Math 589, Spring 2002 **Problem Set 4** **Due April 15, 2002**

Corrected May 7, 2002

1. Let V be a variety with EDPM and let $\boldsymbol{A} \in V$.

   (a) Prove that, for all $a, b, c, d, e, f \in A$ and every $\alpha \in \mathrm{Co}(\boldsymbol{A})$,
   $$\varTheta_{\boldsymbol{A}}(c,d) \subseteq \varTheta_{\boldsymbol{A}}(a,b) \vee \alpha \quad \text{implies} \quad \big(\varTheta_{\boldsymbol{A}}(c,d) \cap \varTheta_{\boldsymbol{A}}(e,f)\big) \subseteq \big(\varTheta_{\boldsymbol{A}}(a,b) \cap \varTheta_{\boldsymbol{A}}(e,f)\big) \vee \alpha.$$

   (b) Prove that, for all $a_1, b_1, \dots, a_n, b_n, c, d, e, f \in A$,
   $$\varTheta_{\boldsymbol{A}}(c,d) \subseteq \bigvee_{i \le n} \varTheta_{\boldsymbol{A}}(a_i,b_i) \quad \text{implies} \quad \varTheta_{\boldsymbol{A}}(c,d) \cap \varTheta_{\boldsymbol{A}}(e,f) \subseteq \bigvee_{i \le n} \big(\varTheta_{\boldsymbol{A}}(a_i,b_i) \cap \varTheta_{\boldsymbol{A}}(e,f)\big).$$

   [*Hint:* Part (a): Let $\boldsymbol{B} = \boldsymbol{A}/\alpha$ and $\varDelta_\alpha\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ be that natural map; let $\bar{x} = x/\alpha$ for each $x \in A$. Show that $\varDelta^*_\alpha\big(\varTheta_{\boldsymbol{A}}(c,d) \cap \varTheta_{\boldsymbol{A}}(e,f)\big) = \varTheta_{\boldsymbol{B}}(\bar{c},\bar{d}) \cap \varTheta_{\boldsymbol{B}}(\bar{e},\bar{f})$. Use this to show that $\varTheta_{\boldsymbol{B}}(\bar{c},\bar{d}) \cap \varTheta_{\boldsymbol{B}}(\bar{e},\bar{f}) \subseteq \varTheta_{\boldsymbol{B}}(\bar{a},\bar{b}) \cap \varTheta_{\boldsymbol{B}}(\bar{e},\bar{f})$. Now apply $(\varDelta^*_\alpha)^{-1}$ and use Lemma 5.14. Prove part (b) by induction on n using part (a).]

2. Prove that every variety with EDPM is congruence distributive.

   [*Hint:* Let V a variety with EDPM. It suffices to prove that, for every $\boldsymbol{A} \in V$ and all $\alpha, \beta, \gamma \in \mathrm{Co}(\boldsymbol{A})$,
   $$\alpha \cap (\beta \vee \gamma) \subseteq (\alpha \cap \beta) \vee (\alpha \cap \gamma). \tag{1}$$
   Use the first problem to prove that if $\{\langle c_1, d_1\rangle, \dots, \langle c_n, d_n\rangle\}$ is a finite subset β and $\{\langle e_1, f_1\rangle, \dots, \langle e_m, f_m\rangle\}$ is a finite subset γ, then for all $a, b \in A$,
   $$\begin{aligned} &\varTheta_{\boldsymbol{A}}(a,b) \subseteq \Big(\bigvee_{i \le n} \varTheta_{\boldsymbol{A}}(c_i,d_i)\Big) \vee \Big(\bigvee_{j \le m} \varTheta_{\boldsymbol{A}}(e_j,f_j)\Big) \\ &\qquad \text{implies} \quad \varTheta_{\boldsymbol{A}}(a,b) \subseteq \Big(\bigvee_{i \le n} \big(\varTheta_{\boldsymbol{A}}(c_i,d_i) \cap \varTheta_{\boldsymbol{A}}(a,b)\big)\Big) \vee \Big(\bigvee_{j \le m} \big(\varTheta_{\boldsymbol{A}}(e_j,f_j) \cap \varTheta_{\boldsymbol{A}}(a,b)\big)\Big). \end{aligned}$$
   Use this to prove (1).]

