SOLUTIONS PROBLEM SET # 4

1. (a) Let \( \varphi(x, y, z, w) \equiv \varphi(x, y, z, w), \) \( \forall x, y, z, w \)

Define principal meets in \( V \). We first prove that for every surjective homomorphism \( \Theta : A \rightarrow B \) and all \( x \in A^2 \)

(\*) \( \varphi^*(\Theta_A(x)) \equiv \Theta_B(\varphi(x)). \)

By definition \( \varphi^*(\Theta_A(x)) \equiv \Theta_B(\varphi(\Theta_A(x))). \)

\( x \in \Theta_A(x) \), so \( \varphi(x) \in \varphi(\Theta_A(x)) \). So \( \Theta_B(\varphi(x)) \in \Theta_B(\varphi(\Theta_A(x))) = \varphi^*(\Theta_A(x)). \)

For the converse (inclusion), note that \( x \in \varphi^{-1}(\Theta_B(\varphi(x))) \in \text{Co}(A) \).

Thus \( \Theta_A(x) \in \varphi^{-1}(\Theta_B(\varphi(x))). \) So by set theory, since \( \varphi \) is surjective,

\( \varphi(\Theta_A(x)) \in \varphi^{-1}(\Theta_B(\varphi(x))) = \Theta_B(\varphi(x)) \) (consequently), \( \varphi^*(\Theta_A(x)) = \Theta_B(\varphi(\Theta_A(x))) \). This gives (\*).

\[ \Delta^*_2(\Theta_A(u, v, d) \land \Theta_A(u, v, e))^2 \]

\[ = \Delta^*_2 \left( \bigvee_{u \leq m} \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \]

\[ = \bigvee_{\leq m} \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \]

\[ = \bigvee_{\leq m} \Theta_B \left( \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \right) \]

\[ = \bigvee_{\leq m} \Theta_B \left( \bigvee_{\leq m} \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \right) \]

\[ = \bigvee_{\leq m} \Theta_B \left( \bigvee_{\leq m} \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \right) \]

\[ = \bigvee_{\leq m} \Theta_B \left( \bigvee_{\leq m} \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \right) \]

\[ = \bigvee_{\leq m} \Theta_B \left( \bigvee_{\leq m} \Delta^*_2 \left( \Theta_A(u, v, d, u) \land \Theta_A(u, v, e, d) \right) \right) \]
\[
\Theta_B(\overline{w}, \overline{a}) \subseteq \Theta_B(\overline{a}, \overline{a}) \\
\Theta_B(\overline{w}, \overline{a}) \subseteq \Delta^*(\Theta_A(w, a)) \leq \Delta^*(\Theta_A(a, e)) \forall \lambda \\
= \Delta^*(\Theta_A(a, e)) \cup \Delta^*(\lambda) = \Theta_B(\overline{a}, \overline{e}) \cup \Delta_B \\
= \Theta_B(\overline{a}, \overline{e})
\]

Thus, \( \Theta_B(\overline{w}, \overline{a}) \subseteq \Theta_B(\overline{a}, \overline{e}) \) and thus,

\[
\Theta_B(\overline{w}, \overline{a}) \cap \Theta_B(\overline{e}, \overline{e}) \subseteq \Theta_B(\overline{a}, \overline{e}) \cap \Theta_B(\overline{e}, \overline{e})
\]

\[
\Delta^*(\Theta_A(w, a)) \cap \Theta_A(e, e) \subseteq \Delta^*(\Theta_A(a, e)) \cap \Theta_A(e, e)
\]

Applying \( \Delta^* \) to both sides and using Lemma 5.14 (3) we get,

\[
(\Theta_A(w, a) \cap \Theta_A(e, e)) \forall \lambda \leq
(\Theta_A(a, e) \cap \Theta_A(e, e)) \forall \lambda
\]

This gives (a).

(b) Assume \( \Theta_A(w, a) \leq \bigvee_{i \leq m} \Theta_A(a, w_i \bullet e_i) \forall \lambda \)

We prove that

\[
\Theta_A(w, a) \cap \Theta_A(e, e) \leq \bigvee_{i \leq m} (\Theta_A(a, w_i \bullet e_i) \cap \Theta_A(e, e)) \forall \lambda
\]

By induction on \( m \).

\( m = 1 \) is just part (a).

