequently, if $\{\, \Theta_i : i \in I \,\}$ is an upward-directed subset of $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$, then $\{\, T_{\Theta'} : i \in I \,\}$ is also upward-directed.

Let $\{\, \Theta_i : i \in I \,\}$ be an arbitrary subset of $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$. By Thm. 4.13 we have $\bigcap_{i\in I} \Theta_i = \bigcap_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}\,T_{\Theta_i} = \boldsymbol{\Omega}_{\mathcal{S}} \bigcap_{i\in I} T_{\Theta_i} \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$. So $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ is closed under intersection.

Suppose now that $\{\, \Theta_i : i \in I \,\}$ is upward-directed, then as observed above $\{\, T_{\Theta'} : i \in I \,\}$ is also upward-directed. Thus by the premise that $\boldsymbol{\Omega}_{\mathcal{S}}$ is continuous we have $\bigcup_{i\in I} \Theta_i = \bigcup_{i\in I} \boldsymbol{\Omega}_{\mathcal{S}}\,T_{\Theta_i} = \boldsymbol{\Omega}_{\mathcal{S}} \bigcup_{i\in I} T_{\Theta_i} \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$. So $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ is closed under directed unions.

Finally, for each $\Theta \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ and each surjective substitution σ, by Thm. 3.6, $\sigma^{-1}(\Theta) = \sigma^{-1}\,\boldsymbol{\Omega}_{\mathcal{S}}\,T_\Theta = \boldsymbol{\Omega}_{\mathcal{S}}\,\sigma^{-1}(T_\Theta) \in \boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$. So $\boldsymbol{\Omega}_{\mathcal{S}}\big(\mathrm{Th}\,\mathcal{S}\big)$ is closed under inverse surjective substitutions. □

This theorem was proved for 1-deductive systems in [13]. See Note 4.3 for a discussion of other notions of algebraizability and their connection with the Leibniz operator.

4.4.3. *Examples.* Cor. 4.17 is useful for showing that a k-deductive system $\mathcal{S}$ fails to be algebraizable. One has only to find some algebra $\boldsymbol{A}$ for which the mapping $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}} : \mathrm{Fi}_{\mathcal{S}}\, \boldsymbol{A} \to \mathrm{Co}\, \boldsymbol{A}$ either fails to be injective or order-preserving. If there is a small finite algebra with this property, this can be a very practical way of verifying nonalgebraizability. We consider several examples of this kind.

The system $\mathrm{S5}^{\mathrm{C}}$ of Sec. 2.2.5 is not algebraizable. Indeed, the 4-element reduced filtered monadic algebra $\mathfrak{A} = \langle \boldsymbol{A}, F\rangle$ presented there has four $\mathrm{S5}^{\mathrm{C}}$-filters: $\{\top^{\boldsymbol{A}}\}$, $\{\top^{\boldsymbol{A}}, a\}$, $\{\top^{\boldsymbol{A}}, b\}$, and $\{\top^{\boldsymbol{A}}, a, b, \bot^{\boldsymbol{A}}\}$. But the underlying algebra $\boldsymbol{A}$ is simple and thus has only two congruences $\boldsymbol{\Delta}_A$ and $\boldsymbol{\nabla}_A$; so $\boldsymbol{\Omega}^{\boldsymbol{A}}$ cannot be one-one. For a proof by this method that the paraconsistent logic C_1 fails to be algebraizable see [96].

Let IPC^* be the $(\wedge, \vee, \neg, \bot, \top)$-fragment of IPC. IPC^* is not algebraizable. Let $\boldsymbol{A}$ be the 4-element chain $\bot^{\boldsymbol{A}} < a < b < \top^{\boldsymbol{A}}$ with $\neg^{\boldsymbol{A}}\top^{\boldsymbol{A}} = \neg^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}} a = \bot^{\boldsymbol{A}}$ and $\neg^{\boldsymbol{A}}\bot^{\boldsymbol{A}} = \top^{\boldsymbol{A}}$. Then $\mathfrak{A} = \langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle$ is an IPC^*-matrix. Let $F_1 = \{\top^{\boldsymbol{A}}\}$ and $F_2 = \{\top^{\boldsymbol{A}}, b\}$. F_1 and F_2 are IPC^*-filters and $a\ \boldsymbol{\Omega}^{\boldsymbol{A}} F_1\, b$, but it is not the case that $a\ \boldsymbol{\Omega}^{\boldsymbol{A}} F_2\, b$. So $F_1 \subseteq F_2$ but $\boldsymbol{\Omega}^{\boldsymbol{A}} F_1 \not\subseteq \boldsymbol{\Omega}^{\boldsymbol{A}} F_2$.

The $(*, \to)$-fragment $\mathrm{LL}^{*,\to}$ of linear logic is not algebraizable. Let $\langle \mathbb{Z}, +, -, 0\rangle$ be the additive group of integers and let $\boldsymbol{A} = \langle \mathbb{Z}, *^{\boldsymbol{A}}, \to^{\boldsymbol{A}}\rangle$, where $m *^{\boldsymbol{A}} n = m + n$ and $m \to^{\boldsymbol{A}} n = n - m$. $F_1 = \{\, \langle m, n\rangle : m \le n \,\}$ and $F_2 = \boldsymbol{\Delta}_{\mathbb{Z}} = \{\, \langle n, n\rangle : n \in \mathbb{Z} \,\}$ are distinct $\mathrm{LL}^{*,\to}$-filters of $\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}} F_1 = \boldsymbol{\Omega}^{\boldsymbol{A}} F_2 = \boldsymbol{\Delta}_{\mathbb{Z}}$.

4.5. **The deduction-detachment theorem and equivalence.** The property of possessing the DDT is respected by the relation of equivalence between deductive systems. Before showing this we prove that, if a deductive system has the DDT, then it also has a deduction-detachment set for any finite set of premises.

Let $\mathcal{S}$ be a k-dimensional deductive system with (finite) deduction-detachment set $E(\mathbf{p}, \mathbf{q}) \subseteq \mathrm{Fm}^k$; here $\mathbf{p} = \langle p_0, \dots, p_{k-1}\rangle$ and $\mathbf{q} = \langle q_0, \dots, q_{k-1}\rangle$. We define for each $n \ge 1$ a finite set of k-formulas $E_n(\mathbf{p}_0, \dots, \mathbf{p}_{n-1}, \mathbf{q})$ in the $(n+1) \times k$ distinct 1-variables

$$\mathbf{p}_i = \langle p_{i,0}, \dots, p_{i,(k-1)}\rangle, \quad i < n, \qquad \text{and} \qquad \mathbf{q} = \langle q_0, \dots, q_{k-1}\rangle$$

as follows:

$$E_1(\mathbf{p}_0, \mathbf{q}) = E(\mathbf{p}_0, \mathbf{q}), \tag{26}$$

and for $n \ge 1$,

$$E_{n+1}(\mathbf{p}_0, \dots, \mathbf{p}_n, \mathbf{q}) = \bigcup \{ E(\mathbf{p}_0, \boldsymbol{\eta}) \ : \ \boldsymbol{\eta} \in E_n(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}, \mathbf{q}) \}. \tag{27}$$

Observe that if $E(\mathbf{p}, \mathbf{q})$ consists of a single formula, then so does $E_n(\mathbf{p}, \mathbf{q})$, $n \ge 1$. In particular, in the familiar case of classical (or intuitionistic) logic, where $k = 1$ and $E(p, q) = \{p \to q\}$, we get

$$E_n(p_0, \dots, p_{n-1}, q) = \{p_0 \to (p_1 \to (\dots \to (p_{n-1} \to q) \dots))\}.$$

Lemma 4.19. *Let $\mathcal{S}$ be a k-dimensional deductive system that has a deduction-detachment set $E(\mathbf{p}, \mathbf{q})$. For $\Gamma \subseteq \mathrm{Fm}^k$, $\Delta = \{\boldsymbol{\delta}_i \ : \ i < n\} \subseteq \mathrm{Fm}^k$, $n \ge 0$, and $\boldsymbol{\psi} \in \mathrm{Fm}^k$ we have*

$$\Gamma, \Delta \vdash_{\mathcal{S}} \boldsymbol{\psi} \qquad \text{iff} \qquad \Gamma \vdash_{\mathcal{S}} E_n(\boldsymbol{\delta}_0, \dots, \boldsymbol{\delta}_{n-1}, \boldsymbol{\psi}). \tag{28}$$