3. Let $\varPhi$ be a set of UDE's. Recall that $\mathsf{P}_{\mathrm{U}}(\mathsf{K})$ is the class of all isomorphic images of ultraproducts of systems of algebras in K.

   (a) Prove that $\mathsf{P}_{\mathrm{U}}\,\mathsf{Mod}(\varPhi) = \mathsf{Mod}(\varPhi)$.

   (b) Use part (a) to obtain short proof of Theorem 6.4 from Jońsson's Lemma (Theorem 5.25). Recall that Theorem 6.4 says that, for any congruence-distributive variety V,
   $$\mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathsf{Mod}(\varPhi) \cap \mathsf{V}) = \mathsf{Mod}_{\mathrm{prim}}(\varPhi) \cap \mathsf{V}.$$

[*Hint:* Part (a) is a very easy consequence of the first problem on Problem Set #2. The key to part (b) is to show, as in the proof of Theorem 6.4, that $\mathsf{Mod}_{\mathrm{prim}}(\Phi) \cap \mathsf{V}$ is variety. Jonsson's Lemma can be used to get a simpler proof of this than the one used in the proof of Theorem 6.4.]

(over)

4. An equation is said to be *absorbing* if it is of the form $t(x_0, \dots, x_{n-1}) \approx x_i$ for some $i < n$. For example, $(x \cdot y) \cdot y^{-1} \approx x$ is an absorbing equation. Prove that if a set E has no absorbing equations, then E is consistent.

   [*Hint:* You can either argue directly that E has a nontrivial model, or you can obtain this indirectly by showing the $E \nvdash x \approx y$; to show the latter you may find it convenient to use the relation $\equiv^*_E$, in particular Theorem 4.15 from the notes for week 4 on the class webpage.]

5. Let V be a variety with a finite signature Σ. For each $n \in \omega$, let $\mathrm{Id}_n(V)$ be the set of all identities of V that contain at most n distinct variables, i.e., $\mathrm{Id}_n(V) = \mathrm{Id}(V) \cap \mathrm{Te}_\Sigma(\{x_0, \dots, x_{n-1}\})^2$. Let $V_n = \mathsf{Mod}\big(\mathrm{Id}_n(V)\big)$.

   (a) Prove that, if V is locally finite, then V_n is finitely based for every $n \in \omega$

   Since $\mathrm{Id}_0(V) \subseteq \mathrm{Id}_1(V) \subseteq \mathrm{Id}_2(V) \subseteq \cdots \subseteq \mathrm{Id}(V)$ and $\mathrm{Id}(V) = \bigcup_{n\in\omega} \mathrm{Id}_n(V)$, we have $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V$ and $V = \bigcap_{n\in\omega} V_n$.

   (b) Prove that if V is finitely based, then $V = V_n$ for some $n \in \omega$, and that the converse holds if V is locally finite.

   [*Hint:* Part (a): Let $\hat{x} = \langle x_0, \dots, x_{n-1} \rangle$. Show that there is a system $t_1(\hat{x}), \dots, t_m(\hat{x})$ of terms in n-variables such that for every term $s(\hat{x})$ there is an $i \leq m$ such that $s(\hat{x}) \approx t_i(\hat{x})$ is an identity of V_n (i.e., of V); without loss of generality one can assume that the first n terms in this sequence are the variables $x_0, \dots, x_{n-1}$. To show such a sequence exists use the fact that, since V is locally finite, the free algebra $\mathbf{Fr}_n(V)$ over V is finite. It follows that, for every $\sigma \in \Sigma_k$ and every sequence $t_{i_1}(\hat{x}), \dots, t_{i_k}(\hat{x})$ there exists a $t_j(\hat{x})$ such that $\sigma\big(t_{i_1}(\hat{x}), \dots, t_{i_k}(\hat{x})\big) \approx t_j(\hat{x})$ is an identity of V_n. Show that the set of all such equations is a finite base for V_n.]