SOLUTIONS PROBLEM SET # 4

#1. (a) LET $t_i(x,y,z,w) \approx s_i(x,y,z,w)$, $i \le m$, DEFINE PRINCIPAL MEETS IN $\mathcal{V}$. WE FIRST PROVE THAT FOR EVERY SURJECTIVE HOMOMORPHISM $h: \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{B}$ AND ALL $X \subseteq A^2$

$$(*) \qquad h^*(\Theta_{\underset{\sim}{A}}(X)) = \Theta_{\underset{\sim}{B}}(h(X)).$$

BY DEFINITION $h^*(\Theta_{\underset{\sim}{A}}(X)) = \Theta_{\underset{\sim}{B}}(h(\Theta_{\underset{\sim}{A}}(X)))$. $X \subseteq \Theta_{\underset{\sim}{A}}(X)$, SO $h(X) \subseteq h(\Theta_{\underset{\sim}{A}}(X))$. SO $\Theta_{\underset{\sim}{B}}(h(X)) \subseteq \Theta_{\underset{\sim}{B}}(h(\Theta_{\underset{\sim}{A}}(X))) = h^*(\Theta_{\underset{\sim}{A}}(X))$. FOR THE REVERSE INCLUSION, NOTE THAT $X \subseteq h^{-1}(\Theta_{\underset{\sim}{B}}(h(X))) \in \mathrm{Co}(\underset{\sim}{A})$. THUS $\Theta_{\underset{\sim}{A}}(X) \subseteq h^{-1}(\Theta_{\underset{\sim}{B}}(h(X)))$. SO BY SET THEORY, SINCE h IS SURJECTIVE,

$$h(\Theta_{\underset{\sim}{A}}(X)) \subseteq h h^{-1}(\Theta_{\underset{\sim}{B}}(h(X))) = \Theta_{\underset{\sim}{B}}(h(X))$$

CONSEQUENTLY, $h^*(\Theta_{\underset{\sim}{A}}(X)) = \Theta_{\underset{\sim}{B}}(h(\Theta_{\underset{\sim}{A}}(X))) \subseteq \Theta_{\underset{\sim}{B}}(h(X))$. THIS GIVES $(*)$.

$$
\begin{aligned}
&\Delta_\alpha^*(\Theta_{\underset{\sim}{A}}(c,d) \cap \Theta_{\underset{\sim}{A}}(e,f)) \\
&= \Delta_\alpha^*\Big(\bigvee_{i \le m} \Theta_{\underset{\sim}{A}}(t_i^{\underset{\sim}{A}}(a,e,c,d), s_i^{\underset{\sim}{A}}(a,e,c,d))\Big) \\
&\underset{\text{LEM } 5.15(1)}{=} \bigvee_{i \le m} \Delta_\alpha^*(\Theta_{\underset{\sim}{A}}(t_i^{\underset{\sim}{A}}(a,e,c,d), s_i^{\underset{\sim}{A}}(a,e,c,d))) \\
&\underset{(*)}{=} \bigvee_{i \le m} \Theta_{\underset{\sim}{B}}(\Delta_\alpha(t_i^{\underset{\sim}{A}}(a,e,c,d)), \Delta_\alpha(s_i^{\underset{\sim}{A}}(a,e,c,d))) \\
&\underset{\Delta_\alpha\ \text{HOMO}}{=} \bigvee_{i \le m} \Theta_{\underset{\sim}{B}}(t_i^{\underset{\sim}{B}}(\bar{a},\bar{e},\bar{c},\bar{d}), s_i^{\underset{\sim}{B}}(\bar{a},\bar{e},\bar{c},\bar{d}))
\end{aligned}
$$

$= \Theta_{\underset{\frown}{B}}(\bar{c}, \bar{d}) \cap \Theta_{\underset{\frown}{B}}(\bar{e}, \bar{f})$

$\Theta_{\underset{\frown}{B}}(\bar{c}, \bar{d}) \overset{(*)}{=} \Delta^*_\alpha(\Theta_{\underset{\frown}{A}}(c,d)) \subseteq \Delta^*_\alpha(\Theta_{\underset{\frown}{A}}(a,b) \vee \alpha)$