Proof. The proof is by induction on n. For $n = 1$ the claim follows by (5) and the definition of a deduction-detachment set. Now assume the claim holds for $n \geq 1$. Let $\Gamma \subseteq \mathrm{Fm}^k$, $\Delta = \{\boldsymbol{\delta}_i : i < n+1\} \subseteq \mathrm{Fm}^k$, and $\boldsymbol{\psi} \in \mathbf{Fm}^k$. Then

$$\begin{aligned}\Gamma, \Delta \vdash_{\mathcal{S}} \boldsymbol{\psi} \quad &\text{iff} \quad \Gamma, \boldsymbol{\delta}_0, \{\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n\} \vdash_{\mathcal{S}} \boldsymbol{\psi} \\ &\text{iff} \quad \Gamma, \boldsymbol{\delta}_0 \vdash_{\mathcal{S}} E_n(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n, \boldsymbol{\psi}) \\ &\text{iff} \quad \Gamma, \boldsymbol{\delta}_0 \vdash_{\mathcal{S}} \boldsymbol{\eta}, \quad \text{for} \quad \boldsymbol{\eta} \in E_n(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n, \boldsymbol{\psi}) \\ &\text{iff} \quad \Gamma \vdash_{\mathcal{S}} E(\boldsymbol{\delta}_0, \boldsymbol{\eta}), \quad \text{for} \quad \boldsymbol{\eta} \in E_n(\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n, \boldsymbol{\psi}) \\ &\text{iff} \quad \Gamma \vdash_{\mathcal{S}} E_{n+1}(\boldsymbol{\delta}_0, \dots, \boldsymbol{\delta}_n, \boldsymbol{\psi}).\end{aligned}$$

□

Theorem 4.20. *Let $\mathcal{S}$ and $\mathcal{S}'$ be equivalent deductive systems. Then $\mathcal{S}$ has the DDT iff $\mathcal{S}'$ has the DDT.*

Proof. Assume that $\mathcal{S}$ is k-dimensional and $\mathcal{S}'$ is l-dimensional. Let $\boldsymbol{\tau}(\mathbf{p}) = \{\boldsymbol{\tau}^i(\mathbf{p}) : i < n\}$ be a (k,l)-translation and $\boldsymbol{\rho}(\mathbf{p}') = \{\boldsymbol{\rho}^j(\mathbf{p}') : j < m\}$ an (l,k)-translation witnessing the equivalence of $\mathcal{S}$ and $\mathcal{S}'$. Here $\boldsymbol{\tau}^i(\mathbf{p}) = \langle \tau^i_0(\mathbf{p}), \dots, \tau^i_{l-1}(\mathbf{p})\rangle$, with $\mathbf{p} = \langle p_0, \dots, p_{k-1}\rangle$, is an l-formula, and $\boldsymbol{\rho}^j(\mathbf{p}') = \langle \rho^j_0(\mathbf{p}'), \dots, \rho^j_{k-1}(\mathbf{p}')\rangle$, with $\mathbf{p}' = \langle p'_0, \dots, p'_{l-1}\rangle$, is a k-formula. Let $E(\mathbf{p}, \mathbf{q})$ be a deduction-detachment set for $\mathcal{S}$. Now, for $\Gamma \subseteq \mathrm{Fm}^l$ and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathrm{Fm}^l$, we have

$$\begin{aligned}\Gamma, \boldsymbol{\varphi} \vdash_{\mathcal{S}'} \boldsymbol{\psi} \quad &\text{iff} \quad \boldsymbol{\rho}(\Gamma), \boldsymbol{\rho}(\boldsymbol{\varphi}) \vdash_{\mathcal{S}} \boldsymbol{\rho}(\boldsymbol{\psi}) \\ &\text{iff} \quad \boldsymbol{\rho}(\Gamma), \{\boldsymbol{\rho}^j(\boldsymbol{\varphi}) : j < m\} \vdash_{\mathcal{S}} \boldsymbol{\rho}^j(\boldsymbol{\psi}), \quad j < m \qquad (29) \\ &\text{iff} \quad \boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}} E_m\big(\boldsymbol{\rho}^0(\boldsymbol{\varphi}), \dots, \boldsymbol{\rho}^{m-1}(\boldsymbol{\varphi}), \boldsymbol{\rho}^j(\boldsymbol{\psi})\big), \quad j < m \\ &\text{iff} \quad \boldsymbol{\rho}(\Gamma) \vdash_{\mathcal{S}} \bigcup_{j<m} E_m\big(\boldsymbol{\rho}^0(\boldsymbol{\varphi}), \dots, \boldsymbol{\rho}^{m-1}(\boldsymbol{\varphi}), \boldsymbol{\rho}^j(\boldsymbol{\psi})\big) \\ &\text{iff} \quad \boldsymbol{\tau}\big(\boldsymbol{\rho}(\Gamma)\big) \vdash_{\mathcal{S}'} \boldsymbol{\tau}\Big(\bigcup_{j<m} E_m\big(\boldsymbol{\rho}^0(\boldsymbol{\varphi}), \dots, \boldsymbol{\rho}^{m-1}(\boldsymbol{\varphi}), \boldsymbol{\rho}^j(\boldsymbol{\psi})\big)\Big). \qquad (30)\end{aligned}$$

If we let

$$E'(\mathbf{p}', \mathbf{q}') = \boldsymbol{\tau}\Big(\bigcup_{j<m} E_m\big(\boldsymbol{\rho}^0(\mathbf{p}'), \dots, \boldsymbol{\rho}^{m-1}(\mathbf{p}'), \boldsymbol{\rho}^j(\mathbf{q}')\big)\Big),$$

where $\mathbf{p}' = \langle p_0, \dots, p_{l-1})$ and $\mathbf{q}' = \langle q_0, \dots, q_{l-1}\rangle$, and we observe that $\Gamma \dashv\vdash_{\mathcal{S}'} \boldsymbol{\tau}\big(\boldsymbol{\rho}(\Gamma)\big)$, then (30) becomes

$$\Gamma \vdash_{\mathcal{S}'} E'(\boldsymbol{\varphi}, \boldsymbol{\psi}).$$

We have thus shown that

$$\Gamma, \boldsymbol{\varphi} \vdash_{\mathcal{S}'} \boldsymbol{\psi} \qquad \text{iff} \qquad \Gamma \vdash_{\mathcal{S}'} E'(\boldsymbol{\varphi}, \boldsymbol{\psi}),$$

i.e., $E'(\mathbf{p}', \mathbf{q}')$ is a deduction-detachment set for $\mathcal{S}'$. Thus, if $\mathcal{S}$ has the DDT, then so does $\mathcal{S}'$; by symmetry the converse also holds. □

4.6. **Notes.**

Note 4.1. The distinction we make between the syntactical notion of translation and the deductive notion of interpretation is not commonly made in the literature where the terms are used more-or-less interchangeably. The interpretations most often considered in the classical literature are between deductive systems with different language types and are based on definitions of the connectives of the source system in terms of those of target system. The notion of equivalence that this gives is often referred to as *definitional* equivalence. In abstract algebraic logic, in contrast, the language type is fixed and it is the consequence relation of the two systems that is the focus of the interpretation; this leads to a different notion of equivalence, that of *deductive* equivalence defined in Def. 4.1(ii). The classic example of definitional equivalence is that between CPC and the other formalization of the classical propositional calculus obtained by taking $\rightarrow$ and $\neg$ as the only primitive connectives. Related to this is the well known definitional equivalence between the various equational theories of Boolean algebras based on different choices of the fundamental operations; this is usually expressed as the definitional equivalence of different varieties, for example between the (standard) variety of Boolean algebras and the variety of Boolean rings. Another example of definitional interpretation, but not equivalence, is Gödel's well known translation of IPC in the modal system S4. The intuitionistic connectives are defined in terms of the modal connectives as follows. $p \rightarrow q \mapsto \Box(\Box p \rightarrow \Box q)$, $p \wedge q \mapsto \Box p \wedge \Box q$, $p \vee q \mapsto \Box p \vee \Box q$, $\neg p \mapsto \Box \neg \Box q$. This is not a deductive interpretation in the sense of Def. 4.1(i). Note however that if we define new connectives $\rightarrow^*$, $\wedge^*$, $\vee^*$, and $\neg^*$ in S4 by taking $p \rightarrow^* q = \Box(\Box p \rightarrow \Box q)$, $p \wedge^* q = \Box p \wedge \Box q$, etc., then the fragment of the resulting definitional extension of S4 is deductively equivalent to IPC under the identity interpretations. An example in the classical literature of a deductive interpretation in the sense of Def. 4.1(i) is Glivenko's interpretation of CPC in IPC that is given by $\tau(p) = \neg\neg p$. Another example in the realm of algebra is the deductive equivalence of the variety of semilattices and the class of semilattices viewed as partially ordered algebras; this example is discussed in Sec. 4.2.2.

The equivalence of the paraconsistent logic J_3 and the 3-valued Łukasiewicz logic $Ł_3$ (Sec. 4.1.2) is a good illustration of the difference between definitional and deductive equivalence. The classical reference to definitional equivalence in first-order logic is [133]. See also [69]. A detailed treatment of a notion of interpretation that combines those of both definitional and deductive interpretations in the context of abstract algebraic logic can be found in [76].