Assume \( m > 1 \).
Taking \( \Delta \) to be \( \Theta_A(c_m, e_m) \) \( \forall k \geq 3 \), we have by the induction hypothesis

\[
\Theta_A(c_m, e_m) \vee \left( \bigvee_{1 \leq m-1} \Theta_A(c_{m-1}, e_{m-1}) \right) \]

Consider any \( \langle x, y \rangle \in \Theta_A(c_m, e_m) \). If \( \Theta_A(x, y) \leq \Theta_A(c_m, e_m) \), then

\[
\Theta_A(x, y) \leq \Theta_A(c_m, e_m) \land \beta.
\]

So \( \Theta_A(x, y) \leq \Theta_A(c_m, e_m) \land \beta \). So

\[
\Theta_A(x, y) \leq (\Theta_A(c_m, e_m) \land \Theta_A(c_m, e_m)) \land \beta
\]

But \( \Theta_A(x, y) \leq \Theta_A(c_m, e_m) \). So

\[
\Theta_A(x, y) \leq (\Theta_A(c_m, e_m) \land \Theta_A(c_m, e_m)) \land \beta
\]

This holds for every \( \langle x, y \rangle \in \Theta_A(c_m, e_m) \). Hence, finally,

\[
\Theta_A(c_m, e_m) \land \Theta_A(c_m, e_m) \leq \left( \bigvee_{1 \leq m} \Theta_A(c_{m-1}, e_{m-1}) \right) \land \beta
\]

Taking \( \Delta = \Delta_A \), we get \( \text{PART}(e) \).

Let $K = \{ \Theta_A(x) : x \in \omega, \beta \in \gamma \}$ be upward-directed. Because if

$\Theta_A(x), \Theta_A(y) \in K$, then $\Theta_A(x) \cup \Theta_A(y) \in \Theta_A(x \cup y)$.

Thus, $UK \in Co(A)$. $x \in \beta \cup \gamma \Rightarrow \Theta_A(x) \subseteq \Theta_A(\beta \cup \gamma) = \beta \cup \gamma$. Clearly

$\beta \cup \gamma \subseteq UK$, so $\beta \cup \gamma \subseteq UK$. Hence

$\beta \cup \gamma = UK$.

Let $\langle a, \psi \rangle \in \chi \wedge (\beta \vee \gamma)$. Then

$\langle a, \psi \rangle \in \chi$ and $\langle a, \psi \rangle \in \beta \vee \gamma = UK$.

Let $x \in K$ such that $\langle a, \psi \rangle \in \Theta_A(x)$.

Let $X = \{ \langle c, d \rangle \cup \langle e, \delta \rangle : u \leq m \} \cup \{ \langle e, \delta \rangle : u \leq m \}$.

With $x_\beta \subseteq \beta$ and $x_\gamma \subseteq \gamma$.

$\Theta_A(a, \psi) \in \Theta_A(x_\beta) \cup \Theta_A(x_\gamma)$, directly.

From 1-8, we get

$\Theta_A(a, \psi) = \Theta_A(x_\beta) \cap \Theta_A(x_\gamma) \subseteq$

$\left( \bigvee_{u \leq m} \Theta_A(a, \psi) \cap \Theta_A(x_\beta) \right) \cap \left( \bigvee_{u \leq m} \Theta_A(a, \psi) \cap \Theta_A(x_\gamma) \right) \subseteq \beta \cup \gamma$.

So $\langle a, \psi \rangle \in (\beta \cup \gamma) \cap (\alpha \cap \gamma)$. Thus

$\chi \cap (\beta \cup \gamma) \subseteq (\beta \cup \gamma) \cap (\alpha \cap \gamma)$.
#3  (10)  By Birkhoff's subdirect product theorem for every Z-algebra A we have A = ∨{ B_i | i ∈ I } where the B_i are subdirectly irreducible (SDI). The algebras B_i, i ∈ I, are called the SDI factors of A. For any class of Z-algebras K, let F(K) be the class of SDI factors of all A ∈ K.

Claim: HSP(K) = HSP_{SDI} F(K).

Proof: By Birkhoff's theorem, K ≤ P_{SDI} F(K). So HSP(K) ≤ HSP_{SDI} F(K) ≤ HSP_{SDI} F(K). F(K) ≤ H(K), so HSP_{SDI} F(K) ≤ HSP_{SDI} H(K) = HSP(K) ⊑ Claim.

Here are some additional facts that were established in the lectures and that we will need.

(*)  HSP_{SDI} (mod_{p} φ(ι)) ≤ mod_{p} φ(ι)

This is an immediate consequence of Lem 6.3.

(**)  HSP (1 mod (ι)) ≤ 1 mod (ι).

In Theorem 6.4 we proved that 1 mod_{p} φ(ι) ∩ V is closed under H and S.

(***)  is much easier to prove.
(***) \( P_{SD} (MOD(I) \cap V) \subseteq MOD_{PHM}(I) \cap V \).

