

Corrected May 7, 2002

1. Prove the Ordered Second Isomorphism Theorem: Let \mathbf{A}^{\leq} be a poalgebra and $\alpha, \beta \in \text{Qord}(\mathbf{A}^{\leq})$ such that $\beta \subseteq \alpha$. Define

$$\alpha/\beta = \alpha/(\beta \cap \check{\beta}) = \{ \langle a/\beta \cap \check{\beta}, b/\beta \cap \check{\beta} \rangle : a \alpha b \}.$$

Prove that α/β is a quasi-order of \mathbf{A}^{\leq}/β and that $\mathbf{A}^{\leq}/\alpha \cong (\mathbf{A}^{\leq}/\beta)/(\alpha/\beta)$.

2. Let $h: \mathbf{A}^{\leq} \rightarrow \mathbf{B}^{\leq}$ be an order homomorphism of ordered algebras, and let α be the order-kernel of h . Prove that there is an ordered subalgebra \mathbf{C}^{\leq} of \mathbf{B}^{\leq} such that $\mathbf{A}^{\leq}/\alpha \cong \mathbf{C}^{\leq}$.

[Note: This result was used at a couple places in the with the remark that it follows from the ordered homomorphism theorem; this is true, but the proof is not trivial.]

3. A *semiring* is an algebra $\langle A, +, \cdot \rangle$ such that $\langle A, + \rangle$ is a commutative semigroup, $\langle A, \cdot \rangle$ is a semigroup, and \cdot distributives over $+$. A *partially ordered semiring* is a poalgebra $\langle \mathbf{A}, \leq \rangle$ such that \mathbf{A} is a semiring. Prove that the class \mathbf{K} of partially ordered semirings form an ordered variety, i.e., find a set E of inequations and prove that $\mathbf{K} = \text{Mod}(E)$

An ordered variety \mathbf{V} is *algebraizable* if there is a finite set of inequations $t_i(x, y) \preceq s_i(x, y)$, $i = 1, \dots, n$, such that, for every $\mathbf{A}^{\leq} \in \mathbf{V}$ and for all $a, b \in A$,

$$a \leq^{\mathbf{A}} b \quad \text{iff} \quad \forall i \leq n (t_i^{\mathbf{A}}(a, b) \leq^{\mathbf{A}} s_i^{\mathbf{A}}(a, b) \text{ and } s_i^{\mathbf{A}}(a, b) \leq^{\mathbf{A}} t_i^{\mathbf{A}}(a, b)).$$

The ordered variety of lattices is algebraizable; take $n = 1$ and $t_1(x, y) \preceq s_1(x, y)$ to be $x \vee y \preceq y$.

4. Let \mathbf{V} be an algebraizable ordered variety. Prove that, for every $\mathbf{A} \in \mathbf{V}$, the mapping $\alpha \mapsto \alpha \cap \check{\alpha}$ from $\text{Qord}(\mathbf{A}^{\leq})$ to $\text{Co}(\mathbf{A})$ is injective. Use this to show that the ordered variety of semirings is not algebraizable.

Most of the definitions and theorems about poalgebras are exact analogues of those about unordered algebras, but there are some exceptions as the next problem shows.

5. Let \mathbf{V} be an algebraizable ordered variety. Prove that, if $\mathbf{A}^{\leq}, \mathbf{B}^{\leq} \in \mathbf{V}$, then an order homomorphism $h: \mathbf{A}^{\leq} \rightarrow \mathbf{B}^{\leq}$ is an order isomorphism iff h is a bijection. Show by example that this may not be true if \mathbf{V} is not algebraizable (take \mathbf{V} to be the class of partially ordered semirings).