Note 4.2. The first systematic treatment of a large class of algebraizable logics can be in found Rasiowa's book [120]. She introduces the notion of a "standard system of implicative extensional propositional calculus", a SIC logic, for short, and associates a class of algebras with each of them by a generalization of the classical Lindenbaum-Tarski process. The class of SIC logics is a wide one and includes almost all the examples mentioned in Sec. 4.2.1. Every SIC logic is algebraizable but not vice versa. SIC logics correspond roughly to the class of *regularly algebraizable deductive systems.* See [13, Theorem 5.2] for more details on the comparison.

A number of more general notions of algebraizability have also been proposed; so far they have been considered only for 1-deductive systems, but their extension to the multi-dimensional case would present no essential problems. The first was introduced and investigated by Herrmann [81, 82]. It differs from Def. 4.4 mainly in that it does not require that the interpretations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ be finite. For this and other reasons it is becoming standard practice to refer to deductive systems algebraizable in this sense as *algebraizable* and those in the sense of Def. 4.4 as *finitely algebraizable*. Since we deal exclusively with the latter notion in this paper we do not add this qualification.

The two notions are very similar in their formal properties and natural examples of logics that are algebraizable in the more general sense but not in the sense of Def. 4.4 are known ([81, 97]). One major difference however is that an equivalent algebraic semantics in the general sense need not be a quasivariety, or even form an elementary class. In fact, it is an elementary class, and hence a quasivariety, just in case the deductive system is algebraizable in the sense of Def. 4.4. A still more general notion, *weak algebraizability*, has been considered [37]. It cannot be easily described however by means of interpretations.

In [121] and independently in [119] the notion of algebraizability in the sense of Def. 4.4 is extended to sequent calculi, i.e., $\boldsymbol{\omega}$-deductive systems (see Note 3.2).

An even more general theory of algebraizability for 1-deductive systems is developed in [59]. It has its origin in the study of the algebraic nature of the disjunction-conjunction fragment of CPC carried out in [65]. This deductive system is the paradigm for an important class of systems that have clear algebraic content but are not protoalgebraic (Sec. 4.4.1) and hence not algebraizable in any of the above senses. The work in [59] is closely related to the abstract algebraic logic of sequent calculi. In fact, one can view a 1-deductive system $\mathcal{S}$ as being algebraizable in the sense of [59] just in case the class of full generalized matrix models of $\mathcal{S}$ is the class of models of a Gentzen system $\mathcal{G}$ for $\mathcal{S}$ (see Note 3.2) and $\mathcal{G}$ is algebraizable in the sense of [121].

Note 4.3. Two of the more general notions of algebraizability for 1-deductive systems that were mentioned in Note 4.2 also have natural characterizations in terms of the $\mathcal{S}$-Leibniz operator. A 1-deductive system $\mathcal{S}$ is by definition weakly algebraizable in the sense of [37] if the $\mathcal{S}$-Leibniz operator is injective. $\mathcal{S}$ is algebraizable in the sense of [81, 82] iff it is weakly algebraizable and equivalential (Sec. 4.3.2). Finally, we mention that $\mathcal{S}$ is algebraizable in the sense of Def. 4.4 iff it is weakly algebraizable and finitely equivalential.

Note that, in contrast to Thm. 4.15, the characterization of algebraizability that Cor. 4.17 and Thm. 4.18 together provide does not depend on a priori knowledge of the equivalent algebraic semantics of $\mathcal{S}$. For further investigations of the Leibniz operator see [57].

Algebraizability in the sense of [59], which is based on the existence of a fully adequate Gentzen system, can be characterized in terms of an important generalization of Thm 4.18 ([59, Theorem 2.30]): for every deductive system $\mathcal{S}$ and algebra $\boldsymbol{A}$, a natural generalization of the Leibniz operator, the *Tarski operator*, establishes an order-preserving injection from the partially ordered set of *full* $\mathcal{S}$*-filters* $\boldsymbol{A}$ (i.e., the filters of the full generalized models with underlying algebra $\boldsymbol{A}$) into the set of all congruences of $\boldsymbol{A}$. See [60, 61]. There is second generalization of the Leibniz congruence, called the *Suszko congruence*. It is closely related to the Tarski congruence; see [32, 33].

A deductive system $\mathcal{S}$ is *strongly algebraizable* if it is algebraizable and its equivalent quasivariety is a variety V. The question whether an algebraizable deductive system is strongly algebraizable is an important one and was discussed briefly in Notes 2.6 and 3.5. Many of the familiar algebraizable deductive systems are in fact strongly algebraizable; in particular CPC, IPC, K, and $\mathrm{S5}^{\mathrm{G}}$ fall in this category. The first-order predicate calculus PR_ω is also strongly algebraizable. If $\mathcal{S}$ is strongly algebraizable with equivalent variety V, then for every algebra $\boldsymbol{A} \in \mathsf{V}$, $\mathrm{Co}_{\mathsf{V}}\,\boldsymbol{A} = \mathrm{Co}\,\boldsymbol{A}$, and thus the $\mathcal{S}$-Leibniz operator is an isomorphism between $\mathbf{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ and the entire congruence lattice of $\boldsymbol{A}$. However, for logics such as BCK that are algebraizable but not strongly algebraizable (because the class BCK is not a variety; see Sec. 2.2.2) the image of $\boldsymbol{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}$ is a proper subset of $\mathrm{Co}\,\boldsymbol{A}$.

Finally, we mention a radically different notion of algebraizability within the context of category theory, in which the *structural* axiom of k-deductive systems (Def. 3.1(iv)) is reflected in the naturalness of a transformation between certain functors. See [45, 46, 136, 137].

5. Quasivarieties with Equationally Definable Principal Relative Congruences

We prove in this section that $\mathrm{EQ}(\mathsf{K})$ has the deduction-detachment theorem (DDT) in the sense of Def. 3.11 if and only if the quasivariety K has equationally definable principal relative congruences (EDPRC), a property that has been extensive investigated in universal algebra. Consequently, since the DDT transfers from a deductive system to any equivalent system (Thm. 4.20), if $\mathcal{S}$ is algebraizable, then it has the DDT iff its equivalent quasivariety has EDPRC. This raises the possibility of applying the theory of EDPRC to obtain new results about the DDT for algebraizable deductive systems, and vice versa.

5.1. **Equationally definable principal relative congruences.** Let K be a quasivariety. Recall that a congruence Θ on an algebra $\boldsymbol{A}$ is said to be *relative to* K, or simply a K-*congruence*, if $\boldsymbol{A}/\Theta \in \mathsf{K}$, i.e., Θ is an $\mathrm{EQ}(\mathsf{K})$-filter of $\boldsymbol{A}$. The set of K-congruence on $\boldsymbol{A}$ is denoted by $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$. $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$ is closed under arbitrary intersections (as is any filter lattice). It contains $\boldsymbol{\nabla}_A = A \times A$, and, if $\boldsymbol{A} \in \mathsf{K}$, it contains $\boldsymbol{\Delta}_A = \{\,\langle a, a\rangle : a \in A\,\}$ as well. In fact, $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A}$ is the universe of an algebraic lattice. If $\boldsymbol{A} \in \mathsf{K}$ and K is a variety, then $\mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A} = \mathrm{Co}\,\boldsymbol{A}$. For $a, b \in A$, let

$$\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) = \bigcap\{\,\Theta \in \mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A} : \langle a, b\rangle \in \Theta\,\},$$

the *principal* K*-congruence of* $\boldsymbol{A}$ *generated by* the pair $\langle a, b\rangle$. More generally, for $X \subseteq A^2$, let

$$\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}\,X = \bigcap\{\,\Theta \in \mathrm{Co}_{\mathsf{K}}\,\boldsymbol{A} : X \subseteq \Theta\,\}.$$

If K is a variety and $\boldsymbol{A} \in \mathsf{K}$, we omit the subscript "K". We also omit the superscript "$\boldsymbol{A}$" on $\boldsymbol{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}\,X$ when the algebra $\boldsymbol{A}$ is clear from context.

Definition 5.1. A quasivariety K has *equationally definable principal relative congruences* (EDPRC) if there is a finite system of equations $\eta_{i,0}(p_0, p_1, q_0, q_1) \approx \eta_{i,1}(p_0, p_1, q_0, q_1)$, $i < m$, in four variables, such that for every $\boldsymbol{A} \in \mathsf{K}$ and all $a, b, c, d \in A$,

$$c \equiv d\ \big(\boldsymbol{\Theta}_{\mathsf{K}}(a, b)\big) \quad \text{iff} \quad \eta^{\boldsymbol{A}}_{i,0}(a, b, c, d) = \eta^{\boldsymbol{A}}_{i,1}(a, b, c, d) \text{ for } i < m.$$

There is a purely algebraic characterization of REDPC, which we now give, that has proved to be a very useful tool in investigations of REDPC, and of the DDT in view of the connection between REDPC and the DDT we get below.