$= \Delta^*_\alpha(\Theta_{\underset{\frown}{A}}(a,b)) \vee \Delta^*_\alpha(\alpha) \overset{(*)}{=} \Theta_{\underset{\frown}{B}}(\bar{a}, \bar{b}) \vee \Delta_B$

$= \Theta_{\underset{\frown}{B}}(\bar{a}, \bar{b})$

THUS $\Theta_{\underset{\frown}{B}}(\bar{c}, \bar{d}) \subseteq \Theta_{\underset{\frown}{B}}(\bar{a}, \bar{b})$ AND THUS

$$\underbrace{\Theta_{\underset{\frown}{B}}(\bar{c}, \bar{d}) \cap \Theta_{\underset{\frown}{B}}(\bar{e}, \bar{f})}_{\Delta^*_\alpha(\Theta_{\underset{\frown}{A}}(c,d) \cap \Theta_{\underset{\frown}{A}}(e,f))} \subseteq \underbrace{\Theta_{\underset{\frown}{B}}(\bar{a}, \bar{b}) \cap \Theta_{\underset{\frown}{B}}(\bar{e}, \bar{f})}_{\Delta^*_\alpha(\Theta_{\underset{\frown}{A}}(a,b) \cap \Theta_{\underset{\frown}{A}}(e,f))}$$

APPLYING Δ^{*-1}_α TO BOTH SIDES AND USING LEM 5.14 (3) WE GET

$$(\Theta_A(c,d) \cap \Theta_{\underset{\frown}{A}}(e,f)) \vee \alpha \subseteq (\Theta_A(a,b) \cap \Theta_{\underset{\frown}{A}}(e,f)) \vee \alpha.$$

THIS GIVES (a).

(b) ASSUME $\Theta_A(c,d) \subseteq \bigvee_{i \le m} \Theta_A(a_i, b_i) \vee \alpha$

WE PROVE THAT

(**) $\Theta_{\underset{\frown}{A}}(c,d) \cap \Theta_{\underset{\frown}{A}}(e,f) \subseteq \bigvee_{i \le m} (\Theta_A(a_i, b_i) \cap \Theta_{\underset{\frown}{A}}(e,f)) \vee \alpha$

BY INDUCTION ON m.

$m = 1$ IS JUST PART (a).

ASSUME $m > 1$.

TAKING α TO BE $\Theta_{A}(a_m, b_m) \vee \alpha$
WE HAVE BY THE INDUCTION HYPOTHESIS

$$\Theta_{A}(c,d) \cap \Theta_{A}(e,f) \subseteq$$
$$\Theta_{A}(a_m, b_m) \vee \underbrace{\bigvee_{i \leq m-1} \left(\Theta_{A}(a_i, b_i) \cap \Theta_{A}(e,f) \right) \vee \alpha}_{\beta}$$

CONSIDER ANY $\langle x, y \rangle \in \Theta_{A}(c,d) \cap \Theta_{A}(e,f)$.

$$\Theta_{A}(x,y) \subseteq \Theta_{A}(a_m, b_m) \vee \beta.$$

SO BY PART (a).

$$\Theta_{A}(x,y) \cap \Theta_{A}(e,f) \subseteq \left(\Theta_{A}(a_m, b_m) \cap \Theta_{A}(e,f) \right) \vee \beta$$

BUT $\Theta_{A}(x,y) \subseteq \Theta_{A}(e,f)$. SO

$$\Theta_{A}(x,y) \subseteq \left(\Theta_{A}(a_m, b_m) \cap \Theta_{A}(e,f) \right) \vee \beta$$
$$= \bigvee_{i \leq m} \left(\Theta_{A}(a_i, b_i) \cap \Theta_{A}(e,f) \right) \vee \alpha$$

THIS HOLDS FOR EVERY $\langle x, y \rangle \in$
$\Theta_{A}(c,d) \cap \Theta_{A}(e,f)$. HENCE, FINALLY,

$$\Theta_{A}(c,d) \cap \Theta_{A}(e,f) \subseteq \bigvee_{i \leq m} \left(\Theta_{A}(a_i, b_i) \cap \Theta_{A}(e,f) \right) \vee \alpha$$

TAKING $\alpha = \Delta_{A}$ WE GET PART (b).

