SOLUTIONS PROBLEM SET # 5

1. WE DENOTE THE SYMMETRIZATIONS OF α AND β RESPECTIVELY BE α^* AND β^*. SO $\alpha^* = \alpha \cap \breve{\alpha}$ AND $\beta^* = \beta \cap \breve{\beta}$.

THUS $\underset{\smile}{A}^{\leq}/\beta = \langle \underset{\smile}{A}/\beta^*, \beta/\beta^* \rangle$ AND

$$\alpha/\beta = \alpha/\beta^* = \{ \langle a/\beta^*, b/\beta^* \rangle : a \alpha b \}$$

WE FIRST PROVE THAT $\forall a, a', b, b' \in A$

$$(*) \quad a' \alpha b',\ a/\beta^* = a'/\beta^*,\ \text{AND } b/\beta^* = b'/\beta^* \implies a \alpha b.$$

(SINCE $\beta \subseteq \alpha$, $a/\beta^* = a'/\beta^* \implies a \beta a' \implies a \alpha a'$. SIMILARLY, $b/\beta^* = b'/\beta^* \implies b \breve{\beta} b' \implies b' \beta b \implies b' \alpha b$. THUS $a \alpha a' \alpha b' \alpha b$, SO $a \alpha b$).

FROM (*) WE GET

$$(**) \quad a/\beta^* \ (\alpha/\beta)\ b/\beta^* \implies a \alpha b.$$

($a/\beta^* \ (\alpha/\beta)\ b/\beta^* \implies \exists\, a', b' \in A$ S.T. $a' \alpha b'$, $a/\beta^* = a'/\beta^*$, AND $b/\beta^* = b'/\beta^*$. SO $a \alpha b$ BY (*)).

WE NOW SHOW THAT α/β IS A QUASI-ORDER OF $\underset{\smile}{A}^{\leq}/\beta$.

<u>REFLEXIVITY</u>: $a/\beta^* \ (\alpha/\beta)\ a/\beta^*$ BECAUSE $a \alpha a$.

TRANSITIVITY.

$$a/\beta^* \ (\alpha/\beta)\ b/\beta^* \text{ AND } b/\beta^* \ (\alpha/\beta)\ c/\beta^* \overset{(**)}{\Longrightarrow}$$
$$a \,\alpha\, b \,\alpha\, c \Rightarrow a \,\alpha\, c \Rightarrow a/\beta^* \ (\alpha/\beta^*)\ c/\beta^*.$$

SUBSTITUTION.

$$\forall i \leq m \ (a_i/\beta^* \ (\alpha/\beta)\ b_i/\beta^*) \overset{(**)}{\Longrightarrow}$$
$$\forall i \leq m \ (a_i \,\alpha\, b_i) \Rightarrow \sigma^{\underline{A}}(\bar{a}) \,\alpha\, \sigma^{\underline{A}}(\bar{b})$$
$$\Rightarrow \sigma^{\underline{A}/\beta^*}(a_1/\beta^*, \ldots, a_m/\beta^*) \ \alpha/\beta \ \sigma^{\underline{A}/\beta^*}(b_1/\beta^*, \ldots, b_m/\beta^*)$$

$\beta \subseteq \alpha \Rightarrow \beta/\beta^* \subseteq \alpha/\beta^* = \alpha/\beta$ SO

α/β IS A QUASI-ORDER OF $\underline{A}^{\leq}/\beta$.

SINCE $\beta \subseteq \alpha$, BY THE ORDERED HOMOMORPHISM THEOREM $\exists\, h : \underline{A}^{\leq}/\beta \twoheadrightarrow \underline{A}^{\leq}/\alpha$ SUCH THAT $h(a/\beta^*) = a/\alpha^* \ \forall a \in A$.

$$\langle a/\beta^*, b/\beta^* \rangle \in h^{-1}(\alpha/\alpha^*)$$
$$\Leftrightarrow h(a/\beta^*) \ (\alpha/\alpha^*)\ h(b/\beta^*)$$
$$\Leftrightarrow a/\alpha^* \ (\alpha/\alpha^*)\ b/\alpha^*$$
$$\underset{(*)}{\Leftrightarrow} a \,\alpha\, b$$
$$\Leftrightarrow a/\beta^* \ (\alpha/\beta^*)\ b/\beta^*$$

SO $h^{-1}(\alpha/\alpha^*) = \alpha/\beta^* = \alpha/\beta$

SO BY THE ORDER ISOMORPHISM THEOREM

$$\underline{A}^{\leq}/\alpha \cong (\underline{A}^{\leq}/\beta)/(\alpha/\beta)$$

2. LET $h : \underset{\frown}{A}^{\leq} \to \underset{\frown}{B}^{\leq}$ BE AN ORDER HOMOMORPHISM. LET $\underset{\frown}{C} = h(\underset{\frown}{A}) \leq \underset{\frown}{B}$ AND LET $\underset{\frown}{C}^{\leq} = \langle \underset{\frown}{C}, \leq^{\underset{\frown}{B}} \cap C^2 \rangle$. THEN $\underset{\frown}{C}^{\leq}$ IS AN ORDERED SUBALGEBRA OF $\underset{\frown}{B}^{\leq}$.