Let $\boldsymbol{L} = \langle L, \vee, 0\rangle$ be a join semilattice with 0. If $a, b \in L$, then the *dual relative pseudo-complement* of b with respect to a is the smallest element c, if it exists, with the property $a \leq c \vee b$; it is denoted by $a \dot{-} b$. The semilattice $\boldsymbol{L}$ is said to be *dually relatively pseudo-complemented* if $a \dot{-} b$ exists for all $a, b \in L$. The $(\wedge, 1, \to)$-reduct of a Heyting algebra is a relatively pseudocomplemented meet semilattice with $a \dot{-} b = b \to a$.

Let K be a quasivariety and $\boldsymbol{A} \in \mathsf{K}$. $\mathbf{Co}_{\mathsf{K}}\, \boldsymbol{A}$ is the lattice of EQ(K)-filters on $\boldsymbol{A}$. Thus a congruence Θ is compact in the lattice $\mathbf{Co}_{\mathsf{K}}\, \boldsymbol{A}$ iff it is finitely generated, i.e., of the form $\Theta = \boldsymbol{\Theta}_{\mathsf{K}}(X)$ for some finite $X \subseteq A^2$. Let $\mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}$ denote the collection of compact relative congruences of $\boldsymbol{A}$. This set is the universe of a join semilattice with 0: $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A\rangle$. The full proof of the following theorem in case K is a variety is given in [90]; a more general version covering the case presented here can be found in [15].

Theorem 5.2. *Let* K *be a quasivariety. Then* K *has* EDPRC *iff, for all* $\boldsymbol{A} \in \mathsf{K}$, $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A\rangle$ *is dually relatively pseudocomplemented.*

Proof. Assume K has EDPRC with respect to equations $t_i(x, y, z, w) \approx s_i(x, y, z, w)$, $i < m$. We will show that

$$\boldsymbol{\Theta}_{\mathsf{K}}(c, d) \dot{-} \boldsymbol{\Theta}_{\mathsf{K}}(a, b) = \bigvee_{i<m} \boldsymbol{\Theta}_{\mathsf{K}}\big(t_i^{\boldsymbol{A}}(a, b, c, d),\ s_i^{\boldsymbol{A}}(a, b, c, d)\big).$$

Let Φ be the congruence on the right side of this equation. We must show that, for any $\Psi \in \mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}$,

$$\boldsymbol{\Theta}_{\mathsf{K}}(c, d) \subseteq \Psi \vee \boldsymbol{\Theta}_{\mathsf{K}}(a, b) \quad \text{iff} \quad \Phi \subseteq \Psi,$$

Using the assumption that K has the EDPRC and some elementary universal algebra we get

$$\begin{aligned}
&\boldsymbol{\Theta}_{\mathsf{K}}(c, d) \subseteq \Psi \vee \boldsymbol{\Theta}_{\mathsf{K}}(a, b)\\
&\qquad \text{iff} \quad \boldsymbol{\Theta}_{\mathsf{K}}(c/\Psi,\ d/\Psi) \subseteq \boldsymbol{\Theta}_{\mathsf{K}}(a/\Psi,\ b/\Psi) \text{ in } \boldsymbol{A}/\Psi, \quad \text{by Lem. 5.3}\\
&\qquad \text{iff} \quad t_i^{\boldsymbol{A}/\Psi}\big(a/\Psi,\ b/\Psi,\ c/\Psi,\ d/\Psi\big) = s_i^{\boldsymbol{A}/\Psi}\big(a/\Psi,\ b/\Psi,\ c/\Psi,\ d/\Psi\big), \quad \text{for } i < m\\
&\qquad \text{iff} \quad \Phi \subseteq \Psi.
\end{aligned}$$

Thus within $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A\rangle$, dual relative pseudocomplements of principal relative congruences exist. A straightforward induction shows that dual relative pseudocomplements of all compact relative congruences exist.

Conversely, assume $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A\rangle$ is dually relatively pseudocomplemented for all $\boldsymbol{A} \in \mathsf{K}$. Let $\mathbf{Fm}(x, y, z, w)$ be the subalgebra of the formula algebra generated by the variables x, y, z, w, and let $\boldsymbol{A} = \mathbf{Fm}(x, y, z, w)/\Phi$, where Φ is the smallest K-congruence on $\mathbf{Fm}(x, y, z, w)$. Then $\boldsymbol{A}$ is freely generated over K by x/Φ, y/Φ, z/Φ, w/Φ. Let

$$\langle t_i(x, y, z, w)/\Phi,\ s_i(x, y, z, w)/\Phi\rangle, \quad i < m,$$

be a set of generators of the K-congruence $\boldsymbol{\Theta}_{\mathsf{K}}(z/\Phi,\ w/\Phi) \dot{-} \boldsymbol{\Theta}_{\mathsf{K}}(x/\Phi,\ y/\Phi)$ on $\boldsymbol{A}$. A straightforward argument shows that K has the EDPRC with respect to the equations $t_i(x, y, z, w) \approx s_i(x, y, z, w)$, $i < m$. □

The next theorem establishes the link between DDT and EDPRC. To prove it We will need the following special case of the correspondence theorem (Lem. 4.12). To simplify notation somewhat in the sequel we take $\vee^{\mathsf{K}} = \vee^{\mathrm{EQ}(\mathsf{K})}$, the join in the in the lattice of K-congruences.

Lemma 5.3. *Let* K *be a quasivariety,* $\boldsymbol{B} \in \mathsf{K}$*, and* $h : \boldsymbol{A} \to \boldsymbol{B}$ *a surjective homomorphism. For* $a, b \in A$ *we have*

$$h^{-1}\big(\mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb)\big) = \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \vee^{\mathsf{K}} \varPhi,$$

where $\varPhi = h^{-1}(\mathbf{\Delta}_A) = \{\, \langle a, b\rangle \in A^2 : ha = hb \,\}$*, the relation kernel of* h*.*

Proof. $\mathrm{EQ}(\mathsf{K})$ is clearly protoalgebraic; in fact, $\mathbf{\Omega}^{\boldsymbol{A}}_{\mathrm{EQ}(\mathsf{K})}$ is the identity function on $\mathrm{Fi}_{\mathrm{EQ}(\mathsf{K})}\, \boldsymbol{A} = \mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}$ and so is trivially monotonic. By the correspondence theorem (Lem. 4.12) we have

$$h^{-1}\big(h^{\boldsymbol{B}}_{\mathrm{EQ}(\mathsf{K})}(\varTheta) \vee^{\mathrm{EQ}(\mathsf{K})} \varPsi\big) = \varTheta \vee^{\mathrm{EQ}(\mathsf{K})} h^{-1}(\varPsi),$$

for every $\varTheta \in \mathrm{Co}_{\mathsf{K}}(\boldsymbol{A})$ and $\varPsi \in \mathrm{Co}_{\mathsf{K}}(\boldsymbol{B})$. Taking $\varTheta = \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)$ and $\varPsi = \mathbf{\Delta}_B$ we get $h^{-1}\big(h^{\boldsymbol{B}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b)\big) = \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \vee^{\mathsf{K}} \varPhi$. It remains only to observe that $h^{\boldsymbol{B}}_{\mathrm{EQ}(\mathsf{K})}\, \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) = \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}\, h\, \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) = \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb)$. We have the first equality by definition of $h_{\mathrm{EQ}(\mathsf{K})}$. For the second note that the inclusion from right to left is obvious. For the opposite inclusion, from $\mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \subseteq h^{-1}\, \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb)$ we get $h\, \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \subseteq hh^{-1}\, \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb) = \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb)$ and hence $\mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}\, h\, \mathbf{\Theta}^{\boldsymbol{A}}_{\mathsf{K}}(a, b) \subseteq \mathbf{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(ha, hb)$. □

Theorem 5.4. *Let* K *be a quasivariety and* $\mathrm{EQ}(\mathsf{K})$ *its associated 2-deductive system. Then* $\mathrm{EQ}(\mathsf{K})$ *has the* DDT *iff* K *has* EDPRC.