In the proof of Lem 6.3 we proved that if \( A \) is congruence-distributive and \( A \equiv \cup_{i \in I} B_i \) with \( B \equiv SDI \) and \( B_i \equiv MOD(I) \cup \cup_i I \) then \( A \equiv MOD_{PHM}(I) \). An examination of the proof shows that the condition that the \( B \) are SDI is not used in this part of proof. This gives (***)

Johnson's Lemma (Thm 5.25) says that for every class \( K \) of \( \mathbb{Z} \)-MoritaTE such that \( HSP(K) \) is congruence-distributive (***) \( HSP(K) = P_{SD} \ HSP_{SD}(K) \).

Now for the solution of the problem.

\( HSP(MOD_{PHM}(I) \cap V) \)

Claim \( HSP(F_{SDI}(MOD_{PHM}(I) \cap V)) \)

Claim \( HSP(MOD(I) \cap V) \)

Claim \( P_{SD} \ HSP_{SD}(MOD(I) \cap V) \)

Part (1) \( P_{SD} HSP(MOD(I) \cap V) \)

Part (2) \( P_{SD} \ HSP_{SD}(MOD(I) \cap V) \)

\( MOD_{PHM}(I) \cap V \).
\[ \text{So } HSP(\text{mod}(\mathcal{F})) \subseteq \text{mod}_{\text{ ultr}}(\mathcal{F}) \] 

The inclusion in opposite direction is proved as in proof of Thm 6.4.

(a) Let \( \mathcal{B} = (TTA_i)/\Phi(\mathcal{Y}) \) with \( \Phi \) an ultraproduct and assume \( A_i \in \text{mod}(\mathcal{F}) \). Then for \( \forall \Phi(\mathcal{Y}) \), \( \{ i \in I : A_i \equiv \Phi \} = I \in \mathcal{Y} \).

Every \( \mathcal{Y} \) is a disjunction of equations or logical negations of equations (where there are no logical negations). So by Problem 1 of Problem Set 2, \( \mathcal{B} \equiv \Phi \). So \( \mathcal{B} \in \text{mod}(\mathcal{F}) \). Thus \( \mathcal{B} \in \text{mod}(\mathcal{F}) \).

4. An equation \( \mathcal{E} \equiv \bot \) is absorbing if either \( \mathcal{E} \) or \( \bot \) is a variable.

Assume \( \mathcal{E} \) is a set of \( \mathcal{Z} \)-equations none of which is absorbing. We want to show \( \mathcal{E} \) is consistent.

We do this two different ways.

We prove that for any \( \text{var}^\mathcal{E} \) and term \( \mathcal{E} \)

\[ E \vdash x \equiv \mathcal{E} \Rightarrow \mathcal{E} \equiv x. \]

Suppose \( E \vdash x \equiv \mathcal{E} \). Then \( x \equiv \mathcal{E} \)

Then either \( \mathcal{E} \) is \( x \) or \( \top \) \( \mathcal{E} \) \( \bot \) \( \mathcal{E} \) such that

\[ x \equiv E \mathcal{E}_1 \equiv E \mathcal{E}_2 \equiv E \mathcal{E}_3 \equiv E \mathcal{E} \mathcal{E}_4 = \mathcal{E} \]
$X \in E \cup \iota$ implies there is a $\hat{U}(9) = N(9) \in E \cup \iota$ and a sequence of terms $\hat{V}$ such that $\hat{U}_1$ is obtained from $X$ by replacing a subterm $CF X$ or the form $\hat{V}(9)$ by $N(9)$. But this is impossible; $\hat{U}(9)$ is not a variable by assumption. So $\hat{U}(\hat{V})$ is not a variable. But $X$ has only itself as a subterm. Thus $\hat{V}$ must be $X$.

So $E \not\models X$ if $X, Y$ are different variables and hence $E$ is consistent.

We give the alternative model-theoretic $E$ by constructing a two-element model of $E$. Let $A = (0, 1)$ for each $\iota \in \Sigma$. Define $\mathcal{J}_A : A^m \rightarrow A$ by setting $\mathcal{J}_A(\alpha) = 0$ if $\alpha \in A^m$. Consider any nontrivial $E$-formula $\mathcal{J}(\alpha)$. Then $\mathcal{J}(\alpha) = \sigma(\mathcal{J}_A(\alpha), \ldots, \mathcal{J}_A(\alpha))$ for some $\sigma \in \Sigma^m$.

Thus $\forall \alpha \in A^m \quad \mathcal{J}_A(\alpha) = \sigma(\mathcal{J}_A(\alpha), \ldots, \mathcal{J}_A(\alpha)) = 0$. Consequently, $\forall \mathcal{J}(\alpha) \models \mathcal{J}(\alpha)$ in $E$ and $\forall \alpha \in A^m$ $\mathcal{J}_A(\alpha) = 0 = \mathcal{J}_A(\alpha)$. So $A \models \mathcal{J}(\alpha, \mathcal{J}_A)$ is a nontrivial model of $E$. 