#2 LET $K = \{\Theta_{\mathbf{A}}(X) : X \subseteq_\omega \beta \cup \gamma\}$

K IS UPWARD-DIRECTED. BECAUSE IF $\Theta_{\mathbf{A}}(X), \Theta_{\mathbf{A}}(Y) \in K$ THEN $\Theta_{\mathbf{A}}(X), \Theta_{\mathbf{A}}(Y) \subseteq \Theta_{\mathbf{A}}(X \cup Y)$.

THUS $\bigcup K \in Co(\mathbf{A})$. $X \subseteq \beta \cup \gamma \Rightarrow \Theta_{\mathbf{A}}(X) \subseteq \Theta_{\mathbf{A}}(\alpha \cup \gamma) = \beta \vee \gamma$. CLEARLY $\beta, \gamma \subseteq \bigcup K$, SO $\beta \vee \gamma \subseteq \bigcup K$. HENCE $\beta \vee \gamma = \bigcup K$

LET $\langle a, b \rangle \in \alpha \cap (\beta \vee \gamma)$. THEN $\langle a, b \rangle \in \alpha$ AND $\langle a, b \rangle \in \beta \vee \gamma = \bigcup K$.

LET $X \in K$ SUCH THAT $\langle a, b \rangle \in \Theta_{\mathbf{A}}(X)$

LET $X = \underbrace{\{\langle c_i, d_i \rangle : i \leq n\}}_{X_\beta} \cup \underbrace{\{\langle e_j, f_j \rangle : i \leq m\}}_{X_\gamma}$

WITH $X_\beta \subseteq \beta$ AND $X_\gamma \subseteq \gamma$.

$\Theta_{\mathbf{A}}(a, b) \in \Theta_{\mathbf{A}}(X_\beta) \vee \Theta_{\mathbf{A}}(X_\gamma)$, DIRECTLY FROM 1 b) WE GET

$$\Theta_{\mathbf{A}}(a, b) = \Theta_{\mathbf{A}}(a, b) \cap \Theta_{\mathbf{A}}(a, b) \subseteq$$

$$\Big(\bigvee_{i \leq m} \underbrace{(\Theta_{\mathbf{A}}(c_i, d_i) \cap \Theta_{\mathbf{A}}(a, b))}_{\subseteq \beta \cap \alpha}\Big) \vee \Big(\bigvee_{j \leq m} \underbrace{(\Theta_{\mathbf{A}}(e_j, f_j) \cap \Theta_{\mathbf{A}}(a, b))}_{\subseteq \gamma \cap \alpha}\Big)$$

SO $\langle a, b \rangle \in (\beta \cap \alpha) \vee (\alpha \cap \gamma)$. THUS

$\alpha \cap (\beta \vee \gamma) \subseteq (\beta \cap \alpha) \vee (\alpha \cap \gamma)$ $\boxtimes$

#3 (b) By Birkhoff's subdirect product theorem for every Σ-algebra $\underline{A}$ we have $\underline{A} \cong_j \subseteq_{SD} \prod_{i \in I} \underline{B}_i$ where the $\underline{B}_i$ are subdirectly irreducible (SDI). The algebras $\underline{B}_i$, $i \in I$, are called the SDI factors of $\underline{A}$. For any class of Σ-algebras K, let $\mathbb{F}_{SDI}(K)$ be the class of SDI factors of all $\underline{A} \in K$.

Claim. $HSP(K) = HSP\,F_{SDI}(K)$.

Proof. By Birkhoff theorem,
$K \subseteq P_{SD}\,F_{SDI}(K)$. So $HSP(K) \subseteq HSPP_{SD}\,F_{SDI}(K)$
$\subseteq HSP\,F_{SDI}(K)$. $F_{SDI}(K) \subseteq H(K)$. So
$HSPF_{SDI}(K) \subseteq HSPH(K) = HSP(K)$ $\boxtimes$ claim.

Here are same additional facts that were established in the lectures and that we will need.

(*) $\mathbb{F}_{SDI}(Mod_{prim}(\Phi) \cap V) \subseteq Mod(\Phi) \cap V$

This is an immediate consequence of Lem 6.3.

(**) $HS(Mod(\Phi)) \subseteq Mod(\Phi)$.

