Math 589, Spring 2002 **Problem Set 4 (Addtional Hints)**

1. (a) Use the following exercise that was given after Lemma 5.14 and eventually proved in class: For any epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and any $X \subseteq A^2$,

$$h^*\big(\Theta_{\boldsymbol{A}}(X)\big) = \Theta_{\boldsymbol{B}}\big(h(X)\big).$$

You will also have to use part (3) of Lemma 5.14 (I said 5.15 in the original hint, but this was a mistake) 5.14(3) says that for any epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and every $\alpha \in \mathrm{Co}(\boldsymbol{A})$,

$$h^{-1}h^*(\alpha) = \alpha \vee \mathrm{rker}(h).$$

(b) This part is proved by induction on n, but it is not a straightforward induction. First of all you do have to prove the stronger result

$$\Theta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i\leq n} \Theta_{\boldsymbol{A}}(a_i,b_i) \vee \alpha \quad \text{implies} \quad \Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\leq n}\big(\Theta_{\boldsymbol{A}}(a_i,b_i) \cap \Theta_{\boldsymbol{A}}(e,f)\big) \vee \alpha.$$

By the induction hypothesis we get

$$\Theta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i\leq n-1} \Theta_{\boldsymbol{A}}(a_i,b_i) \vee \big(\Theta_{\boldsymbol{A}}(a_n,b_n) \vee \alpha\big)$$

$$\text{implies} \quad \Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\leq n-1} \big(\Theta_{\boldsymbol{A}}(a_i,b_i) \cap \Theta_{\boldsymbol{A}}(e,f)\big) \vee \big(\Theta_{\boldsymbol{A}}(a_n,b_n) \vee \alpha\big).$$

Now consider any set $X \subset A^2$ such that $\Theta_{\boldsymbol{A}}(X) = \Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f)$ (can take X to be all of $\Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f)$). Use part (a) to prove that, for each $\langle x,y\rangle \in X$,

$$\Theta(x,y) \subseteq \bigvee_{i\leq n-1} \big(\Theta_{\boldsymbol{A}}(a_i,b_i) \cap \Theta_{\boldsymbol{A}}(e,f)\big) \vee \big((\Theta_{\boldsymbol{A}}(a_n,b_n) \cap \Theta_{\boldsymbol{A}}(e,f)) \vee \alpha\big).$$

Finally, use this to get

$$\Theta_{\boldsymbol{A}}(c,d) \cap \Theta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i\leq n} \big(\Theta_{\boldsymbol{A}}(a_i,b_i) \cap \Theta_{\boldsymbol{A}}(e,f)\big).$$

3. (b) By Birkhoff's Subdirect Product Theorem, for every Σ-algebra $\boldsymbol{A}$ we have $A \cong ; \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$ where the $\boldsymbol{B}_i$ are subdirectly irreducible (SDI). The algebras $\boldsymbol{B}_i$, $i \in I$, are called the SDI *factors* of $\boldsymbol{A}$. For any class K of Σ-algebras, let $\mathbf{F}_{\mathrm{SDI}}(\mathsf{K})$ be the class of SDI factors of all $\boldsymbol{A} \in \mathsf{K}$. Prove that, for any class K is Σ-algebras,

$$\mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K}) = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathbf{F}_{\mathrm{SDI}}(\mathsf{K}).$$

The following facts were all either established in the lectures at various places, or they are easy consequences of results that were established. You may use them in the proof of part (b) without justifying them.

$$\mathbf{F}_{\mathrm{SDI}}\big(\mathsf{Mod}_{\mathrm{prim}}(\Phi) \cap \mathsf{V}\big) \subseteq \mathsf{Mod}(\Phi) \cap \mathsf{V}$$
$$\mathbf{H}\,\mathbf{S}\big(\mathsf{Mod}(\Phi)\big) \subseteq \mathsf{Mod}(\Phi)$$
$$\mathbf{P}_{\mathrm{SD}}\big(\mathsf{Mod}(\Phi) \cap V\big) \subseteq \mathsf{Mod}_{\mathrm{prim}}(\Phi) \cap V.$$

Use the the above facts together with Jońsson's Lemma to show that $\mathbf{H}\,\mathbf{S}\,\mathbf{P}\big(\mathsf{Mod}_{\mathrm{prim}}(\Phi) \cap V\big) \subseteq \mathsf{Mod}_{\mathrm{prim}}(\Theta) \cap K$. Part (b) now follows as in the proof of Thm. 6.4.