Proof. $\Longrightarrow$: If $E(\mathbf{p}, \mathbf{q}) = \{\boldsymbol{\eta}_i(\mathbf{p}, \mathbf{q}) : i < m\}$ is a deduction-detachment set for $\mathrm{EQ}(\mathsf{K})$, then the pairs $\eta_{i,0}(p_0, p_1, q_0, q_1)$, $\eta_{i,1}(p_0, p_1, q_0, q_1)$, $i < m$, define the principal K-congruences. It suffices to verify this for all countably generated algebras in K. This is so because the principal K-congruence generated by $\langle a, b\rangle$ in $\boldsymbol{A}$ is the union of the principal K-congruences generated by $\langle a, b\rangle$ in all finitely generated subalgebras of $\boldsymbol{A}$ that contain a and b. So let $\boldsymbol{A} \in \mathsf{K}$ be countably generated and $h : \mathbf{Fm} \to \boldsymbol{A}$ a surjection. Let $a, b, c, d \in A$ and $\varphi_0, \varphi_1, \psi_0, \psi_1 \in \mathrm{Fm}$ such that $h(\varphi_0) = a$, $h(\varphi_1) = b$, $h(\psi_0) = c$, $h(\psi_1) = d$. Thus, if $h(p_i) = a_i \in A$, $i = 0, 1, 2, \ldots$, and $\bar{a} = \langle a_0, a_1, \ldots\rangle$, then $a = \varphi_0^{\boldsymbol{A}}(\bar{a})$, $b = \varphi_1^{\boldsymbol{A}}(\bar{a})$, $c = \psi_0^{\boldsymbol{A}}(\bar{a})$, $d = \psi_1^{\boldsymbol{A}}(\bar{a})$. Let $\varPhi$ be the relation kernel of h, i.e., $\varPhi = \{\langle\varphi, \psi\rangle \in \mathrm{Fm}^2 : h(\varphi) = h(\psi)\}$. Then $\varPhi$ is a K-congruence on $\mathbf{Fm}$. Now by Lem. 5.3,

$$c \equiv d \;\big(\mathbf{\Theta}_{\mathsf{K}}(a, b)\big) \quad \text{iff} \quad \psi_0 \equiv \psi_1 \;\big(\varPhi \vee^{\mathsf{K}} \mathbf{\Theta}_{\mathsf{K}}(\varphi_0, \varphi_1)\big). \tag{31}$$

But $\varPhi \vee^{\mathsf{K}} \mathbf{\Theta}_{\mathsf{K}}(\varphi_0, \varphi_1) = \mathbf{\Theta}_{\mathsf{K}}\big(\varPhi \cup \{\langle\varphi_0, \varphi_1\rangle\}\big)$. Hence, if we write $\boldsymbol{\varphi} = \langle\varphi_0, \varphi_1\rangle$ and $\boldsymbol{\psi} = \langle\psi_0, \psi_1\rangle$, then by Lem. 3.3 the right-hand side of (31) is equivalent to

$$\varPhi, \boldsymbol{\varphi} \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\psi},$$

which by the DDT for $\mathrm{EQ}(\mathsf{K})$ is equivalent to

$$\varPhi \vdash_{\mathrm{EQ}(\mathsf{K})} \eta_i(\boldsymbol{\varphi}, \boldsymbol{\psi}), \quad i < m.$$

Since $\varPhi$ is a K-congruence, using Lem. 3.3 this in turn is equivalent to the congruence

$$\eta_{i,0}(\varphi_0, \varphi_1, \psi_0, \psi_1) \equiv \eta_{i,1}(\varphi_0, \varphi_1, \psi_0, \psi_1)\ (\varPhi),$$

in **Fm**, which holds iff

$$\eta^{\boldsymbol{A}}_{i,0}(a,b,c,d) = \eta^{\boldsymbol{A}}_{i,1}(a,b,c,d).$$

Thus K has EDPRC.

$\Longleftarrow$: Let $\eta_{i,0}(p_0,p_1,q_0,q_1)$, $\eta_{i,1}(p_0,p_1,q_0,q_1)$, with $i < m$, be the terms defining the principal relative congruences in K. Set

$$\boldsymbol{\eta}_i(\mathbf{p},\mathbf{q}) = \langle \eta_{i,0}(\mathbf{p},\mathbf{q}), \eta_{i,1}(\mathbf{p},\mathbf{q})\rangle, \qquad i < m,$$

where $\mathbf{p}$ and $\mathbf{q}$ are the 2-variables $\langle p_0,p_1\rangle$ and $\langle q_0,q_1\rangle$, respectively. We claim that $E(\mathbf{p},\mathbf{q}) = \{\, \boldsymbol{\eta}_i(\mathbf{p},\mathbf{q}) : i < m \,\}$ is a deduction-detachment set for $\mathrm{EQ}(\mathsf{K})$. To verify this, let $\Gamma \subseteq \mathrm{Fm}^2$, $\boldsymbol{\varphi},\boldsymbol{\psi} \in \mathrm{Fm}^2$. Now by Lem. 3.3

$$\Gamma,\boldsymbol{\varphi} \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\psi} \quad \text{iff} \quad \boldsymbol{\psi} \in \boldsymbol{\Theta}_{\mathsf{K}}(\Gamma \cup \{\boldsymbol{\varphi}\}) = \boldsymbol{\Theta}_{\mathsf{K}}(\Sigma) \vee^{\mathsf{K}} \boldsymbol{\Theta}_{\mathsf{K}}(\varphi_0,\varphi_1). \tag{32}$$

Let $\boldsymbol{B} = \mathbf{Fm}/\boldsymbol{\Theta}_{\mathsf{K}}(\Gamma)$, and let $h : \mathbf{Fm} \to \boldsymbol{B}$ be the canonical homomorphism. Then $\boldsymbol{B} \in \mathsf{K}$, h is surjective and the relation kernel of h is $\boldsymbol{\Theta}_{\mathsf{K}}(\Gamma)$. So by the previous lemma we have

$$\boldsymbol{\Theta}_{\mathsf{K}}(\Gamma) \vee^{\mathsf{K}} \boldsymbol{\Theta}_{\mathsf{K}}(\varphi_0,\varphi_1) = h^{-1}(\boldsymbol{\Theta}_{\mathsf{K}}(h(\varphi_0),h(\varphi_1)).$$

It follows by Lem. 3.3 and the last equality that the statement $\Gamma,\boldsymbol{\varphi} \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\psi}$ is equivalent to

$$h(\psi_0) \equiv h(\psi_1)\big(\boldsymbol{\Theta}_{\mathsf{K}}\big(h(\varphi_0),h(\varphi_1)\big)\big).$$

Since K has EDPRC, this in turn is equivalent to

$$\eta^{\boldsymbol{B}}_{i,0}(h(\varphi_0),h(\varphi_1),h(\psi_0),h(\psi_1)) = \eta^{\boldsymbol{B}}_{i,1}(h(\varphi_0),h(\varphi_1),h(\psi_0),h(\psi_1)),$$

for $i < m$, or equivalently, to

$$\eta_{i,0}(\varphi_0,\varphi_1,\psi_0,\psi_1) \equiv \eta_{i,1}(\varphi_0,\varphi_1,\psi_0,\psi_1)\ \big(\boldsymbol{\Theta}_{\mathsf{K}}(\Gamma)\big), \quad i < m,$$

and thus, since $\boldsymbol{\Theta}_{\mathsf{K}}(\Gamma) = D(\Gamma)$,

$$\Gamma \vdash_{\mathrm{EQ}(\mathsf{K})} \ (\boldsymbol{\eta}_i)_0(\varphi_0,\varphi_1,\psi_0,\psi_1) \approx (\boldsymbol{\eta}_i)_1(\varphi_0,\varphi_1,\psi_0,\psi_1), \quad i < m,$$

or $\Gamma \vdash_{\mathrm{EQ}(\mathsf{K})} \boldsymbol{\eta}_i(\boldsymbol{\varphi},\boldsymbol{\psi})$, $\quad i < m$. Thus $E(\mathbf{p},\mathbf{q}) = \{\, \boldsymbol{\eta}_i(\mathbf{p},\mathbf{q}) : i < m \,\}$ is a deduction-detachment set for $\mathrm{EQ}(\mathsf{K})$.

□

Theorem 5.5. *Let $\mathcal{S}$ be an algebraizable deductive system, with equivalent quasivariety K. Then $\mathcal{S}$ has the* DDT *iff K had* EDPRC.

Proof. Let $\mathcal{S}$ be equivalent to $\mathrm{EQ}(\mathsf{K})$, for K a quasivariety. By Thm. 4.20, $\mathcal{S}$ has the DDT iff $\mathrm{EQ}(\mathsf{K})$ does. But by Thm. 5.4, $\mathrm{EQ}(\mathsf{K})$ has the DDT iff K has EDPRC. □

Assume $\mathcal{S}$ is algebraizable with equivalent quasivariety K. Then the for every algebra $\boldsymbol{A}$ the Leibniz operator induces an isomorphism between the semilattices of compact K-congruences and of compact $\mathcal{S}$-filters of $\boldsymbol{A}$. Consequently as corollary of the previous two theorems we have that $\mathcal{S}$ has the DDT iff, for every $\boldsymbol{A} \in \mathsf{K}$, the semilattice of compact $\mathcal{S}$-filters of $\boldsymbol{A}$ is dually relatively pseudo-complemented, and it is not difficult to show that this holds for all algebras $\boldsymbol{A}$, not just the members of K. This characterization of DDT holds for every protoalgebraic deductive system [30].

If K is a variety with EDPRC and $\boldsymbol{A} \in \mathsf{K}$, then every congruence of $\boldsymbol{A}$ is a K-congruence, and we say that K has *equationally definable principal congruences* (EDPC). For varieties the property was introduced in [66], in a purely algebraic context. The structure of varieties with EDPC was investigated in that paper and later in [9, 10, 18, 17, 90].