In Theorem 6.4 we proved that $Mod_{prim}(\Phi) \cap V$ is closed under H and S
(**) is much easier to prove.

$$(***)\quad P_{SD}(\mathrm{Mod}(\Phi)\cap V) \subseteq \mathrm{Mod}_{\mathrm{PRIM}}(\Phi)\cap V.$$

IN THE PROOF OF LEM 6.3 WE PROVED THAT IF $\underset{\sim}{A}$ IS CONGRUENCE-DISTRIBUTIVE AND $\underset{\sim}{A} \cong; \subseteq_{SD} \prod_{i\in I} \underset{\sim}{B}_i$ WITH $\underset{\sim}{B}_i$ SDI AND $\underset{\sim}{B}_i \in \mathrm{Mod}(\Phi)$ $\forall i \in I$, THEN $A \in \mathrm{Mod}_{\mathrm{PRIM}}(\Phi)$. AN EXAMINATION OF THE PROOF SHOWS THAT THE CONDITION THAT THE B_i ARE SDI IS NOT USED IN THIS PART OF PROOF. THIS GIVES (***).

JONSSON'S LEMMA (THM 5.25) SAYS THAT FOR EVERY CLASS K OF Σ-ALGEBRAS SUCH THAT $HSP(K)$ IS CONGRUENCE-DISTRIBUTIVE

$$(****)\quad HSP(K) = P_{SD}\, HSP_U(K).$$

NOW FOR THE SOLUTION OF THE PROBLEM.

$$\begin{aligned}
& HSP(\mathrm{Mod}_{\mathrm{PRIM}}(\Phi)\cap V) \\
& \overset{\text{CLAIM}}{=} HSP(F_{SDI}(\mathrm{Mod}_{\mathrm{PRIM}}(\Phi)\cap V)) \\
& \overset{(*)}{\subseteq} HSP(\mathrm{Mod}(\Phi)\cap V) \\
& \overset{(****)}{=} P_{SD}\, HSP_U(\mathrm{Mod}(\Phi)\cap V) \\
& \overset{\text{PART (a)}}{=} P_{SD}\, HS(\mathrm{Mod}(\Phi)\cap V) \\
& \overset{(**)}{\subseteq} P_{SD}(\mathrm{Mod}(\Phi)\cap V) \\
& \overset{(***)}{\subseteq} \mathrm{Mod}_{\mathrm{PRIM}}(\Phi)\cap V.
\end{aligned}$$

SO $HSP(Mod(\Phi) \cap K) \subseteq Mod_{PRIM}(\Phi) \cap K$.
THE INCLUSION IN OPPOSITE DIRECTION IS PROVED AS IN PROOF OF THM 6.4.

(a) LET $\underset{\frown}{B} = (\prod \underset{\frown}{A_i}) / \Phi(\mathcal{U})$ WITH $\mathcal{U}$ AN ULTRAFILTER AND ASSUME $\underset{\frown}{A_i} \in Mod(\Phi)$.
THEN $\forall \varphi \in \Phi$, $\{ i \in I : \underset{\frown}{A_i} \models \varphi \} = I \in \mathcal{U}$.
EVERY UDE IS A DISJUNCTION OF EQUATIONS OR LOGICAL NEGATIONS OF EQUATIONS (WHERE THERE ARE NO LOGICAL NEGATIONS). SO BY PROBLEM 1 OF PROBLEM SET 2.
$\underset{\frown}{B} \models \varphi$. SO $\underset{\frown}{B} \in Mod(\Phi)$. THUS
$P_U(Mod(\Phi)) \subseteq Mod(\Phi)$.

#4. AN EQUATION $\sigma \approx \mu$ IS <u>ABSORBING</u> IF EITHER σ OR μ IS A VARIABLE
ASSUME E IS A SET OF Σ-EQUATIONS NONE OF WHICH IS ABSORBING. WE WANT TO SHOW E IS CONSISTENT.
WE DO THIS TWO DIFFERENT WAYS.
1. WE PROVE THAT, FOR ANY VARIABLE x, AND TERM σ

$$E \vdash x \approx \sigma \implies \sigma \text{ IS } x.$$

SUPPOSE $E \vdash x \approx \sigma$. THEN $x \equiv^*_E \sigma$
THEN EITHER σ IS x OR $\exists\ \mu_1, \ldots, \mu_n$ SUCH THAT