LET $h^* : \underset{\frown}{A} \to \underset{\frown}{C}$ BE THE SURJECTIVE HOMOMORPHISM WITH THE SAME GRAPH AS h. SINCE h IS AN ORDER HOMOMORPHISM,

$a \leq^{\underset{\frown}{A}} a' \Rightarrow h(a) \leq^{\underset{\frown}{B}} h(a') \Rightarrow$
$h^*(a) \; (\leq^{\underset{\frown}{B}} \cap C^2) \; h^*(a')$

SO $h^* : \underset{\frown}{A}^{\leq} \twoheadrightarrow \underset{\frown}{C}^{\leq}$ IS A SURJECTIVE ORDER HOMOMORPHISM.

$h^*(a) \leq^{\underset{\frown}{C}} h^*(a') \iff h(a) \leq^{\underset{\frown}{B}} h(a')$

SO $h^{*-1}(\leq^{\underset{\frown}{C}}) = h^{-1}(\leq^{\underset{\frown}{B}}) = \alpha$

SO BY THE ORDERED ISOMORPHISM THM $\underset{\frown}{A}^{\leq}/\alpha \cong \underset{\frown}{C}^{\leq}$.

3. LET E BE THE SET OF INEQUATIONS

$$(*) \quad (x+y)+z \preccurlyeq x+(y+z), \quad (x+y)+z \succcurlyeq x+(y+z)$$
$$x+y \preccurlyeq y+z, \quad y+x \preccurlyeq x+y$$
$$(x\cdot y)\cdot z \preccurlyeq x\cdot(y\cdot z), \quad x\cdot(y\cdot z) \preccurlyeq (x\cdot y)\cdot z$$
$$x\cdot(y+z) \preccurlyeq x\cdot y + x\cdot z, \quad x\cdot y + x\cdot z \preccurlyeq x\cdot(y+z)$$
$$(y+z)\cdot x \preccurlyeq y\cdot x + z\cdot x, \quad y\cdot x + z\cdot x \preccurlyeq (y+z)\cdot x$$

LET $\mathbb{K} = \text{MOD}(E)$ AND LET $\underset{\frown}{A}^{\leq} = \langle \underset{\frown}{A}, \leq \rangle \in \mathbb{K}$. BY DEFINITION OF MODEL $\leq$ IS A PARTIAL ODERING OF A WITH THE SUBSTITUTION PROPERTY

BY $(*)$, $\forall a, b, c \in A$, WE HAVE
$(a +^{A} b) +^{A} c \leq^{A} a +^{A} (b +^{A} c)$ AND
$a +^{A} (b +^{A} c) \leq^{A} (a +^{A} b) +^{A} c$.
SO $(a +^{A} b) +^{A} c = a +^{A} (b +^{A} c)$
SINCE $\leq^{A}$ IS A PARTIAL ORDERING.
SO $+^{A}$ IS ASSOCIATIVE. SIMILARLY, $+^{A}$ IS COMMUTATIVE, $\cdot^{A}$ IS ASSOCIATIVE, AND $\cdot^{A}$ DISTRIBUTES OVER $+^{A}$. THUS $\underset{\frown}{A}^{\leq}$ IS A PARTIALLY ORDERED SEMIRING.
CONVERSELY, IF $\underset{\frown}{A}^{\leq}$ IS A PARTIALLY ORDERED SEMIRING, THEN $\underset{\frown}{A}^{\leq} \in \text{MOD}(E)$.

4. LET $\underset{\frown}{A}^{\leq} \in V$, AND CONSIDER THE MAPPING $\alpha \mapsto \alpha \cap \breve{\alpha} = \alpha^*$ FROM $\text{QORD}(\underset{\frown}{A}^{\leq})$ INTO $\text{Co}(\underset{\frown}{A})$. ASSUME V IS ALGEBRAIZABLE.
LET $\alpha \in \text{QORD}(\underset{\frown}{A}^{\leq})$

<u>CLAIM</u>: $\forall a, b \in A$
$\langle a, b \rangle \in \alpha \iff \forall i \leq m \left(\delta_i^{A}(a,b) \; \alpha^* \; \varepsilon_i^{A}(a,b) \right)$

<u>PROOF</u>.
$\langle a, b \rangle \in \alpha \iff \langle a/\alpha^*, b/\alpha^* \rangle \in \alpha/\alpha^*$