We will discuss some of the salient properties of varieties and quasivarieties with EDPRC. These will then be applied to obtain results about the deduction-detachment theorem for algebraizable logics. In accordance with common practice in universal algebra we will from now on normally omit the superscript indicating the algebra from symbols for algebraic operations when there is not uncertainty; thus we write $\wedge$ for $\wedge^{\boldsymbol{A}}$, etc.

5.2. **Examples.**

5.2.1. *Boolean algebras* (BA). By the last theorem we know that BA has EDPC since CPC as the DDT. Moreover, an analysis of the proof of Thm. 4.20 will actually produce a deduction-detachment system for BA. It is easier however to establish the fact directly.

Theorem 5.6. *Let* $\boldsymbol{A} = \langle A, \rightarrow, \wedge, \vee, \neg, \bot, \top \rangle$ *be a Boolean algebra and* $a, b, c, d \in A$. *Then*

$$c \equiv d\,\big(\Theta(a,b)\big) \quad \textit{iff} \quad (a \leftrightarrow b) \wedge c = (a \leftrightarrow b) \wedge d,$$

where $a \leftrightarrow b$ *abbreviates* $(a \rightarrow b) \wedge (b \rightarrow a)$. *Thus* BA *has* EDPC.

Proof. Let $m = a \leftrightarrow b$ and define $c \equiv d$ iff $m \wedge c = m \wedge d$. $\equiv$ is obviously an equivalence relation. If $c \equiv d$, then $m \wedge \neg c = m \wedge (\neg m \vee \neg c) = m \wedge \neg(m \wedge c) = m \wedge \neg(m \wedge d) = m \wedge \neg d$. So $\neg c \equiv \neg d$. Similarly for the other operations of $\boldsymbol{A}$. Thus $\equiv$ is a congruence relation of $\boldsymbol{A}$. Since $m \wedge a = a \wedge b = m \wedge b$, we have $a \equiv b$, and hence $\Theta(a,b) \subseteq \equiv$.

For the reverse inclusion, note that $m = a \leftrightarrow b\ \Theta(a,b)\ a \leftrightarrow a = \top$. Thus $c \equiv d$ implies $c\ \Theta(a,b)\ m \wedge c = m \wedge d\ \Theta(a,b)\ d$. So $\equiv \subseteq \Theta(a,b)$. □

5.2.2. *Heyting algebras* (HA). Heyting algebras have EDPC, too, and the same defining equation can be used as was used for BA.

5.2.3. *Modal algebras* (MA). Modal algebras $\boldsymbol{A} = \langle A, \rightarrow, \wedge, \vee, \neg, \bot, \top, \Box \rangle$ will turn out not to have EDPC (see Sec. 5.3.1 below), but the subvariety MO of monadic algebras does have the property, and for $a, b, c, d \in A$,

$$c \equiv d\,\big(\Theta(a,b)\big) \quad \text{iff} \quad \Box(a \leftrightarrow b) \wedge c = \Box(a \leftrightarrow b) \wedge d.$$

5.2.4. *Distributive lattices* (DL). For $\boldsymbol{A} = \langle A, \wedge, \vee \rangle$ a distributive lattice and $a, b, c, d \in A$ we have

$$c \equiv d\,\big(\Theta(a,b)\big) \quad \text{iff} \quad (a \wedge b) \wedge c = (a \wedge b) \wedge d \ \text{ and } \ (a \vee b) \vee c = (a \vee b) \vee d.$$

The proof is similar to the proof for Boolean algebras.

5.2.5. *Discriminator varieties.* The *ternary discriminator* function t on a set A is defined by

$$t(a,b,c) = \begin{cases} c, & \text{if } a = b \\ a, & \text{otherwise.} \end{cases}$$

A variety V is a *discriminator variety* if there is a class $\mathsf{K} \subseteq \mathsf{V}$ such that $\mathsf{V} = \mathbf{HSP}(\mathsf{K})$ and a term-function $t(x,y,z)$ of V that coincides with the ternary discriminator on all algebras in K. Boolean algebras form a discriminator variety: the term $t(x,y,z) = \big((\neg y) \wedge (x \vee z)\big) \vee (x \wedge z)$ defines the discriminator on the 2-element Boolean algebra **2**. Other examples of discriminator varieties are monadic algebras, relation algebras, n-dimensional cylindric algebras with n finite, and n-valued Post and n-valued Łukasiewicz algebras with n finite.

If t is a ternary discriminator term for a variety V, and $\boldsymbol{A} \in \mathsf{V}$, $a, b, c, d \in A$, then

$$c \equiv d \left(\Theta(a, b)\right) \quad \text{iff} \quad t^{\boldsymbol{A}}(a, b, c) = t^{\boldsymbol{A}}(a, b, d).$$

Thus V has EDPC.

Discriminator varieties constitute a natural generalization of a wide class of varieties arising from logic and have been extensively investigated in the univeral algebra literature; see [10] for references.

5.3. **First-order definable principal relative congruences.** If a quasivariety K of type $\mathcal{L}$ has EDPRC, then in particular it has (*first-order*) *definable principal relative congruences* (DPRC), i.e., there is a first-order formula $\varphi(x, y, z, w)$, in the language of the first-order predicate logic (with equality), whose only nonlogical symbols are the operation symbols of $\mathcal{L}$, with the property: for all $\boldsymbol{A} \in \mathsf{K}$ and all $a, b, c, d \in A$,

$$c \equiv d \left(\boldsymbol{\Theta}_{\mathsf{K}}(a, b)\right) \quad \text{iff} \quad \vDash_{\boldsymbol{A}} \varphi[a, b, c, d].$$

For example, the variety of commutative rings with 1 has DPRC, since for any such ring $\boldsymbol{R}$ and $a, b, c, d \in R$, we have

$$c \equiv d \left(\boldsymbol{\Theta}_{\mathsf{K}}(a, b)\right) \quad \text{iff} \quad \exists z[c - d = z \cdot (a - b)].$$

Quasivarieties with EDPRC are just quasivarieties with DPRC such that the defining first-order formula is particularly simple, viz., a conjunction of equations in four variables. If K is a variety we say that it has *first-order definable principal congruences* (DPC).

Let K be a quasivariety and let $\boldsymbol{A} \in \mathsf{K}$. $\boldsymbol{A}$ is *relatively simple* if it is nontrivial and $\mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A} = \{\boldsymbol{\Delta}_A, \boldsymbol{\nabla}_A\}$, and $\boldsymbol{A}$ is *relatively subdirectly irreducible* if $(\mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}) \setminus \{\boldsymbol{\Delta}_A\}$ has a smallest element under inclusion. If K is a variety and $\boldsymbol{A} \in \mathsf{K}$ is nontrivial, then $\boldsymbol{A}$ is relatively simple (relatively subdirectly irreducible) iff it is simple (subdirectly irreducible) in the usual sense. Let K_{S} denote the class of relatively simple members of K and K_{SI} the class of relatively subdirectly irreducible members. The following result provides a useful tool for showing that a quasivariety fails to have DPRC.

Theorem 5.7. *Suppose* K *is a quasivariety with* DPRC, *in particular with* EDPRC. *Then the classes* K_{S} *and* K_{SI} *are both closed under ultraproducts.*

Proof. It suffices to show that both classes are first-order definable. Let $\varphi(x, y, z, w)$ be a first-order formula defining the principal relative congruences in K. Then

$$\boldsymbol{A} \in \mathsf{K}_{\mathrm{S}} \quad \text{iff} \quad \vDash_{\boldsymbol{A}} \exists xy[\neg(x \approx y)] \wedge \forall xyzw[\neg(x \approx y) \to \varphi(x, y, z, w)] \quad \text{and}$$

$$\boldsymbol{A} \in \mathsf{K}_{\mathrm{SI}} \quad \text{iff} \quad \vDash_{\boldsymbol{A}} \exists x \exists y \forall z \forall w[\neg(x \approx y) \wedge \left(\neg(z \approx w) \to \varphi(z, w, x, y)\right)].$$

□

5.3.1. *Examples.* We use the theorem to show the quasivariety BCK of BCK algebras does not have DPRC and hence does not have EDPRC. Consider $\boldsymbol{A} = \langle \omega, \to^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle$, where $\omega = \{0, 1, 2, \dots\}$, $1^{\boldsymbol{A}} = 0$, and $m \to^{\boldsymbol{A}} n = n - m$ if $m \leq n$; otherwise $m \to^{\boldsymbol{A}} n = 0$. It is easy to check that $\boldsymbol{A}$ satisfies the defining identities and quasi-identity of BCK algebras (see Sec 2.2). Also, $\boldsymbol{A}$ is simple and hence relatively simple. On the other hand, the ultrapower $\boldsymbol{B} = \boldsymbol{A}^{\omega}/U$ is not relatively simple for any nonprincipal ultrafilter U. For, if we identify $\boldsymbol{A}$

with a subalgebra of $\boldsymbol{B}$ in the standard way and define $b \equiv b'$ if $b \to^{\boldsymbol{B}} b', b' \to^{\boldsymbol{B}} b \in A$, then $\equiv$ is a BCK-congruence of $\boldsymbol{B}$ that identifies each element of A with $1^{\boldsymbol{B}}$ but no element of $B \setminus A$ with $1^{\boldsymbol{B}}$.