$$x \equiv_E \mu_1 \equiv_E \mu_2 \equiv_E \cdots \equiv_E \mu_n = \sigma$$

$x \equiv_E u_1$ IMPLIES THERE IS A $\nu(\hat{y}) \approx \nu'(\hat{y}) \in \breve{E} \cup \check{E}$ AND A SEQUENCE OF TERMS $\hat{w}$ SUCH THAT u_1 IS OBTAINED FROM x BY REPLACING A SUBTERM OF x OF THE FORM $\nu(\hat{y})$ BY $\nu'(\hat{y})$. BUT THIS IS IMPOSSIBLE; $\nu(\hat{y})$ IS NOT A VARIABLE BY ASSUMPTION. SO $\nu(\hat{w})$ IS NOT A VARIABLE. BUT x HAS ONLY ITSELF AS A SUBTERM. THUS τ MUST BE x. SO $E \nvdash x \approx y$ IF x, y ARE DIFFERENT VARIABLES, AND HENCE E IS CONSISTENT.

WE GIVE THE ALTERNATIVE MODEL-THEORETIC ~~BY~~ BY CONSTRUCTING A TWO-ELEMENT MODEL OF E. LET $A = \{0, 1\}$ AND, FOR EACH $\sigma \in \Sigma_m$, DEFINE $\sigma^{\underline{A}}: A^m \to A$ BY SETTING $\sigma^{\underline{A}}(\hat{a}) = 0 \ \forall \hat{a} \in A^m$. CONSIDER ANY NONVARIABLE Σ-TERM $\tau(\hat{x})$. THEN $\tau(\hat{x}) = \sigma(u_1(\hat{x}), \dots, u_m(\hat{x}))$ FOR SOME $\sigma \in \Sigma_m$. THUS $\forall \hat{a} \in A^m$, $\tau^{\underline{A}}(\hat{a}) = \sigma^{\underline{A}}(u_1^{\underline{A}}(\hat{a}), \dots, u_m^{\underline{A}}(\hat{a})) = 0$. CONSEQUENTLY, $\forall \tau(\hat{x}) \approx \nu(\hat{x})$ IN E AND $\forall \hat{a} \in A^m$ $\tau^{\underline{A}}(\hat{a}) = 0 = \nu^{\underline{A}}(\hat{a})$. SO $\underline{A} = \langle A, \sigma^{\underline{A}} \rangle_{\sigma \in \Sigma}$ IS A NONTRIVIAL MODEL OF E.

#5. LET $\underset{\sim}{F} = Fw_m(V)$, THE FREE ALGEBRA OVER V WRT THE FREE GENERATORS $\hat{x}^{\underset{\sim}{F}} = \langle x_0^{\underset{\sim}{F}}, \ldots, x_{m-1}^{\underset{\sim}{F}} \rangle$. SINCE $\underset{\sim}{F}$ IS FINITELY GENERATED, IT IS FINITE. LET $m = |F|$. CHOOSE TERMS

$t_0(\hat{x}), \ldots, t_{m-1}(\hat{x}) \in Te_\Sigma(\hat{x})$ SUCH THAT $t_0^{\underset{\sim}{F}}(\hat{x}^{\underset{\sim}{F}}), \ldots, t_{m-1}^{\underset{\sim}{F}}(\hat{x}^{\underset{\sim}{F}})$ IS A ENUMERATION WITHOUT REPETITIONS OF ALL m ELEMENTS OF $\underset{\sim}{F}$. ASSUME ALSO THE FIRST m ARE THE VARIABLES $x_0, \ldots, x_{m-1}$, I.E., $t_i(\hat{x}) = x_i$ FOR EACH $i < m$. ASSUME ALSO THAT ALL CONSTANT SYMBOLS IN Σ_0 OCCUR AMONG THE $t_i(\hat{x})$

LET $\sigma \in \Sigma_k$. BY THE <u>MULTIPLICATION TABLE</u> FOR $\sigma^{\underset{\sim}{F}}$ WE MEAN THE SET OF ALL EQUATIONS OF THE FORM

$$\sigma(t_{i_1}(\hat{x}), \ldots, t_{i_k}(\hat{x})) \approx t_j(\hat{x})$$

WHERE $\langle i_1, \ldots, i_k \rangle \in \{0, \ldots, m-1\}^k$ AND j IS THE UNIQUE $j < m$ SUCH THAT

$$t_j^{\underset{\sim}{F}}(\hat{x}^{\underset{\sim}{F}}) = \sigma^{\underset{\sim}{F}}(t_{i_1}^{\underset{\sim}{F}}(\hat{x}^{\underset{\sim}{F}}), \ldots, t_{i_k}^{\underset{\sim}{F}}(\hat{x}^{\underset{\sim}{F}}))$$

LET E_m BE THE UNION OF THESE MULTIPLICATION TABLES AS σ RANGES OVER Σ. WE SHOW THAT

$$V_m = \mathrm{Mod}(E_m)$$

SINCE E_m IS FINITE, THIS SOLVES PROBLEM.