$\iff$ (SINCE $\underline{A}/\alpha \in V$) $\quad t_u^{\underline{A}/\alpha^*}(a/\alpha^*, b/\alpha^*) \ (\alpha/\alpha^*) \ s_u^{\underline{A}/\alpha^*}(a/\alpha^*, b/\alpha^*)$
AND $s_u^{\underline{A}/\alpha^*}(a/\alpha^*, b/\alpha^*) \ (\alpha/\alpha^*) \ t_u^{\underline{A}/\alpha^*}(a/\alpha^*, b/\alpha^*)$

$\iff \quad t_u^{\underline{A}}(a,b)/\alpha^* \ (\alpha/\alpha^*) \ s_u^{\underline{A}}(a,b)/\alpha^*$
AND $s_u^{\underline{A}}(a,b)/\alpha^* \ (\alpha/\alpha^*) \ t_u^{\underline{A}}(a,b)/\alpha^*$

$\iff$ (LEM 7.4) $\quad t_u^{\underline{A}}(a,b) \ \alpha \ s_u^{\underline{A}}(a,b)$ AND
$s_u^{\underline{A}}(a,b) \ \alpha \ t_u^{\underline{A}}(a,b)$

$\iff \quad t_u^{\underline{A}}(a,b) \ \alpha^* \ s_u^{\underline{A}}(a,b) \quad \forall u \le m$

$\boxtimes$ CLAIM

SUPPOSE $\alpha \cap \breve{\alpha} = \beta \cap \breve{\beta}$. BY CLAIM $\alpha = \beta$.
SO THE MAPPING $\alpha \mapsto \alpha \cap \breve{\alpha}$ IS INJECTIVE

LET $\underline{\omega}^{\Delta\omega} = \langle \langle \omega, +, \cdot \rangle, \Delta_\omega \rangle$ WHERE $\langle \omega, +, \cdot \rangle$ IS THE SEMIRING OF NATURAL NUMBERS. THEN $\underline{\omega}^{\Delta\omega}$ IS A PARTIALLY ORDERED SEMIRING. LET $\le$ BE THE STANDARD ORDER ON THE INTEGERS. THEN $\alpha = \le$ AND $\beta = \ge = \breve{\alpha}$ ARE DISTINCT QUASI-ORDERS OF $\underline{\omega}^{\Delta\omega}$ SUCH THAT $\alpha \cap \breve{\alpha} = \beta \cap \breve{\beta} = \Delta_\omega$. SO THE ORDERED VARIETY OF PARTIALLY ORDERED SEMIRINGS IS NOT ALGEBRAIZABLE BY PROBLEM 3

5. ASSUME $\mathbb{V}$ IS ALGEBRAIZABLE; LET $h : \underset{\sim}{A}^{\leq} \to \underset{\sim}{B}^{\leq}$ BE AN ORDER HOMOMORPHISM. IF h IS AN ORDER ISOMORPHISM, THEN h IS A BIJECTION BY DEFINITION. ASSUME IT IS A BIJECTION. TO SHOW h IS AN ORDER ISOMORPHISM IT SUFFICE TO SHOW THAT $h^{-1} : \underset{\sim}{B} \twoheadrightarrow \underset{\sim}{A}$ IS ORDER-PRESERVING, I.E., $\forall a, a' \in A \ (h(a) \leq^{B} h(a') \Rightarrow a \leq^{A} a')$ SUPPOSE NOT. THEN $\exists\, a, a' \in A$ S.T. $a \not\leq^{A} a'$ BUT $h(a) \leq^{B} h(a')$, I.E., $\langle a, a' \rangle \in h^{-1}(\leq^{B}) \setminus \leq^{A}$. SO $h^{-1}(\leq^{B})$ AND $\leq^{A}$ ARE DISTINCT QUASI-ORDERS ON $A^{\leq}$ AND $h^{-1}(\leq^{B}) \cap h^{-1}(\geq^{B}) = h^{-1}(\leq^{B} \cap \geq^{B}) = h^{-1}(\Delta_B) \underset{\uparrow}{=} \Delta_A = \leq^{A} \cap \geq^{A}$.

SINCE h IS INJECTIVE

SINCE $\mathbb{V}$ IS ALGEBRAIZABLE, THIS IS IMPOSSIBLE BY PROBLEM 4.

LET $\underset{\sim}{\omega}^{\Delta_\omega}$ AND $\underset{\sim}{\omega}^{\leq}$ BE THE PARTIALLY ORDERED SEMIRINGS OF NATURAL NUMBERS WITH THE IDENTITY ORDER Δ_ω AND NATURAL ORDER $\leq$, RESPECTIVELY. THEN THE IDENTITY $\Delta_\omega : \underset{\sim}{\omega}^{\Delta_\omega} \twoheadrightarrow \underset{\sim}{\omega}^{\leq}$ IS A BIJECTIVE ORDER HOMOMORPHISM THAT IS NOT AN ORDER ISOMORPHISM.