Theorem 5.7 can also be used to show that the variety MA of modal algebras fails to have EDPC. We first present a way of constructing modal algebras. For S a set and $R \subseteq S \times S$ a binary relation on S, let $\langle S, R\rangle^+$ be the modal algebra $\langle 2^S, \to, \cap, \cup, \sim, \emptyset, S, \Box\rangle$ of all subsets of S endowed with the unary operation defined for all $X \subseteq S$ by

$$\Box X = \{u \in S : \forall v \in S\big(\langle u, v\rangle \in R \Longrightarrow v \in X\big)\}.$$

An easy computation shows that

(i) $\Box S = S$,
(ii) $\Box(X \cap Y) = \Box X \cap \Box Y$,

so $\langle S, R\rangle^+$ is a modal algebra.

Now for each $n < \omega$, let $\boldsymbol{C}_n$ be an n-element cycle, i.e., $\boldsymbol{C}_n = \langle C, R\rangle$ where $C = \{0, 1, \dots, n-1\}$ and

$$R = \{\, \langle i, i+1\rangle : i = 0, \dots, n-2\} \cup \{\langle n-1, 0\rangle\} \cup \{\, \langle i, i\rangle : i = 0, \dots, n-1\,\}.$$

Then $\boldsymbol{C}_n^+$ is a simple modal algebra. However, if $\boldsymbol{A} = \prod_{n<\omega} \boldsymbol{C}_n^+/U$, for any nonprincipal ultrafilter U over ω, then $\boldsymbol{A}$ is not simple. Thus BCK and K do not have the DDT.

For a detailed study of the EDPC property in the context of a large family of varieties that includes both BCK and MA see [17].

5.4. **Relative congruence extension and relative congruence distributivity.** Let K be a quasivariety. K has the *relative congruence extension property* (RCEP) if, for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $\Theta \in \mathrm{Co}_{\mathsf{K}}\, \boldsymbol{B}$, then there is $\Theta' \in \mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}$ such that $\Theta' \cap B^2 = \Theta$. If K is a variety, we say that it has the *congruence extension property* (CEP).

Theorem 5.8. *Let* K *be a quasivariety with* EDPRC*. Then* K *has* RCEP*.*

Proof. Assume $\boldsymbol{A} \in \mathsf{K}$ and $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$. We will prove that every principal K-congruence on $\boldsymbol{B}$ extends to a K-congruence on $\boldsymbol{A}$. It is shown in [15] that this ensures that K has RCEP; for K a variety this was first proved by Day [41]. We show that, if $a, b \in B$, then $\Theta_{\mathsf{K}}^{\boldsymbol{B}}(a, b) = \Theta_{\mathsf{K}}^{\boldsymbol{A}}(a, b) \cap B^2$. Since $\Theta_{\mathsf{K}}^{\boldsymbol{A}}(a, b) \cap B^2$ is a K-congruence of $\boldsymbol{B}$ containing $\langle a, b\rangle$, the inclusion $\Theta_{\mathsf{K}}^{\boldsymbol{B}}(a, b) \subseteq \Theta_{\mathsf{K}}^{\boldsymbol{A}}(a, b) \cap B^2$ is obvious. For the opposite inclusion, let $\langle c, d\rangle \in \Theta_{\mathsf{K}}^{\boldsymbol{A}}(a, b) \cap B^2$. Suppose the equations $t_i(x, y, z, w) \approx s_i(x, y, z, w)$, $i < m$, define the relative principal congruences in K. Then $t_i^{\boldsymbol{A}}(a, b, c, d) = s_i^{\boldsymbol{A}}(a, b, c, d)$, $\quad i < m$. But since $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $a, b, c, d \in B$, $t_i^{\boldsymbol{B}}(a, b, c, d) = t_i^{\boldsymbol{A}}(a, b, c, d)$ and $s_i^{\boldsymbol{B}}(a, b, c, d) = s_i^{\boldsymbol{A}}(a, b, c, d)$ for $i < m$. Thus $t_i^{\boldsymbol{B}}(a, b, c, d) = s_i^{\boldsymbol{B}}(a, b, c, d)$, $i < m$, as well, and hence $c \equiv d \,\big(\Theta_{\mathsf{K}}^{\boldsymbol{B}}(a, b)\big)$. $\Box$

The RCEP is actually equivalent to the local form of the EDPRC property, i.e., a quasivariety K has the RCEP iff $\mathrm{EQ}(\mathsf{K})$ has the local DDT [15, 34]; more details can be found in Note 5.1.

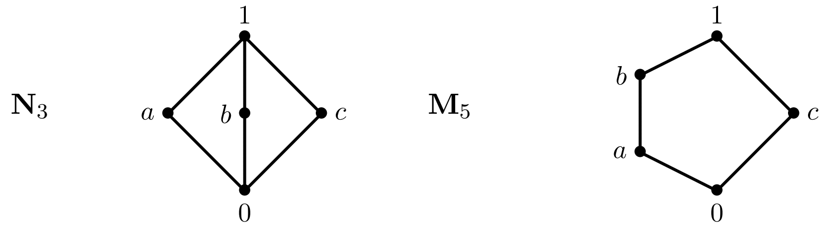

FIGURE 5

5.4.1. *Example.* let K be a quasivariety of lattices that properly contains the variety of distributive lattices. Then $\boldsymbol{M}_3 \in \mathsf{K}$ or $\boldsymbol{N}_5 \in \mathsf{K}$, where $\boldsymbol{M}_3$ and $\boldsymbol{N}_5$ are the lattices shown in Figure 5.

It follows that K does not have RCEP. For if $\boldsymbol{M}_3 \in \mathsf{K}$, let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{M}_3$ with universe $\{0, a, b, 1\}$. $\mathit{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(0, a) = \mathit{\Theta}^{\boldsymbol{B}}(0, a)$ since $\boldsymbol{B}/\mathit{\Theta}^{\boldsymbol{B}}(0, a)$ is the two-element lattice, which must be a member of K. Thus $\mathit{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(0, a) \neq \boldsymbol{\nabla}_B$ and hence cannot be extended to $\boldsymbol{M}_3$ since $\boldsymbol{M}_3$ is simple. Also, if $\boldsymbol{N}_5 \in \mathsf{K}$, let $\boldsymbol{B}$ be the subalgebra of $\boldsymbol{N}_5$ again with universe $\{0, a, b, 1\}$. Then by essentially the same argument $\mathit{\Theta}^{\boldsymbol{B}}_{\mathsf{K}}(b, 1)$ cannot be extended. Thus the variety of distributive lattices is the only quasivariety of lattices with EDPRC; see the remarks following Def. 3.11.

For more information on the RCEP see [15, 31]. See also Note 5.1.

The following result in the variety case is due independently to [67] and [90].

Theorem 5.9. *Let* K *be a quasivariety with* EDPRC*. Then* K *is relatively congruence-distributive.*

Proof. Suppose K is a quasivariety with EDPRC. We need to show $\langle \mathrm{Co}_{\mathsf{K}}\, \boldsymbol{A}, \wedge, \vee \rangle$ is a distributive lattice for every $\boldsymbol{A} \in \mathsf{K}$. Now the map

$$\mathit{\Phi} \mapsto \{\, \mathit{\Theta} \in \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A} : \mathit{\Theta} \subseteq \mathit{\Phi} \,\}$$

establishes a bijection between the K-congruences of $\boldsymbol{A}$ and the ideals of the join semilattice $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A \rangle$ (an *ideal* here is a downward closed subset that is also closed under join). Both this bijection and its inverse preserve the partial order $\subseteq$; thus the K-congruence lattice of $\boldsymbol{A}$ is isomorphic to the lattice of ideals of $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A \rangle$. The join semilattice $\langle \mathrm{Cp}_{\mathsf{K}}\, \boldsymbol{A}, \vee, \boldsymbol{\Delta}_A \rangle$ is dually relatively pseudocomplemented by Thm. 5.2, and the lattice of ideals of a dually relatively pseudocomplemented semilattice is known to be distributive. See [90] for this result and for references to papers on pseudocomplemented semilattices. □

5.4.2. *Example.* Let BG be the variety of *Boolean groups* $\langle A, \cdot, 1 \rangle$, where $\boldsymbol{A}$ is a commutative monoid with 1 satisfying the identity $x \cdot x \approx 1$. This variety is term-equivalent with $\mathbf{HSP}(\mathbb{Z}_2)$, where $\mathbb{Z}_2 = \langle \{0, 1\}, +_{\mathrm{Mod2}}, 0 \rangle$ is the two element group. Although it is therefore congruence modular, it is not congruence distributive; the lattice of congruences of the 4-element group $\mathbb{Z}^2$ is isomorphic $\boldsymbol{M}_3$, the (modular) but non-distributive 5-element lattice containing three atoms (see Figure 5). Thus BG does not have EDPC.