WE FIRST NOTE THAT $E_m \subseteq Id_m$:
EACH EQUATION IN E_m IS AN IDENTITY OF E_m BECAUSE, BY DEF 3.21 AND THM 3.25, IF F IS A FREE ALGEBRA OVER V WITH FREE GENERATORS $x_0^F, \ldots, x_{m-1}^F$, THEN $t(\hat{x}) \approx s(\hat{x})$ IS AN IDENTITY OF V IFF $t^F(\hat{x}^F) = s^F(\hat{x}^F)$.
THUS $V_m = \mathrm{Mod}(Id_m(V)) \subseteq \mathrm{Mod}(E_m)$.

TO SHOW $\mathrm{Mod}(E_m) \subseteq V_m$ IT SUFFICES TO SHOW THAT

(*) $E_m \vdash \varepsilon$ FOR EACH $\varepsilon \in Id_m(V)$

FOR THIS PURPOSE WE PROVE THAT

$$(**)\left\{\begin{array}{l}\forall\, s(\hat{x}) \in Te_Z(\hat{x})\ \exists\, l < m \text{ SUCH THAT} \\ \qquad E_m \vdash s(\hat{x}) \approx t_l(\hat{x})\end{array}\right.$$

THE PROOF GOES BY STRUCTURAL INDUCTION. SUPPOSE $s(\hat{x})$ IS A VARIABLE OR CONSTANT SYMBOL. THEN $s(\hat{x})$ IS ACTUALLY ONE OF THE $t_l(\hat{x})$ AND SO $s(\hat{x}) \approx t_l(\hat{x})$ IS A TAUTOLOGY. ASSUME NOW THAT $s(\hat{x}) = \sigma(s_1(\hat{x}), \ldots, s_h(\hat{x}))$. BY THE INDUCTION HYPOTHESIS $\exists\, l_1, \ldots, l_h < m$ SUCH THAT $E_m \vdash s_i(\hat{x}) \approx t_{l_i}(\hat{x})$ $\forall\, i \leq h$.
THUS $E_m \vdash \sigma(t_{l_1}(\hat{x}), \ldots, t_{l_h}(\hat{x}))$ BY (REPL).
BUT $\sigma(t_{l_1}(\hat{x}), \ldots, t_{l_h}(\hat{x})) \approx t_l(\hat{x}) \in E_m$ FOR SOME $l < m$. SO

SO $E_m \vdash \lambda(\hat{x}) \approx t_\ell(\hat{x})$ BY (TRAN)

THIS GIVES $(*)$.

~~SUPPOSE~~ $\varepsilon \in Id_m(V)$. LET $\varepsilon = \left(u(\hat{x}) \approx v(\hat{x})\right)$

BY $(**)$ $E_m \vdash u(\hat{x}) \approx t_{\ell_1}(\hat{x}),\ v(\hat{x}) \approx t_{\ell_2}(\hat{x})$.

SINCE F IS A MODEL OF E_m

$t_{\ell_1}^F(\hat{x}^F) = u^F(\hat{x}^F) = v^F(\hat{x}^F) = t_{\ell_2}^F(\hat{x}^F)$.

THUS $\ell_1 = \ell_2$ AND $E_m \vdash \varepsilon$ BY (TRAN).

SO $(*)$ HOLDS AND $Mod(E_m) \subseteq Mod(I_m(V))$

THIS GIVES PART (a)

(b). ASSUME V IS FINITELY BASED.

LET E BE A BASE FOR THE IDENTITIES OF V. THEN CLEARLY $E \subseteq E_m$ FOR SOME m. THUS $V = Mod(E) \supseteq Mod(E_m) = V_m$. SO $V = V_m$.

ASSUME V IS LOCALLY FINITE AND $V = V_m$. THEN V IS FINITELY BASED SINCE V_m IS.