#5. Let \( F = F_{\omega_1}(\mathbb{Z}) \) be the free algebra over \( \mathbb{Z} \) with the free generators \( \hat{F} = \langle x_0^F, \ldots, x_m^F \rangle \). Since \( F \) is finitely generated, it is finite. Let \( m = |F| \). Choose terms

\[\mathcal{X}_0(x), \ldots, \mathcal{X}_{m-1}(x) \in \mathcal{T}_{\omega_0}(\mathbb{Z})\] such that \( \mathcal{X}_0^F(\hat{F}), \ldots, \mathcal{X}_{m-1}^F(\hat{F}) \) is a enumeration without repetitions of all the elements of \( F \). Assume also the first \( m \) are the variables \( x_0, \ldots, x_{m-1} \), i.e., \( \mathcal{X}_i(x) = x_i \) for each \( i < m \). Assume also that all constant symbols in \( \mathbb{Z}_0 \) occur among the \( \mathcal{X}_i(x) \).

Let \( \mathcal{T} \in \mathcal{T}_{\omega_0} \) be the multiplication table for \( \mathcal{F} \). We mean the set of all equations of the form

\[\mathcal{T}(\mathcal{X}_i(x), \ldots, \mathcal{X}_j(x)) = \mathcal{X}_j(x)\]

where \( \mathcal{X}_i(x), \ldots, \mathcal{X}_j(x) \in \{0, \ldots, m-1\} \). And \( j \) is the unique \( j < m \) such that

\[\mathcal{T}_j^F(\hat{F}) = \mathcal{T}_j^F(\mathcal{X}_i^F(\hat{F}), \ldots, \mathcal{X}_j^F(\hat{F}))\]

Let \( E_m \) be the union of these multiplication tables as \( \mathcal{T} \) ranges over \( \mathcal{T}_{\omega_0} \). We show that

\[\forall m = \text{mod} \left( \left| E_m \right| \right)\]

Since \( E_m \) is finite, this solves problem.
We first note that $E_m \subseteq \text{Id}_m$.

Each equation in $E_m$ is an identity of $E_m$ because by Def 3.21 and Thm 3.25 if $E$ is a free algebra over $Y$ with free generators $x_0, \ldots, x_{m-1}$, then $x(x) = \lambda(x)$ is an identity of $Y$ if $E(xE) = \lambda (xE)$. Thus $V_m = \text{Mod} (\text{Id}_m(Y)) \subseteq \text{Mod} (E_m)$.

To show $\text{Mod} (E_m) \subseteq V_m$ it suffices to show that

\((*)\) $E_m \vdash E$ for each $E \in \text{Id}_m(Y)$

For this purpose we prove that

\((**)*\) \(\forall \lambda(x) \in T_{E_m}^{E_m}(x) \exists \lambda \leq m \text{ such that } E_m \vdash \lambda(x) = \lambda(x)\)

The proof goes by structural induction.

Suppose $\lambda(x)$ is a variable or constant symbol. Then $\lambda(x)$ is actually one of the $T_{E_m}^{E_m}(x)$ and so $\lambda(x) = \lambda(x)$ is a tautology. Assume now that

$\lambda(x) = \sigma(\lambda_1(x), \ldots, \lambda_n(x))$. By the induction hypothesis $\forall \lambda_1, \ldots, \lambda_n \leq m$ such that $E_m \vdash \lambda_i(x) = \lambda_i(x) \forall x \in A$.

Thus $E_m \vdash \sigma(\lambda_1(x), \ldots, \lambda_n(x))$ by (refl).

But $\sigma(\lambda_1(x), \ldots, \lambda_n(x)) = \lambda(x) \in E_m$ for some $\lambda \leq m$. So
So $E_m \vdash \lambda (x) = \xi_1 (x)$ by (2). This gives (\textit{a}).

Suppose $E \in \text{Id}_m (V)$. Let $E = (\eta (x) = \nu (x))$.

Since $E$ is a model of $E_m$,

$\xi_1^E (\xi_1 E) = \eta^E (\xi_1 E) = \nu^E (\xi_1 E) = \xi_2^E (\xi_1 E)$.

Thus $\xi_1 = \xi_2$ and $E \vdash E$ by (3) by (2).

So (\textit{a}) holds and $\text{mod} (E_m) \leq \text{mod} (E_1 (V))$.

This gives part (\textit{a}).

\textbf{(b): Assume $V$ is finitely based.}

Let $E$ be a base for the identities of $V$. Then clearly $E \subseteq E_m$ for some $m$. Thus $V = \text{mod} (E) \leq \text{mod} (E_m) = V_m$. So $V = V_m$.

Assume $V$ is locally finitely based and $V = V_m$. Then $V$ is finitely based since $V_m$ is.