5.5. **Algebraizable deductive systems and the DDT.**

Theorem 5.10. *Let* $\mathcal{S}$ *be an algebraizable deductive system with equivalent quasivariety* K*. If* $\mathcal{S}$ *has the* DDT*, then*

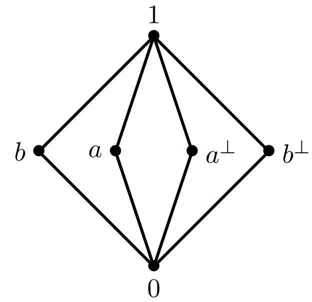

FIGURE 6

(i) K *has* DPRC*;*
(ii) K *has* RCEP*;*
(iii) K *is relatively congruence-distributive.*

Proof. By Thm. 5.5 K has EDPRC. Then K trivially has DPRC, and it has the RCEP and is relatively congruence-distributive by Thms. 5.8 and 5.9. □

The converse also holds. The equivalence of the DDT with the conjunction of the three conditions of Thm. 5.10 is a special of more general results in [15] and in [34]; see Note 5.1.

5.6. **Examples.**

5.6.1. BA*,* HA*, and* MO*.* The fact that the varieties BA, HA, and MO have EDPC (Secs. 5.2.1, 5.2.2, and 5.2.3) reflects the fact that CPC, IPC, and $\mathrm{S5}^{\mathrm{G}}$ have the DDT (Thms. 2.5 and 2.9,2.15).

5.6.2. BCK *and* K*.* On the other hand, the algebraizable logics BCK and K both fail to have the DDT, since the corresponding equivalent quasivariety BCK and variety MA fail to have DPRC and DPC, respectively (Sec. 5.3.1). A similar argument applies to first-order predicate logic PR_ω and its equivalent variety, the class of cylindric algebras of dimension ω. All of these logics do possess a local version of the deduction theorem, however. See Thms. 2.11 and 2.20; see Note 5.1.

5.6.3. *Linear logic.* The equivalent quasivariety of the linear logic system $\mathrm{LL}^{*,\to,\wedge}$ (Sec. 3.3.6) consists of all semilattice ordered residuated commutative semigroups. See [109, 111, 123, 135]. This quasivariety is a variety, which fails to possess the CEP. The 2-dimensional deductive system $\mathrm{LL}^{*,\to,\wedge}$ therefore does not have the DDT. (The $\to$-introduction rule for linear logic, Sec. 3.3.5(LLR2), should not be confused with the DDT theorem for $\mathrm{LL}^{*,\to,\wedge}$.)

5.6.4. *Equivalential logic.* The equivalential fragment of the classical propositional calculus, $\mathrm{CPC}^{\leftrightarrow}$, was investigated by Łukasiewicz [100]. He showed it can be axiomatized by the single axiom

$$(p \leftrightarrow q) \leftrightarrow \big((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r)\big)$$

and the rule

$$\frac{p,\, p \leftrightarrow q}{q}.$$

This deductive system is algebraizable, and its equivalent quasivariety is the variety BG of Boolean groups. In Sec. 5.4.2 we saw that BG is not congruence-distributive, and therefore does not possess EDPC. Hence the calculus $\mathrm{CPC}^{\leftrightarrow}$ does not have the DDT.

5.6.5. *Orthomodular logic.* Let V be the variety of orthomodular lattices $\langle L, \vee, \wedge, {}^{\perp}, 0, 1\rangle$.

There are several different 1-deductive systems $\mathcal{S}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_{\mathsf{V}}$, any one of which is a candidate for orthomodular logic. But none of them can have the DDT since V does not have CEP, as is easily verified by considering the 4-element orthomodular lattice shown in Figure 6 (This result was announced in [15] and obtained independently by Malinowski [102].)

5.7. **Notes.**

Note 5.1. We have seen that EDPRC implies all three properties DPRC (trivially), RCEP, and relative congruence-distributivity. And in the case of algebraizable deductive systems, the DDT implies that the equivalent quasivariety has all these properties. It turns out that the connection between the DDT and each of these properties can be considerably refined by passing to the local DDT. Moreover, by considering natural matrix-model analogues of algebraic properties such as the RCEP, these results can be extended to a much wider class of deductive systems than the algebraizable ones.

Recall that the local deduction theorem was defined informally in Note 2.5; we refine the definition and extend it to k-deductive systems. Let $\mathcal{S}$ be a k-deductive system. Let I be a set and, for each $i \in I$, let $E_i(\mathbf{p}, \mathbf{q}) = \{\boldsymbol{\eta}_0(\mathbf{p}, \mathbf{q}), \dots, \boldsymbol{\eta}_{m_i-1}(\mathbf{p}, \mathbf{q})\}$ be a finite set of k-formulas in $2k$ variables $\mathbf{p} = \langle p_0, \dots, p_{k-1}\rangle$ and $\mathbf{q} = \langle q_0, \dots, q_{k-1}\rangle$. The system $\mathcal{E}(\mathbf{p}, \mathbf{q}) = \{\, E_i(\mathbf{p}, \mathbf{q} : i \in I \,\}$ is called a *local deductive-detachment system* for $\mathcal{S}$ if, for all $\Gamma \subseteq \mathrm{Fm}^k$ and $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathrm{Fm}^k$, we have

$$\Gamma, \boldsymbol{\varphi} \vdash_{\mathcal{S}} \boldsymbol{\psi} \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} E_i(\mathbf{p}, \mathbf{q}) \text{ for some } i \in I.$$

A deductive system $\mathcal{S}$ has the *local deduction-detachment theorem* (LDDT) if a local deduction-detachment system for $\mathcal{S}$ exists.

Let M be a class of k-matrices closed under the formation of submatrices. M has the *filter extension property* (FEP) if every $\mathfrak{A} \in \mathsf{M}$ has the property that every $\mathcal{S}$-filter on a submatrix $\mathfrak{A}$ can be extended to an $\mathcal{S}$-filter on $\mathfrak{A}$. A protoalgebraic k-deductive system has the LDDT iff its class of matrix-models has the FEP iff its class of reduced matrix-models has the FEP. Specialized to $\mathrm{EQ}(K)$, for any quasivariety K, the equivalence of the first and third conditions says that K has the local EDPRC property iff it has the RCEP.

A local deduction-detachment system $\mathcal{E} = \{\, E_i : i \in I \,\}$ for a k-deductive system $\mathcal{S}$ is *directed* if, for all $i, j \in I$ there exists a $k \in I$ such that $E_i \vdash_{\mathcal{S}} E_k$ and $E_j \vdash_{\mathcal{S}} E_k$. Let $\mathcal{S}$ be a protoalgebraic k-deductive system with the LDDT, and let $\mathcal{E}$ be any local deduction-detachment system for $\mathcal{S}$. Then $\mathcal{S}$ is *filter-distributive*, i.e., the lattice of $\mathcal{S}$-filters on any algebra is distributive, iff $\mathcal{E}$ is directed.

A class M of matrix-models of a k-deductive system $\mathcal{S}$ has *definable principal filters* (DPF) if there is a first-order formula that defines the principally generated $\mathcal{S}$-filters of every $\mathfrak{A} \in \mathsf{M}$, i.e., all $\mathcal{S}$-filters of the form $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F^{\mathfrak{A}} \cup \{\mathbf{a}\})$ with $\mathbf{a} \in A^k$. Assume $\mathcal{S}$ has the LDDT. If the class of all matrix-models of $\mathcal{S}$ has DPF, then a simple compactness argument shows that any local deduction-detachment system for $\mathcal{S}$ must include a finite subset which is also a local deduction-detachment system for $\mathcal{S}$; moreover, this system can be taken to be a singleton if the original system is directed. Thus a k-deductive system $\mathcal{S}$ has the DDT iff the class of all matrix-models of $\mathcal{S}$ has the FEP, DPF, and is filter-distributive. The equivalence holds also for the class of all reduced matrix-models of $\mathcal{S}$. Applied to equational deductive systems this gives Thm. 5.10 and its converse.

All the results on the LDDT mentioned here together with many others can be found in [15, 31]. In addition, Czelakowski in [31] considers a parameterized version of the LDDT for 1-deductive systems in which the sets of formulas E_i of the local deduction-detachment system $\mathcal{E}$ may contain variables, called *parameters*, distinct from p and q. Among other things he shows that an arbitrary deductive system has the parameterized LDDT iff it is protoalgebraic, and that it filter-distributive iff it has a directed parameterized deduction-detachment system.

For other results related to the deduction theorem in an algebraic context see also [30, 34, 35]. The monograph [32] contains a comprehensive survey the results in this area.

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